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EXISTENCE RESULTS FOR NONLINEAR SECOND–ORDER q–DIFFERENCE EQUATIONS WITH q–INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we present some new existence and uniqueness results for the q-integral boundary value problem of nonlinear q-difference equation. Our results are based on Banach's contraction principle and Krasnowselskii's fixed point theorem. An example is given to illustrate the advantage of our results.

1. Introduction

The analysis of q-calculus was initiated at the beginning of the last century by Jackson [1] and was developed in intensive work especially by Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5], etc. From that time onwards, this theory has evolved in various research disciplines and their applications. For some recent works on q-difference equations, we refer the reader to [9]-[21] and references cite therein.

Ahmad et al. [15] proposed the boundary value problem of nonlinear second order q-difference equations with nonlocal boundary conditions given by

$$\begin{cases} D_q^2 x(t) = f(t,x(t)), \ \ t \in [0,T], \\ \alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \\ \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2), \end{cases} \tag{1.1}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and $q \in (0,1)$ is a constant. By employing Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative, Ahmad et al derived the existence of solutions of such a problem.

Later, Yu and Wang [18] proposed the boundary value problem of nonlinear second order q-difference equation given by

$$\begin{cases} D_q^2 u(t) + f(t,u(t),D_q u(t)) = 0, \ \ t \in [0,T], \\ D_q u(0) = 0, \ \ D_q u(1) = \alpha u(1), \end{cases} \tag{1.2}$$

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where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and $\alpha \neq 0$ is a fixed constant. They discussed the existence and uniqueness of solution of this problem by using Banach's contraction principle, the Leray-Schauder nonlinear alternative and Leray-Schauder continuation theorem.

In this article, we consider the following the q-integral boundary value problem of nonlinear second order q-difference equation provide by

$$\begin{cases} D_q^2 x(t) = f(t, x(t)), & t \in [0,T], \\ x(0) = x(T), & \displaystyle\int_0^T x(s) d_p s = 0, \end{cases} \tag{1.3}$$

where $0 < p, q < 1$ and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$.

Our overall goal is to prove an existence and uniqueness of solutions of the problem 1.3 by employing Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 2, we briefly discuss about the basic definitions, some properties of q-difference and present a lemma that will be used throughout the paper. In Sections 3 and 4, we give the main results and an example, respectively.

2. Preliminaries

The basic definitions and some properties of q-calculus [6]-[8] are as follows.

DEFINITION 1. For $0 < q < 1$, we define *the q-derivative* of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

The higher order q-derivatives are given by

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \ \ n \in \mathbb{N}.$$

where $D_q^0 f(t) = f(t)$.

The definite q-integral of a function f defined on the interval $[0,T]$ is given by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n)$$

where last term is convergent series.

If $a \in [0,T]$, then

$$\int_a^b f(s) d_q s = I_q f(b) - I_q f(a) = (1-q) \sum_{n=0}^{\infty} q^n \left[b f(bq^n) - a f(aq^n) \right].$$

We note that

$$D_q I_q f(x) = f(x),$$

while if f is continuous at $x=0$, then

$$I_q D_q f(x) = f(x) - f(0).$$

The property of product rule and integration by parts formula are

$$D_q(gh)(t) = (D_q g(t))h(t) + g(qt)D_q h(t),$$

$$\int_0^x f(t)D_q g(t)d_q t = \Big[f(t)g(t)\Big]_0^x - \int_0^x D_q f(t)g(qt)d_q t.$$

For reversing the order of integration is given by

$$\int_0^t \int_0^s f(r)d_q r d_q s = \int_0^t \int_{qr}^t f(r)d_q s d_q r.$$

In the limit $q \to 1$ the above results correspond to their counterparts in standard calculus.

LEMMA 1. *Let $0 < p,q < 1$. Then for any $h \in C[0,T]$, the boundary value problem*

$$D_q^2 x(t) = h(t), \qquad t \in [0,T], \tag{2.1}$$

$$x(0) = x(T), \qquad \int_0^T x(s)d_p s = 0, \tag{2.2}$$

has a unique solution

$$x(t) = \int_0^t (t-qs)h(s)d_q s + \frac{T-(1+p)t}{(1+p)T}\int_0^T (T-qs)h(s)d_q s - \frac{1}{T}\int_0^T \int_0^s (s-qu)h(u)d_q u d_p s. \tag{2.3}$$

Proof. For $t \in [0,T]$, taking double q-integrating (2.1) from 0 to t, we get

$$x(t) = \int_0^t \int_0^s h(u)d_q u d_q s + c_1 t + c_2. \tag{2.4}$$

By changing the order of q-integration, we have

$$\begin{aligned} x(t) &= \int_0^t \int_{qu}^t h(u)d_q s d_q u + c_1 t + c_2 \\ &= \int_0^t (t-qs)h(s)d_q s + c_1 t + c_2. \end{aligned} \tag{2.5}$$

From (2.5), we take the p-integral of $x(t)$ from 0 to t, we have

$$\int_0^t x(s)d_p s = \int_0^t \int_0^s (s-qu)h(u)d_q u d_p s + c_1 \frac{t^2}{(1+p)} + c_2 t. \tag{2.6}$$

By substituting $t = T$ in (2.6) and employing the second condition of (2.2), we find that

$$\frac{T^2}{1+p}c_1 + Tc_2 = -\int_0^T \int_0^s (s-qu)h(u)d_q u d_p s. \tag{2.7}$$

In particular, for $t = 0$ and T in (2.5), we obtain

$$x(0) = c_2, \tag{2.8}$$

$$x(T) = \int_0^T (T-qs)h(s)d_q s + c_1 T + c_2. \tag{2.9}$$

Now, using the first condition of (2.2) with (2.8), (2.9); we have

$$c_1 = -\frac{1}{T}\int_0^T (T-qs)h(s)d_q s. \tag{2.10}$$

Substituting c_1 in (2.7), we get

$$c_2 = \frac{1}{1+p}\int_0^T (T-qs)h(s)d_q s - \frac{1}{T}\int_0^T \int_0^s (s-qu)h(u)d_q u d_p s. \tag{2.11}$$

Substituting c_1 and c_2 in (2.5), we obtain (2.3).
Therefore the proof is completes. □

3. Main results

To accomplish the main results, we transform the boundary value problem 1.3 into a fixed point problem. From Lemma 1, we let $\mathscr{C} = C([0,T],\mathbb{R})$ denote the Banach space of all functions x. The norm is defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. The operator $F : \mathscr{C} \to \mathscr{C}$ is define by

$$\begin{aligned}(Fx)(t) = &\int_0^t (t-qs)f(s,x(s))d_q s + \frac{T-(1+p)t}{(1+p)T}\int_0^T (T-qs)f(s,x(s))d_q s \\ &-\frac{1}{T}\int_0^T \int_0^s (s-qu)f(u,x(u))d_q u d_p s.\end{aligned}$$

For $u \in \mathscr{C}$ and $t \in [0,T]$. It should be noticed that the problem 1.3 has solutions if and only if the operator F has fixed points.

Now we are in position to establish the main results. Our first result is based on Banach's fixed point theorem.

THEOREM 1. *Assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying the conditions*

(H_1) $|f(t,x) - f(t,y)| \leqslant L|x-y|$, *for all* $t \in [0,T]$, $L > 0$ *and* $x, y \in \mathbb{R}$,

(H_2) $L\Omega < 1$,

where L is a Lipschitz constant, and

$$\Omega = T^2\left[\frac{1}{(1+q)}+\frac{1}{(1+p)(1+q)}+\frac{1}{(1+q)(1+p+p^2)}\right]. \tag{3.1}$$

Then the boundary value problem 1.3 has a unique solution.

Proof. Firstly, we set an operator $F:\mathscr{C}\to\mathscr{C}$ is defined by (3.1). Let

$$\sup_{t\in[0,T]}|f(t,0)| = M,$$

and choose a constant R satisfied

$$R \geqslant \frac{M\Omega}{1-L\Omega}.$$

Now we shall show that $FB_R \subset B_R$, where $B_R = \{x\in\mathscr{C} : \|x\| \leqslant R\}$. For any $x\in B_R$, we have

$$\begin{aligned}
&\|Fx\| \\
&= \sup_{t\in[0,T]}\left|\int_0^t (t-qs)f(s,x(s))d_qs + \frac{T-(1+p)t}{(1+p)T}\int_0^T (T-qs)f(s,x(s))d_qs\right. \\
&\quad \left. -\frac{1}{T}\int_0^T\int_0^s (s-qu)f(u,x(u))d_qud_ps\right| \\
&\leqslant \sup_{t\in[0,T]}\left\{\int_0^t (t-qs)(|f(s,x(s))+f(s,0)|+|f(u,0)|)d_qs\right. \\
&\quad +\frac{|T-(1+p)t|}{(1+p)T}\int_0^T (T-qs)(|f(s,x(s))+f(s,0)|+|f(s,0)|)d_qs \\
&\quad \left. +\frac{1}{T}\int_0^T\int_0^s (s-qu)(|f(u,x(u))+f(u,0)|+|f(u,0)|)d_qud_ps\right\} \\
&\leqslant \sup_{t\in[0,T]}\left\{(L\|x\|+M)\int_0^t (t-qs)d_qs + (L\|x\|+M)\frac{|T-(1+r)t|}{(1+r)T}\int_0^T (T-qs)d_qs\right. \\
&\quad \left. +\frac{1}{T}(L\|x\|+M)\int_0^T\int_0^s (s-qu)d_qud_ps\right\} \\
&= \sup_{t\in[0,T]}\left\{(L\|x\|+M)\frac{t^2}{1+q} + (L\|x\|+M)\frac{|T-(1+p)t|}{(1+p)T}\cdot\frac{T^2}{1+q}\right. \\
&\quad \left. +\frac{1}{T}(L\|x\|+M)\frac{T^3}{(1+q)(1+p+p^2)}\right\} \\
&= \sup_{t\in[0,T]}\left\{(L\|x\|+M)\left(\frac{t^2}{1+q}+\frac{T|T-(1+p)t|}{(1+p)(1+q)}+\frac{T^2}{(1+q)(1+p+p^2)}\right)\right\} \\
&\leqslant (LR+M)\left(\frac{T^2}{(1+q)}+\frac{T^2}{(1+p)(1+q)}+\frac{T^2}{(1+q)(1+p+p^2)}\right)\Bigg\} \\
&= (LR+M)\Omega \\
&\leqslant R.
\end{aligned}$$

Therefore, $FB_R \subset B_R$.

We next show that F is a contraction. For any $x, y \in \mathscr{C}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\| \\
&= \sup_{t\in[0,T]} \Big|(Fx)(t) - (Fy)(t)\Big| \\
&= \sup_{t\in[0,T]} \Bigg| \int_0^t (t-qs)\,|f(s,x(s)) - f(s,y(s))|\,d_qs \\
&\qquad + \frac{T-(1+p)t}{(1+p)T} \int_0^T (T-qs)\,|f(s,x(s)) - f(s,y(s))|\,d_qs \\
&\qquad - \frac{1}{T}\int_0^T \int_0^s (s-qu)\,|f(u,x(u)) - f(u,y(u))|\,d_qud_ps \Bigg| \\
&\leqslant \sup_{t\in[0,T]} \Bigg\{ L\|x-y\| \int_0^t (t-qs)d_qs + L\|x-y\| \frac{|T-(1+p)t|}{(1+p)T} \int_0^T (T-qs)d_qs \\
&\qquad + \frac{1}{T} L\|x-y\| \int_0^T \int_0^s (s-qu)d_qud_ps \Bigg\} \\
&\leqslant L\|x-y\| \left(\frac{T^2}{(1+q)} + \frac{T^2}{(1+p)(1+q)} + \frac{T^2}{(1+q)(1+p+p^2)} \right) \\
&= L\Omega\|x-y\|.
\end{aligned}$$

As $L\Omega < 1$, F is a contraction. Note that we complete this proof by using Banach's contraction mapping principle. □

Further, we consider the existence of a solution to the boundary value problem 1.3. We shall use the Krasnoselskii's fixed point theorem [22].

THEOREM 2. *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*

(i) $Ax + By \in K$ *whenever* $x, y \in K$

(ii) A *is compact and continuous*

(iii) B *is a contraction mapping.*

Then there exists $z \in K$ such that $z = Az + Bz$.

THEOREM 3. *Assume that (H_1) and (H_2) hold. In addition we suppose that:*

(H_3) $|f(t,x)| \leqslant \mu(t)$, *for all* $(t,x) \in [0,T] \times \mathbb{R}$, *with* $\mu \in C([0,T], \mathbb{R}^+)$.

If

$$\Omega < 1, \tag{3.2}$$

where Ω is given by (3.1), then the boundary value problem 1.3 has at least one solution on $[0,T]$.

Proof. We let $\sup_{t\in[0,T]} |\mu(t)| = \|\mu\|$ and choosing a constant

$$R \geqslant \|\mu\|\Omega, \tag{3.3}$$

we consider $B_R = \{x \in \mathscr{C} : \|x\| \leqslant R\}$.

In view of Lemma 1, we define the operators $\mathscr{F}_1$ and $\mathscr{F}_2$ on the ball B_R as

$$\begin{aligned}(\mathscr{F}_1 x)(t) &= \int_0^t (t-qs) f(s,x(s)) d_q s,\\ (\mathscr{F}_2 x)(t) &= \frac{T-(1+p)t}{(1+p)T}\int_0^T (T-qs) f(s,x(s)) d_q s\\ &\qquad - \frac{1}{T}\int_0^T \int_0^s (s-qu) f(u,x(u)) d_q u d_p s.\end{aligned}$$

For $x,y \in B_R$, we have

$$\begin{aligned}\|\mathscr{F}_1 x + \mathscr{F}_2 y\| &\leqslant \|\mu\| \int_0^t (t-qs) d_q s + \|\mu\| \frac{|T-(1+p)t|}{(1+p)T}\int_0^T (T-qs) d_q s\\ &\quad + \frac{\|\mu\|}{T}\int_0^T \int_0^s (s-qu) d_q u d_p s\\ &\leqslant \|\mu\|\Omega\\ &\leqslant R,\end{aligned}$$

which implies that $\mathscr{F}_1 x + \mathscr{F}_2 y \in B_R$. The condition (3.2) implies that $\mathscr{F}_2$ is a contraction mapping. Next, we will show that $\mathscr{F}_1$ is compact and continuous. The continuity of f together with the assumption (H_3) implies that an operator $\mathscr{F}_1$ is continuous and uniformly bounded on B_R. We let $\sup_{(t,x)\in[0,T]\times B_R} |f(t,x)| = N$. For $t_1, t_2 \in [0,T]$ with $t_1 \leqslant t_2$ and $x \in B_R$, we have that

$$\begin{aligned}|\mathscr{F}_1 x(t_2) - \mathscr{F}_1 x(t_1)| &= \left| \int_0^{t_2} (t_2-qs) f(s,x(s)) d_q s - \int_0^{t_1} (t_1-qs) f(s,x(s)) d_q s \right)\right|\\ &= \left| \int_0^{t_1} (t_2-t_1) f(s,x(s)) d_q s + \int_{t_1}^{t_2} (t_2-qs) f(s,x(s)) d_q s \right|\\ &\leqslant N \frac{|t_2^2 - t_1^2|}{(1+q)}.\end{aligned}$$

As $t_2 - t_1 \to 0$, the above inequality, which is independent of x, tends to zero. Thus, $\mathscr{F}_1$ is relatively compact on B_R. Hence, we can conclude by the Arzelá-Ascoli Theorem that $\mathscr{F}_1$ is compact on B_R. Therefore, all the assumptions of Theorem 3 are satisfied and the conclusion of Theorem 3 implies that the boundary value problem 1.3 has at least one solution on $[0,T]$. This completes the proof. □

4. Example

In this section, we give an example to illustrate our main results. Consider the following boundary value problem of nonlinear second-order q-difference equations with three-point boundary conditions

$$\begin{cases} D^2_{\frac{1}{2}} x(t) = \dfrac{e^{sin^2(\pi t)}|x(t)|}{(5+t)^2(1+|x(t)|)}, & t \in [0,2], \\ x(0) = x(2), \quad \displaystyle\int_0^2 x(s) d_{\frac{3}{4}} s = 0. \end{cases} \tag{4.1}$$

Set $q = 1/2$, $p = 3/4$ and $T = 2$, since

$$|f(t,x) - f(t,y)| \leqslant \frac{1}{25}|x-y|$$

then (H_1) is satisfied with $L = \frac{1}{25}$. Form Theorem 1, we find that

$$\Omega = 4\left[\tfrac{1}{(3/2)} + \tfrac{1}{(7/4)(3/2)} + \tfrac{1}{(3/2)(37/16)}\right] \approx 5.344.$$

Hence, we get

$$L\Omega \approx 0.214 < 1.$$

Therefore, by Theorem 1, problem (4.1) has at least one solution on $[0,2]$.

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