Differential
Equations
& Applications
Volume 8, Number 4 (2016), 411–427 doi:10.7153/dea-08-23

# ON THE RAYLEIGH–PLATEAU INSTABILITY. THE REGULARITY IN $H^3_{per}$

A. ALRIYABI

(*Communicated by Jean-Michel Rakotoson*)

*Abstract.* In this paper, we study the Rayleigh-Plateau instability of a cylindrical pore. We are interested in the model developed by Spencer et al. [20], Kirill et al. [12] and Boutat et al. [3] in absence of the stress. We obtain a nonlinear parabolic PDE of fourth order. We obtain the local existence and uniqueness of the solution of this problem. The global existence of the solution and the convergence to the mean value of the initial data for long time, represent the main results of this work. In this study, we give also a numerical tests in order to validate the theoretical results.

## 1. Introduction

It is known that a liquid jet, initially of constant radius, is falling vertically under gravity. The liquid length increases and reaches a critical value. At this moment the jet loses its cylindrical shape and it decomposes into a stream of droplets. This phenomenon occurs primarily as a result of surface tension. This effect of surface tension is called Rayleigh-Plateau instability [24, 25, 26]. All the streams fluid contain perturbations, which are small changes in a physical system (such as a stream). These perturbations are sometimes compounded into sinusoidal functions and appear as waves. In the lower section of the wave, where the radius of stream is smaller, the surface tensions creates a higher pressure. At the crest of the wave where the radius is larger, the pressure will be lower. As a result of the pressure difference the amplitude of some waves will increase. It will eventually align waves in what looks like a destructive interference patterns, however they do not cancel each other out. When the waves become large enough, stream will be in a bottleneck and spherical droplets will be formed. The droplets are spherical because the liquid is more stable at a lower energy level and by decreasing the surface area, number of higher energy molecules is decreased and thus lowers the droplets energy. The Rayleigh-Plateau instability quantifies this phenomenon and explains why and how a falling stream of fluid breaks up into smaller packets with the same volume but less surface area.

In this paper, we are interested on a cylindrical and axisymmetric crystalline structure in the case without stress. We study the morphological instabilities at cylindrical pore surface [20], [12]. Such phenomena are observed for example in materials science

*Mathematics subject classification* (2010): 26D15, 26A51, 32F99, 41A17.
*Keywords and phrases*: Rayleigh-Plateau instability, nonlinear parabolic PDE, thin films.

© ELEMENT, Zagreb
Paper DEA-08-23

[16], since the manufacture of materials introduced distributions of pores that affect their mechanical and physical properties. The evolution equation of the surface instability is written in the form (see [3]):

$$\frac{\partial h}{\partial t} = -\frac{1}{h}\frac{\partial}{\partial Z}\left[h\frac{\partial}{\partial Z}\left((1-\theta\ln(h))h_{ZZ} - \frac{\theta}{2}h^{-2}h_Z^2\right)\right], \tag{1.1}$$

where $\partial h/\partial Z$ (or $h_Z$), $h_{ZZ}$ are the partial derivatives of $h$ of order one and two, respectively; $\theta$ is a physical parameter ($\theta = \sigma_0^2/\gamma\mu$) which depends on the stress applied to material $\sigma_0$, on shear modulus $\mu$ and on the free energy of the surface $\gamma$. In this work, we consider the absence of stress case ($\sigma_0 = 0$), which is called the Rayleigh-Plateau instability

$$\frac{\partial h}{\partial t} = -\frac{1}{h}\frac{\partial}{\partial x}\left(h\,\frac{\partial^3 h}{\partial x^3}\right). \tag{1.2}$$

We are interested in the periodic problem

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\dfrac{1}{h}\dfrac{\partial}{\partial x}\left(h\,\dfrac{\partial^3 h}{\partial x^3}\right) \text{ on } (0,T)\times(0,1), \\ h(t,.) \text{ is a periodic function on } (0,1), \\ h(0,.) = h_0 > 0 \text{ is a periodic function given on } (0,1). \end{cases} \tag{1.3}$$

Arguments of longitudinal perturbation of small amplitude of the cylinder and numerical calculations are used in [7, 8, 2] to study the nonlinear equation which governs the morphological change of the cylinder surface. Stability analysis of cylinder surface using linear theory in the case of uniaxial stress is studied in [11, 12]. In [18, 19], the authors study the morphological stability of the surface of a pore under the action of the artificial tension (with zero stress), they show the transformation of cylindrical pores in the spheres and that the distance between the spheres depends on the diffusion of surface and on the diffusion of volume; they also show that the surface becomes unstable if the wavelength is larger than the cylinder circumference. In particular, they analyze the external evolution due to the perturbations of small amplitudes. Some recent interesting studies on thin film equations can be found in [9, 14, 15, 27, 28].

In this study, we use the model developed in [20], [12], the radius of cylinder satisfies a parabolic partial differential equation. Under assumptions of formal asymptotic and with a changes of appropriate scale, we simplify the PDE satisfied by the cylinder radius [3].

REMARK 1.1. *1. Equation (1.2) can be written in general form of evolution equations of thin films* [10]*:*

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}\left(v^n\frac{\partial^3 v}{\partial x^3} + \alpha v^{n-1}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \beta v^{n-2}(\frac{\partial v}{\partial x})^3\right), \tag{1.4}$$

*with* $n = 0,\ \alpha = -3/2,\ \beta = 3/4$ *and* $v = h^2$.

2. *We introduce, for* $m \in \mathbb{N}$ *the periodic Sobolev space* $H^m_{per}(0,1)$ *by:*

$$H^m_{per}(0,1) = \{u \in H^m(0,1); \quad \forall j \in \{0,1,\dots,m-1\}, \quad u^{(j)}(0) = u^{(j)}(1)\},$$

*where* $H^m(0,1)$ *is the usual Sobolev space. We also consider the functional space* $\mathscr{X}$ *defined by:*

$$\mathscr{X} = L^2\big(0,T;H^3_{per}(\Omega)\big) \cap L^\infty\big(0,T;H^1_{per}(\Omega)\big).$$

*We endow* $\mathscr{X}$ *with the norm:*

$$\| u \|_{\mathscr{X}} = \left[ \int_0^T \int_0^1 |u^{(3)}(t,x)|^2 \, dx\, dt + \sup_{t\in(0,T)} \left( \int_0^1 |u'(t,x)|^2 \, dx + \int_0^1 |u(t,x)|^2 \, dx \right) \right]^{\frac{1}{2}}.$$

*For simplicity, we shall assume that* $\Omega = (0,1)$, $\Omega_T = (0,T) \times (0,1)$.

DEFINITION 1.1. *We say that the solution* $h \in L^2\big(0,T;H^3_{per}(\Omega)\big)$ *of (1.3) is weak if:*

1. $\dfrac{1}{h}\dfrac{\partial h}{\partial x}\dfrac{\partial^3 h}{\partial x^3} \in L^1(\Omega_T)$.
2. $h_t \in L^2\big(0,T;(H^3_{per}(\Omega))'\big)$.
3. *For all* $\varphi \in H^3_{per}(\Omega)$, *we have, in* $\mathscr{D}'(0,T)$, *the following equality:*

$$\frac{d}{dt}\int_0^1 h(t,x)\varphi(x)dx = -\int_0^1 h_{xx}\varphi_{xx}\, dx - \int_0^1 \frac{h_x}{h} h_{xxx}\varphi\, dx. \tag{1.5}$$

4. $h(0) = h_0 > 0$ *(given in* $H^1_{per}(\Omega)$ *).*

LEMMA 1.1. *Suppose that* $h \in \mathscr{X}$, $h_0 \in \mathscr{C}[0,1]$, *and* $h_0 \geqslant \varepsilon > 0$. *Then, for all local weak solution* $h$ *of (1.5), there is a moment* $T_*(\varepsilon) > 0$, *where* $h(t,x) \geqslant \varepsilon/2, \forall t \in [0,T_*]$ *and* $for all x \in [0,1]$.

In equation (1.2), it is appropriate to add the hypotheses (see [3]):

$$R_0 \leqslant h \qquad \text{and} \qquad h/\ell \ll 1. \tag{1.6}$$

$R_0$, $\ell$ are respectively the radius and the length of the cylinder at the initial state. Thanks to lemma 1.1, we make the change of variable $h(t,x) = e^{u(t,x)}$. Equation (1.2) becomes under simplified form:

$$\frac{\partial u}{\partial t} = -u^{(4)} - (5u'u^{(3)} + 9u'^2u'' + 3u''^2 + 2u'^4), \tag{1.7}$$

where $u^{(k)}(t,x) = \dfrac{\partial^k u}{\partial x^k}(t,x)$. Therefore, we consider the boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u^{(4)} - F(u', u'', u^{(3)}) \text{ on } \Omega_T, \\ u(t,.) \text{ is a periodic function on } \Omega, \\ u(0,.) = u_0 = \ln h_0 \text{ is given a periodic function on } \Omega, \end{cases} \tag{1.8}$$

where

$$F(u', u'', u^{(3)}) = 5u'u^{(3)} + 9u'^2u'' + 3u''^2 + 2u'^4.$$

## 2. Local Existence

In this section, we prove the existence of solution of Problem (1.8) using Faedo-Galerkin method [13, 22].

DEFINITION 2.1. *A function $u$ defined on $[0,T] \times [0,1]$ is a weak solution of Problem (1.8) if:*

$$(\mathscr{P}_0) \begin{cases} 1.\ u \in L^2(0,T; H^3_{per}(\Omega)). \\ 2.\ \textit{For all } \varphi \in H^3_{per}(\Omega), \textit{ the following equality holds in } \mathscr{D}'(0,T) \\ \quad \dfrac{d}{dt}\displaystyle\int_0^1 u(t,x)\,\varphi(x)\,dx + \int_0^1 u''(t,x)\,\varphi''(x)dx + \int_0^1 F(u', u'', u^{(3)})\varphi(x)dx = 0. \\ 3.\ u(0,.) = u_0 \in H^1_{per}(\Omega) \ \textit{ and } \dfrac{\partial u}{\partial t} \in L^2(0,T; (H^3_{per}(\Omega))'). \end{cases}$$

THEOREM 2.1. *There exists a constant $\Lambda_0 > 0$, such that if $|u_0'|_{L^2(\Omega)} < \Lambda_0$, then the problem ($\mathscr{P}_0$) accept a unique global solution $u$ in $\mathscr{X}$, such that:*

$$|u_x(t)|_{L^2(\Omega)} \leqslant C_0\, e^{-\nu t}, \qquad t > 0, \tag{2.1}$$

*with $C_0 > 0$, $\nu > 0$ are constants which are independent of $u_0$.*

**Proof of Theorem 2.1**
In order to prove theorem 2.1, we proceed in three steps.

**First step**: In this step, we construct a function $u_m$ solving the approximated problem on a finite dimensional space.
Let $\{\varphi_j, j \in \mathbb{N}^*\}$ be the basis of $H^1_{per}(\Omega)$, satisfying:

1. $\varphi_j \in \mathscr{C}^\infty_{per}(\Omega) = \displaystyle\bigcap_{m \geqslant 1} H^m_{per}(\Omega)$.

2. $\forall \varphi \in H^1_{per}(\Omega)$; $\quad (\varphi_j', \varphi')_{L^2(\Omega)} = \mu_j(\varphi_j, \varphi)_{L^2(\Omega)}$; where the sequence $\mu_j$ converges to $+\infty$ when $j$ converges to $+\infty$ and satisfies $0 = \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_j \cdots$.

3. $(\varphi_j, \varphi_k)_{L^2(\Omega)} = \delta_{jk}$; where $\delta_{jk}$ is the Kronecker symbol.

We set $H_m$ the finite-dimensional space generated by $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. $P_m$ denotes the orthogonal projection of $L^2(\Omega)$ into $H_m$. By the Cauchy–Peano theorem, we can introduce a time $T^* > 0$ and a function $u_m \in \mathscr{C}^1\big([0,T^*]; H_m\big)$, such that, for all $\psi \in H_m$

$$\frac{d}{dt}\int_0^1 u_m(t,x)\psi(x)dx + \int_0^1 u_m''(t,x)\psi''(x)dx + \int_0^1 F(u_m', u_m'', u_m^{(3)})(t,x)\psi(x)dx = 0, \tag{2.2}$$

where

$$F(u_m', u_m'', u_m^{(3)})(t,x) = 5u_m'(t,x)u_m^{(3)}(t,x) + 9u_m'^2(t,x)u_m''(t,x) + 3u_m''^2(t,x) + 2u_m'^4(t,x),$$

with the initial data $u_m(0) = P_m(u_0)$.

**Second step :** In this step, we give a uniform estimates in $m$, in space, and in time We need the following lemmas:

LEMMA 2.1. *There are strictly positive constants* $\Lambda_0, c_0$, *and* $\mu$, *such that if the initial data satisfies* $\sup_{m\geqslant 1} |u_{0m}'(t)|_{L^2(\Omega)} < \Lambda_0$ *(say the norm in* $H_p^1 er$ *is small enough), then:*

$$|u_m'(t)|_{L^2(\Omega)} \leqslant c_0 e^{-\mu t}, \quad \forall t > 0. \tag{2.3}$$

*Proof.* In equation (2.2), we set $\psi = -u_m''$. An integration by parts gives:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u_m'(t,x)|^2\,dx + \int_0^1 |u_m^{(3)}(t,x)|^2\,dx = \underbrace{\int_0^1 F(u', u'', u_m^{(3)})u_m''\,dx}_{I}. \tag{2.4}$$

Where

$$I = \frac{1}{2}\int_0^1 u_m''^3(t,x)dx - 3\int_0^1 u_m'^3(t,x)u_m^{(3)}(t,x)dx := I_1 + I_2. \tag{2.5}$$

By interpolation inequalities, one has :

$$|u_m''(t)|^3_{L^3(\Omega)} \leqslant c_3 |u_m'(t)|^{\frac{5}{4}}_{L^2(\Omega)} |u_m^{(3)}(t)|^{\frac{7}{4}}_{L^2(\Omega)}.$$

Applying Young's inequality, we deduce that

$$|u_m''(t)|^3_{L^3(\Omega)} \leqslant \frac{c_3^8}{8}\big(|u_m'(t)|^{\frac{5}{4}}_{L^2(\Omega)}\big)^8 + \frac{7}{8}\big(|u_m^{(3)}(t)|^{\frac{7}{4}}_{L^2(\Omega)}\big)^{\frac{8}{7}}.$$

Hence:

$$|I_1| \leqslant c_4 |u_m'(t)|^{10}_{L^2(\Omega)} + \frac{7}{8}|u_m^{(3)}(t)|^2_{L^2(\Omega)} \tag{2.6}$$

On the other hand, by the Young's inequality, for all $\varepsilon > 0$, one has:

$$|I_2| \leqslant c_\varepsilon \int_0^1 |u_m'|^6(t,x)dx + \varepsilon \int_0^1 |u_m^{(3)}|^2(t,x)dx.$$

Applying interpolation inequalities, one deduce that

$$|I_2| \leqslant c_\varepsilon |u_m'(t)|^5_{L^2(\Omega)} |u_m^{(3)}(t)|_{L^2(\Omega)} + \varepsilon \int_0^1 |u_m^{(3)}|^2(t,x)dx.$$

Once again by Young's inequality, for all $\varepsilon > 0$, we have:

$$|I_2| \leqslant c_5 |u_m'(t)|^{10}_{L^2(\Omega)} + 2\varepsilon \int_0^1 |u_m^{(3)}|^2(t,x)dx. \tag{2.7}$$

Both inequalities (2.6) and (2.7) give:

$$|I| \leqslant c_6 |u_m'(t)|^{10}_{L^2(\Omega)} + (2\varepsilon + \frac{7}{8}) \int_0^1 |u_m^{(3)}|^2(t,x)dx.$$

Therefore, for $4\varepsilon < \dfrac{1}{8}$, one has:

$$\frac{d}{dt}|u_m'(t)|^2_{L^2(\Omega)} + \frac{1}{8}|u_m^{(3)}(t)|^2_{L^2(\Omega)} \leqslant c\,|u_m'(t)|^{10}_{L^2(\Omega)}. \tag{2.8}$$

By Poincaré inequality, there exist a constant $k > 0$, such that

$$k|u_m'(t)|^2_{L^2(\Omega)} \leqslant \frac{1}{8}|u_m^{(3)}(t)|^2_{L^2(\Omega)}.$$

By relation (2.8), we deduce that:

$$\frac{d}{dt}|u_m'(t)|^2_{L^2(\Omega)} + k\,|u_m'(t)|^2_{L^2(\Omega)} \leqslant c\,|u_m'(t)|^{10}_{L^2(\Omega)}. \tag{2.9}$$

Now, multiplying by $e^{kt}$, one has:

$$\frac{d}{dt}[e^{kt}|u_m'(t)|^2_{L^2(\Omega)}] \leqslant c\,e^{-4kt}[e^{kt}|u_m'(t)|^2_{L^2(\Omega)}]^5.$$

We set $Z(t) = e^{kt}|u_m'(t)|^2_{L^2(\Omega)}$ and we integrate between $0$ and $t$:

$$-Z^{-4}(t) + Z^{-4}(0) \leqslant -\frac{c}{k}(e^{-4kt} - 1) \leqslant \frac{c}{k}, \quad t > 0.$$

If $\sup\limits_{m\geqslant 1} |u_{0m}'|^8_{L^2(\Omega)} < \dfrac{k}{c} := \Lambda_0^8$, then there exist $c_0 > 0$ and $\mu = \frac{k}{2}$ such that:

$$|u_m'(t)|_{L^2(\Omega)} \leqslant c_0\, e^{-\mu t}, \quad \forall t > 0. \tag{2.10}$$

REMARK 2.1. *From (2.10), we deduce that $T^* = T$, that is $u_m$ is a global solution (because $c_0$ and $\mu$ are independent of $T^*$ and $m$).*

LEMMA 2.2. *There exist a constant $c(u_0) > 0$ independent of $T$, such that:*

$$\int_0^T \int_0^1 |u_m^{(3)}|^2(\tau,x)dxd\tau \leqslant c(u_0). \tag{2.11}$$

*Proof.*
We integrate the inequality (2.8) between 0 and $T$:

$$\int_0^1 |u_m'|^2(T,x)dx+\frac{1}{8}\int_0^T\int_0^1 |u_m^{(3)}|^2(\tau,x)dxd\tau \leqslant c\int_0^T |u_m'(\tau)|^{10}_{L^2(\Omega)}d\tau + \int_0^1 |(P_m u_0)'|^2dx. \tag{2.12}$$

Hence

$$\frac{1}{8}\int_0^T\int_0^1 |u_m^{(3)}|^2(\tau,x)dxd\tau \leqslant c\int_0^T e^{-10\mu\tau}d\tau+|u_0|^2_{H^1_{per}(\Omega)} \leqslant \frac{c}{10\mu}(1-e^{-10\mu\tau T})+|u_0|^2_{H^1_{per}(\Omega)}. \tag{2.13}$$

Therefore

$$\int_0^T\int_0^1 |u_m^{(3)}|^2(\tau,x)dxd\tau \leqslant c(u_0), \qquad \text{(independent of } T). \tag{2.14}$$

LEMMA 2.3. *There exists a contant $C(u_0)$ independent of $T$, such that:*

$$\left|\int_0^1 u_m(t,x)\,dx\right| \leqslant C(u_0), \qquad \forall t\in[0,T]. \tag{2.15}$$

*Proof.*
In (2.2), we take $\psi=1$:

$$\frac{d}{dt}\int_0^1 u_m(t,x)\,dx+\int_0^1 F(u_m'(t,x),u_m''(t,x),u_m^{(3)}(t,x))dx=0. \tag{2.16}$$

An integration by parts (2.16) gives:

$$\frac{d}{dt}\int_0^1 u_m(t,x)dx=2\int_0^1 u_m''^2(t,x)dx-2\int_0^1 u_m'^4(t,x)dx. \tag{2.17}$$

We integrate (2.17) between 0 and $t$:

$$\int_0^1 u_m(t,x)dx=2\int_0^t\int_0^1 u_m''^2(\tau,x)d\tau dx-2\int_0^t\int_0^1 u_m'^4(\tau,x)d\tau dx+\int_0^1 u_{0m}(x)dx. \tag{2.18}$$

From which:

$$\left|\int_0^1 u_m(t,x)dx\right| \leqslant 2\int_0^T\int_0^1 |u_m''|^2(\tau,x)d\tau dx+2\int_0^T\int_0^1 |u_m'|^4(\tau,x)d\tau dx + \left|\int_0^1 u_{0m}(x)dx\right|. \tag{2.19}$$

We have:

$$\int_0^T\int_0^1 |u_m''|^2(\tau,x)d\tau dx \leqslant \int_0^T\int_0^1 |u_m^{(3)}|^2(\tau,x)d\tau dx \leqslant c_1(u_0). \tag{2.20}$$

On the other hand, by the interpolation inequalities, one has:

$$|u_m'(t)|^4_{L^4(\Omega)} \leqslant c_1\, |u_m'(t)|^3_{L^2(\Omega)} |u_m''(t)|_{L^2(\Omega)}. \tag{2.21}$$

Applying the Young's inequality, we deduce:

$$|u_m'(t)|^4_{L^4(\Omega)} \leqslant \alpha |u_m'(t)|^6_{L^2(\Omega)} + c_\alpha |u_m''(t)|^2_{L^2(\Omega)}, \quad \forall \alpha > 0. \tag{2.22}$$

Hence:

$$\begin{aligned}\int_0^T |u_m'(\tau)|^4_{L^4(\Omega)} d\tau &\leqslant \alpha \int_0^T |u_m'(\tau)|^6_{L^2(\Omega)} d\tau + c_\alpha \int_0^T |u_m''(\tau)|^2_{L^2(\Omega)} d\tau \\ &\leqslant c(1-e^{-6\mu T}) + c_2(u_0) \leqslant c_3(u_0).\end{aligned} \tag{2.23}$$

Since:

$$\left| \int_0^1 (P_m u_0) dx \right| \leqslant \int_0^1 |P_m u_0|^2 dx \leqslant |u_0|_{L^2(\Omega)} < \infty, \tag{2.24}$$

therefore:

$$\left| \int_0^1 u_m(t,x) dx \right| \leqslant C(u_0), \qquad \forall t \in [0,T]. \tag{2.25}$$

REMARK 2.2. *we know from Poincare-Sobolev inequality (see for instance* [4, 21]*), that for all* $v \in H^1(\Omega)$*, we can write:*

$$|v|_{L^2(\Omega)} \leqslant c\Big( \left| \int_0^1 v dx \right| + |v'|_{L^2(\Omega)} \Big).$$

*Then, thanks to lemmas 2.1 and 2.3, there exists a constant* $c_1(u_0) > 0$*, independent of* $T$*, such that:*

$$|u_m(t)|_{L^2(\Omega)} \leqslant c_1(u_0). \tag{2.26}$$

PROPOSITION 2.2. *(The uniforme estimates in* $m$*). We have the following assertions:*

1. *For all* $\alpha \geqslant 1$*, there exists a constant* $c_\alpha(u_0) > 0$*, such that:*

$$\sup_{m \geqslant 1} |u_m|_{L^\infty(0,T;L^\alpha(\Omega))} \leqslant c_\alpha(u_0) < +\infty.$$

2. *For all* $r \geqslant 2$ *and* $k \in [1,\infty[$ *which satisfy the condition*

$$\frac{k}{4} - \frac{k}{2r} \leqslant 2, \tag{2.27}$$

*there exists a constant* $c_{k,r}(u_0) > 0$*, such that:*

$$\sup_{m \geqslant 1} |u_m'|_{L^k(0,T;L^r(\Omega))} \leqslant c_{k,r}(u_0) < +\infty.$$

3. *For all* $\bar{r} \geqslant 2$ *and* $\bar{k} \in [1,\infty[$ *which satisfy the condition*

$$\frac{3\bar{k}}{4} - \frac{\bar{k}}{2\bar{r}} \leqslant 2, \tag{2.28}$$

*there exists a constant* $c_{\bar{k},\bar{r}}(u_0) > 0$, *such that:*

$$\sup_{m\geqslant 1} |u_m''|_{L^{\bar{k}}(0,T;L^{\bar{r}}(\Omega))} \leqslant c_{\bar{k},\bar{r}}(u_0) < +\infty.$$

**Third step.** (Time-Derivative Estimats) By the obtained estimates, we deduce that $u_m$ remains in a bounded set of $L^2(0,T;H^3_{per}(\Omega)) \cap L^\infty(0,T;H^1_{per}(\Omega))$. For all $\psi \in H^3_{per}(\Omega)$, we take the function $P_m\psi$ as a test function:

$$\begin{aligned}\int_0^1 \frac{\partial u_m}{\partial t}(t,x)(P_m\psi(x))dx = &-\int_0^1 u_m''(t,x)(P_m\psi(x))''dx - 2\int_0^1 u_m'^4(t,x)(P_m\psi(x))dx \\ &+3\int_0^1 u_m'^3(t,x)(P_m\psi(x))'dx - 3\int_0^1 u_m''^2(t,x)(P_m\psi(x))dx \\ &-5\int_0^1 u_m'(t,x)u_m^{(3)}(t,x)(P_m\psi(x))dx.\end{aligned} \tag{2.29}$$

Applying the Hölder's inequality, one has:

$$\begin{aligned}\left|\int_0^1 \frac{\partial u_m}{\partial t}(t,x)(P_m\psi(x))dx\right| \leqslant &|u_m''(t)|_{L^2(\Omega)}|(P_m\psi)''|_{L^2(\Omega)} + 2|u_m'(t)|^4_{L^8(\Omega)}|P_m\psi|_{L^2(\Omega)} \\ &+3|u_m'(t)|^3_{L^6(\Omega)}|(P_m\psi)'|_{L^2(\Omega)} + 2|P_m\psi|_\infty|u''(t)|^2_{L^2(\Omega)} \\ &+5|P_m\psi|_\infty|u_m'(t)|_{L^2(\Omega)}|u_m^{(3)}(t)|_{L^2(\Omega)}.\end{aligned} \tag{2.30}$$

We have:

$$\int_0^1 \frac{\partial u_m}{\partial t}(t,x)(P_m\psi(x))dx = \int_0^1 P_m\big(\frac{\partial u_m}{\partial t}(t,x)\big)\psi(x)dx = \int_0^1 \frac{\partial u_m}{\partial t}(t,x)\psi(x)dx,$$

and since $|P_m\psi|_\infty \leqslant c_1|(P_m\psi)'|_{L^2(\Omega)}$ and $|(P_m\psi)^{(k)}|_{L^2(\Omega)} \leqslant c_2|\psi|_{H^3_{per}(\Omega)}$, where $k = 0,1,2$, then:

$$\left|\int_0^1 \frac{\partial u_m}{\partial t}(t,x)\psi(x)dx\right| \leqslant c_3\, Y(u_m'(t),u_m''(t),u_m^{(3)}(t))\,|\psi|_{H^3_{per}(\Omega)}, \tag{2.31}$$

with

$$\begin{aligned}Y(u_m'(t),u_m''(t),u_m^{(3)}(t)) = &|u_m''(t)|_{L^2(\Omega)} + |u_m'(t)|^4_{L^8(\Omega)} + |u_m'(t)|^3_{L^6(\Omega)} \\ &+|u''(t)|^2_{L^2(\Omega)} + |u_m'(t)|_{L^2(\Omega)}|u_m^{(3)}(t)|_{L^2(\Omega)}.\end{aligned} \tag{2.32}$$

Hence:

$$\int_0^T |\frac{\partial u_m}{\partial t}(t)|^2_{H^3_{per}(\Omega)}dt \leqslant c_3 \int_0^T Y^2(u_m'(t),u_m''(t),u_m^{(3)}(t))dt. \tag{2.33}$$

By (2.3), we find $\sup_{t\geqslant 0}|u'_m(t)|_{L^2(\Omega)} \leqslant c_0$. Then:

$$Y^2(u'_m(t),u''_m(t),u_m^{(3)}(t)) \leqslant c_4\big(|u''_m(t)|^2_{L^2(\Omega)} + |u'_m(t)|^8_{L^8(\Omega)} + |u'_m(t)|^6_{L^6(\Omega)} + |u''_m(t)|^4_{L^2(\Omega)} + |u_m^{(3)}(t)|^2_{L^2(\Omega)}\big). \tag{2.34}$$

We have:

$$\int_0^T |u''_m(t)|^2_{L^2(\Omega)}dt \leqslant c\int_0^T |u_m^{(3)}(t)|^2_{L^2(\Omega)} \leqslant c(u_0), \quad \text{(by (2.11))}. \tag{2.35}$$

On the other hand, for $k=r=8$ or $k=r=6$, the condition (2.27) is valid. Then $|u'_m|^n_{L^n(\Omega)}$ remains in a bounded set of $L^1(]0,T[)$, with $n=6,8$. In the same way, the condition (2.28) remains valid for $\bar{k}=4$ and $\bar{r}=2$, which affirms that $|u''_m|^4_{L^2(\Omega)}$ remains also in a bounded set of $L^1(]0,T[)$. Then, $Y^2(u'_m,u''_m,u^{(3)})$ remains in a bounded set of $L^1(]0,T[)$. Hence $\dfrac{\partial u_m}{\partial t}$ remains in a bounded set of $L^2(0,T;(H^3_{per}(\Omega))')$ when $m\to+\infty$.

REMARK 2.3. We have $u_m(0)\xrightarrow[m\longrightarrow+\infty]{} u(0)$ in $H^1_{per}(\Omega)$. Since

$$u_m(0)=P_m u_0 \xrightarrow[m\longrightarrow+\infty]{} u_0,$$

then $u(0)=u_0$ in the meaning of $H^1_{per}(\Omega)$.

Since $u_m$ belongs to a bounded set of $L^\infty(0,T;H^1_{per}(\Omega))$, $\dfrac{\partial u_m}{\partial t}$ remains in a bounded set of $L^2(0,T;(H^3_{per}(\Omega))')$ when $m\to+\infty$, and using the identification

$$H^1_{per}(\Omega)\subset L^2(\Omega)\cong (L^2(\Omega))'\subset (H^3_{per}(\Omega))',$$

we deduce, by the Aubin–Lions theorem, that $u_m$ converges in $\mathscr{C}\big([0,T];L^2(\Omega)\big)$. Therefore, by the previous estimates, there exists a subsequence of $u_m$ which is denoted also by $u_m$ and a function $u$, such that:

(a) $\dfrac{\partial u_m}{\partial t}\xrightarrow[m\longrightarrow+\infty]{}\dfrac{\partial u}{\partial t}$ in $L^2(0,T;(H^3_{per}(\Omega))')$–weak.

(b) $u_m\xrightarrow[m\longrightarrow+\infty]{} u$ in $L^2(0,T;H^3_{per}(\Omega))$–weak.

(c) $u_m\xrightarrow[m\longrightarrow+\infty]{} u$ weak $L^\infty(0,T;H^1_{per}(\Omega))$–weak star.

(d) $u_m\xrightarrow[m\longrightarrow+\infty]{} u$ strongly in $\mathscr{C}\big([0,T];L^2(\Omega)\big)$.

(e) $u$ is a weak solution of the problem (1.8).

REMARK 2.4. *Under similar conditions of the main theorem 2.1, if $u_0$ is fixed in $H^s_{per}(\Omega)$, $s<3$, We can take the space $H^s_{per}(\Omega)$ as a pivot space:*

$$H^3_{per}(\Omega)\subset H^s_{per}(\Omega)\cong\left(H^s_{per}(\Omega)\right)'\subset(H^3_{per}(\Omega))',$$

*with continuous and dense injections, then, we have*

$$u_m\to u \text{ in } \mathscr{C}\left([0,T];H^s_{per}(\Omega)\right)$$

.

COROLLARY 2.1. (of Theorem 2.1) *There are two constants $c>0$, $\mu>0$, such that, for all $x\in[0,1]$, one has:*

$$|u(x,t)-\bar{u}(t)|_\infty\leqslant c\,e^{-\mu t},\qquad t>0,$$

*where* $\bar{u}(t)=\displaystyle\int_0^1 u(t,x)dx.$

*Proof.* For all $x\in[0,1]$, by the interpolation inequalities and the estimation (2.3), we deduce that:

$$|u(x,t)-\bar{u}(t)|_\infty\leqslant c|u(x,t)-\bar{u}(t)|^{\frac{1}{2}}_{L^2(\Omega)}|u_x(t)|^{\frac{1}{2}}_{L^2(\Omega)}\leqslant ce^{-\mu t},\qquad t>0.$$

## 3. Uniqueness

We show in this section the uniqueness of the solution of the system (1.8) on the space $\mathscr{X}$.

THEOREM 3.1. *The problem (1.8), has at most one solution $u\in\mathscr{X}$, with the initial data $u(0)=u_0\in H^1_{per}(\Omega)$.*

*Proof.* Let $u_1$, $u_2\in\mathscr{X}$ be two solutions of (1.8) and we set $\delta u=u_1-u_2$. Since $u_1$ and $u_2$ satisfy the equation (1.7), then, making the difference between the two equations, multiplying the resulting equation by $(\delta u)''$ and integrating between 0 and 1, one has:

$$\frac{1}{2}\frac{d}{dt}\int_0^1|(\delta u)'(t,x)|^2dx+\int_0^1|(\delta u)^{(3)}(t,x)|^2dx=P_1+P_2+P_3+P_4, \tag{3.1}$$

with

$$\begin{cases} P_1=5\displaystyle\int_0^1\left(u_1'(t,x)u_1^{(3)}(t,x)-u_2'(t,x)u_2^{(3)}(t,x)\right)(\delta u)''(t,x)dx,\\ P_2=9\displaystyle\int_0^1\left({u_1'}^2(t,x)u_1''(t,x)-{u_2'}^2(t,x)u_2''(t,x)\right)(\delta u)''(t,x)dx,\\ P_3=3\displaystyle\int_0^1\left({u_1''}^2(t,x)-{u_2''}^2(t,x)\right)(\delta u)''(t,x)dx,\\ P_4=2\displaystyle\int_0^1\left({u_1'}^4(t,x)-{u_2'}^4(t,x)\right)(\delta u)''(t,x)dx. \end{cases} \tag{3.2}$$

We can rewrite the term $P_1$ under the following form:

$$\begin{aligned} P_1 &= 5\int_0^1 (\delta u)'(t,x)u_1^{(3)}(t,x)(\delta u)''(t,x)dx + 5\int_0^1 u_2'(t,x)(\delta u)^{(3)}(t,x)(\delta u)''(t,x)dx \\ &= P_{11} + P_{12}. \end{aligned}$$

To estimate the term $P_{11}$, we use the interpolation inequalities and the estimation (2.3). Since $\int_0^1 (\delta u)''(t,x)dx = 0$, then

$$|(\delta u)''(t,x)| \leqslant \int_0^1 |(\delta u)^{(3)}(t,x)|dx \leqslant |(\delta u)^{(3)}(t)|_{L^2(\Omega)}. \tag{3.3}$$

Thus, by the Cauchy–Schwarz and Young inequalities, and for all $\varepsilon > 0$, one has:

$$\begin{aligned} |P_{11}| &\leqslant 5|(\delta u)^{(3)}(t)|_{L^2(\Omega)} \int_0^1 |(\delta u)'(t,x)||u_1^{(3)}(t,x)|dx \\ &\leqslant c_1|(\delta u)^{(3)}(t)|_{L^2(\Omega)} \left(\int_0^1 |u_1^{(3)}(t,x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |(\delta u)'(t,x)|^2 dx\right)^{\frac{1}{2}} \\ &\leqslant c_\varepsilon \left(\int_0^1 |u_1^{(3)}(t,x)|^2 dx\right)\left(\int_0^1 |(\delta u)'(t,x)|^2 dx\right) + \frac{\varepsilon}{8}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx \\ &= c_2|u_1^{(3)}(t)|^2_{L^2(\Omega)}|(\delta u)'(t)|^2_{L^2(\Omega)} + \frac{\varepsilon}{8}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx. \end{aligned} \tag{3.4}$$

In the same way, using also the interpolation inequalities, and for all $\varepsilon > 0$, we deduce that:

$$\begin{aligned} |P_{12}| &\leqslant c_3|u_2'(t)|_\infty \int_0^1 |(\delta u)^{(3)}(t,x)||(\delta u)''(t,x)|dx \\ &\leqslant c_\varepsilon c_5^2|u_2'(t)|^2_\infty|(\delta u)''(t)|^2_{L^2(\Omega)} + \frac{\varepsilon}{24}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx \\ &\leqslant c_4|u_2'(t)|^2_\infty|(\delta u)'(t)|_{L^2(\Omega)}|(\delta u)^{(3)}(t,x)|_{L^2(\Omega)} + \frac{\varepsilon}{24}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx \\ &\leqslant c_5|u_2'(t)|^4_\infty(t)|(\delta u)'(t)|^2_{L^2(\Omega)} + \frac{\varepsilon}{8}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx. \end{aligned} \tag{3.5}$$

Then,

$$|P_1| \leqslant f_1(t)|(\delta u)'(t)|^2_{L^2(\Omega)} + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx, \quad \forall \varepsilon > 0, \tag{3.6}$$

with, $f_1(t) = c_5\big(|u_1^{(3)}(t)|^2_{L^2(\Omega)} + |u_2'(t)|^4_\infty\big)$. We show that $f_1 \in L^1(]0,T[)$. Thanks to proposition 2.2, with $k = 4$ et $r = \infty$, we deduce that $|u_2'|^4_\infty \in L^1(]0,T[)$, and since $u_1 \in L^2\big(0,T;H^3_{per}(\Omega)\big)$, then $f_1 \in L^1(]0,T[)$.
An integration by parts allows us to write the term $P_2$ under the following form:

$$\begin{aligned} P_2 &= -3\int_0^1 \big(u_1'^3(t,x) - u_2'^3(t,x)\big)(\delta u)^{(3)}(t,x)dx \\ &= \int_0^1 \big(u_1'(t,x) - u_2'(t,x)\big)\big(u_1'^2(t,x) + u_1'(t,x)u_2'(t,x) + u_2'^2(t,x)\big)(\delta u)^{(3)}(t,x)dx. \end{aligned}$$

Applying the Young inequality for $\varepsilon > 0$, one has:

$$\begin{aligned}
|P_2| &\leqslant c_6\big(|u_1'(t)|_\infty^2 + |u_1'(t)|_\infty |u_2'(t)|_\infty + |u_2'(t)|_\infty^2\big) \int_0^1 |(\delta u)'(t,x)||(\delta u)^{(3)}(t,x)|dx \\
&\leqslant c_7\big(|u_1'(t)|_\infty^2 + |u_2'(t)|_\infty^2\big) \int_0^1 |(\delta u)'(t,x)||(\delta u)^{(3)}(t,x)|dx \\
&\leqslant c_\varepsilon\big(|u_1'(t)|_\infty^2 + |u_2'(t)|_\infty^2\big)^2 \int_0^1 |(\delta u)'(t,x)|^2 dx + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx \\
&\leqslant c_8\big(|u_1'(t)|_\infty^4 + |u_2'(t)|_\infty^4\big) \int_0^1 |(\delta u)'(t,x)|^2 dx + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx.
\end{aligned} \tag{3.7}$$

Thus,

$$|P_2| \leqslant f_2(t)|(\delta u)'(t)|_{L^2(\Omega)}^2 + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx, \quad \forall \varepsilon > 0, \tag{3.8}$$

with $f_2(t) = c_8\big(|u_1'(t)|_\infty^4 + |u_2'(t)|_\infty^4\big)$. Applying proposition 2.2, with $k = 4$ and $r = \infty$, one has $f_2 \in L^1(]0,T[)$.
We can write the term $P_3$ as following:

$$\begin{aligned}
P_3 &= 3\int_0^1 \big(u_1''(t,x) + u_2''(t,x)\big)\big(u_1''(t,x) - u_2''(t,x)\big)(\delta u)''(t,x)dx \\
&= 3\int_0^1 \big(u_1''(t,x) + u_2''(t,x)\big)(\delta u)''^2(t,x)dx.
\end{aligned} \tag{3.9}$$

By the intepolation inequalities, we deduce that:

$$\begin{aligned}
|P_3| &\leqslant c_9\big(|u_1''(t)|_\infty + |u_2''(t)|_\infty\big) \int_0^1 |(\delta u)''(t,x)|^2 dx \\
&\leqslant c_{10}\big(|u_1''(t)|_\infty + |u_2''(t)|_\infty\big)|(\delta u)'(t)|_{L^2(\Omega)}|(\delta u)^{(3)}(t)|_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

Then, the Young inequality gives us

$$|P_3| \leqslant f_3(t)|(\delta u)'(t)|_{L^2(\Omega)}^2 + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx, \quad \forall \varepsilon > 0, \tag{3.11}$$

with

$$f_3(t) = c_{11}\big(|u_1''(t)|_\infty + |u_2''(t)|_\infty\big)^2 \leqslant c_{12}\big(|u_1''(t)|_\infty^2 + |u_2''(t)|_\infty^2\big).$$

Applying proposition 2.2, with $\bar{k} = 2$ and $\bar{r} = \infty$, we deduce that $f_3 \in L^1(]0,T[)$.
Finally, we can write the term $P_4$ as:

$$P_4 = 2\int_0^1 (\delta u)''(t,x)(\delta u)'(t,x)\big(u_1'(t,x) + u_2'(t,x)\big)\big(u_1'^2(t,x) + u_2'^2(t,x)\big)dx,$$

then

$$|P_4| \leqslant c_{13}(|u_1'(t)|_\infty + |u_2'(t)|_\infty)(|u_1'(t)|_\infty^2 + |u_2'(t)|_\infty^2)\int_0^1 (\delta u)''(t,x)(\delta u)'(t,x)dx. \tag{3.12}$$

Thus,

$$|P_4| \leqslant f_4(t)|(\delta u)'(t)|^2_{L^2(\Omega)} + \frac{\varepsilon}{4}\int_0^1 |(\delta u)^{(3)}(t,x)|^2 dx, \quad \forall \varepsilon > 0, \tag{3.13}$$

with

$$f_4(t) = c_{14}(|u_1'(t)|_\infty + |u_2'(t)|_\infty)^2(|u_1'(t)|^2_\infty + |u_2'(t)|^2_\infty)^2.$$

Hence

$$f_4(t) \leqslant c_{15}(|u_1'(t)|^6_\infty + |u_2'(t)|^6_\infty).$$

By using proposition 2.2, with $k = 6$ and $r = \infty$, we deduce that $f_4 \in L^1(]0,T[)$. Therefore $f := \sum_{i=1}^4 f_i \in L^1(]0,T[)$. For $\varepsilon < 1$, the estimations (3.6), (3.8), (3.11), and (3.13) allow us to write

$$\frac{d}{dt}\int_0^1 |(\delta u)'(t,x)|^2 dx \leqslant c\, f(t)|(\delta u)'(t)|^2_{L^2(\Omega)}, \tag{3.14}$$

Setting $G(t) = |(\delta u)'(t)|^2_{L^2(\Omega)}$, one has:

$$\big(G(t)\big)'_t \leqslant c f(t)\, G(t). \tag{3.15}$$

The Gronwall inequality gives:

$$G(t) \leqslant G(0)\exp\left(c\int_0^t f(\tau)d\tau\right) = 0.$$

Therefore $u_1 = u_2$, then the solution (1.8) is unique in $\mathscr{X}$.

## 4. Numerical Validation

In this section we review the numerical verification of previous theoretical study. We use pseudo-spectral method conjugated with an exponential scheme that breaks down to a classical forward Euler time scheme, for zero wave number [23].

### 4.1. Spatial discretization

We consider the periodic problem (1.3), where $h(x,t)$ supposed $2\pi$-periodic. Eq. (1.3) can be written in the form

$$\frac{\partial h}{\partial t} + \mathscr{L}(h) = \mathscr{N}(h) \tag{4.1}$$

where $\mathscr{L}$ and $\mathscr{N}$ are respectively the linear and nonlinear operators of (1.3), i.e.,

$$\mathscr{L} \equiv \frac{\partial^4}{\partial x^4} \tag{4.2}$$

and

$$\mathscr{N}(h) = -h^{-1}h'h^{(3)}. \tag{4.3}$$

The periodic boundary conditions and given initial data yield

$$\begin{aligned} h(0,t) &= h(2\pi,t), \quad t \in \mathbb{R}_+ \\ h(x,0) &= h_0(x), \quad x \in (0,2\pi). \end{aligned}$$

The solution of (1.3) is approximated as a truncated series in the Fourier basis functions $\{(\Phi_k)_{k\in\mathbb{Z}},\ \Phi_k(x) \equiv e^{ikx}\}$:

$$h_N(x,t) = P_N(h(x,t)) = \sum_{k\in\mathbb{I}_N} \hat{h}_k(t)\Phi_k(x),$$

where $\mathbb{I}_N = [1-\frac{N}{2}, \frac{N}{2}]$; the $\hat{h}_k$ are the spectral coefficients. We require the orthogonality of the residue for all functions of $S_N$ who make up the vectorial space generated by $(\Phi_k)_{k\in\mathbb{Z}}$. In Fourier space, we can write

$$\frac{\partial \hat{h}_k}{\partial t} = \mathscr{L}_k \hat{h}_k + \mathscr{N}_k,$$

where $\mathscr{N}_k$ is the $k$-th Fourier coefficient of the nonlinear term of (4.1).

### 4.2. Time discretization

To approach the solution of (1.3), we adopt a pseudo-spectral method which is associated with an exponential scheme in time [23]. Let $\delta t = t_{n+1} - t_n$ be the time step (constant) then we obtain the exponential scheme in time

$$\hat{h}_k^{n+1} = \hat{h}_k^n \exp(\mathscr{L}_k \delta t) + \mathscr{N}_k \frac{\exp(\mathscr{L}_k \delta t) - 1}{\mathscr{L}_k}.$$

This scheme is based on a discrete version of the "variation of constants method". The nonlinear term $\mathscr{N}_k$ is calculated at each step time in the direct space then in the Fourier space by a fast discrete transform.

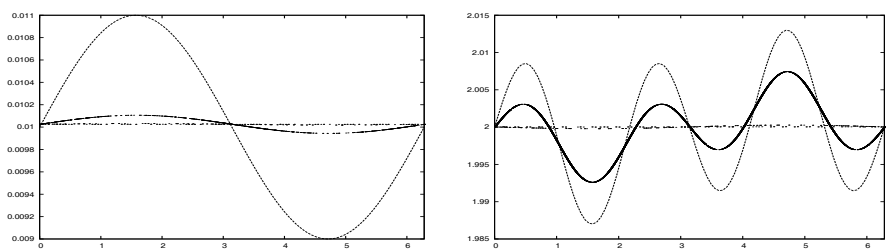


Figure 1: On the left, solution $h(x,t)$ of system (1.3) with the initial data $h_0(x) = 0.01 + 0.001\sin(x)$ and for $\delta t = 10^{-2}$ and $N = 8192$. On the right, solution (1.3) of $h_0(x) = 2 + 0.01\sin(3x) - 0.003\sin(x)$. The initial solution is given by a dotted line. We obtain the convergence of the solution to the mean value of the initial data.

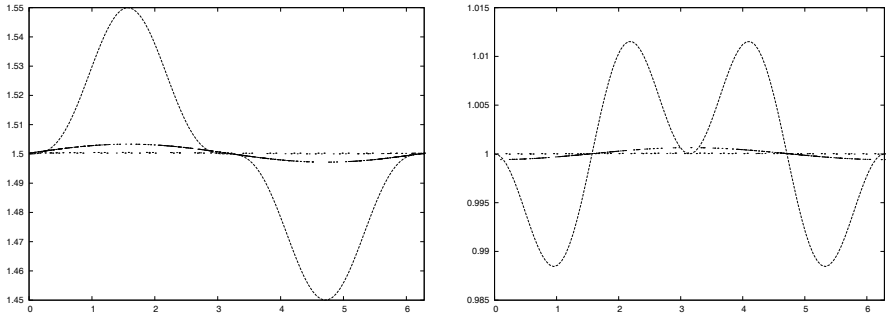


Figure 2: On the left, solution $h(x,t)$ of system (1.3) with the initial data $h_0(x) = 1.5 + 0.05\sin^3(x)$ and for $\delta t = 10^{-3}$ and $N = 8192$. On the right, solution (1.3) for $h_0(x) = 1 + 0.03\cos^3(x) - 0.03\cos(x)$. The initial solution is given by a dotted line. Again we obtain the convergence of the solution to the mean value of the initial data.

## Remark

I would like to draw your attention that this article is a part of my thesis [1].

*Acknowledgements.* The author would like to express his gratitude to Professor J.M. Rakotoson for helpful discussions, ideas and remarks.

REFERENCES

[1] A. Alriyabi, *Analysis of some delay differential equations and PDE modelling the surface instabilities*, University of Poitiers - France - 2013.

[2] S Asvadurov, B D Coleman, R S Falk, M Moakher. *Similarity solutions in the theory of curvature driven diffusion along planar curves: I. Symmetric curves expanding in time*, Physica D, 1998, 121: 263–274.

[3] M. Boutat, Y. D'Angelo, S. Hilout, V. Lods, *An interface evolution problem for axisymmetric stressed problem*, Asymptotic Analysis, 2005, 44: 131–150.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer; 1st Edition, 2010.

[5] J.A. Carrillo, J. Dolbeault, I. Gentil, A Jüngel, *Entropy-energy inequalities and improved convergence rates for nonlinear parabolic equations*, Discrete Contin. Dyn. Syst. Ser. B, 2006, 6: 1027–1050.

[6] M.J. Cáseres, J.A. Carrillo, J. Dolbeault, I. Gentil, *Nonlinear stability in $L^p$ for a confined system for charged particules*, SIAM J. Math. Anal., 2002, 34: 478–494.

[7] B.D. Coleman, R.S. Falk, M. Moakher, *Stability of cylindrical bodies in the theory of surface diffusion*, Physica D, 1995, 89: 123–135.

[8] B.D. Coleman, R.S. Falk, M. Moakher, *Space-time finite element methods for surface diffusion with applications to the theory of the stability of cylinders*, SIAM J. Sci. Comp., 1996, 17: 1434–1448.

[9] J.Y. Guo, *Existence of weak solutions for a generalized thin film equation*, (Chinese), Math. Practice Theory, 2008, 38: 155–161.

[10] J.R. King, *Two generalisations of the thin film equation, Mathematical and Computer Modelling*, 2001, 34: 737–756.

[11] D.J. Kirill, S.H. Davis, M.J. Miksis, A.W. Voorhees, *Morphological instability of a whisker*, Proc. R. Soc. Lond. A, 1999, 455: 3825–3844.

[12] D.J. Kirill, S.H. Davis, M.J. Miksis, A.W. Voorhees, *Morphological instability of pores and tubules, Interfaces and Free Boundaries*, 2002, 4: 371–394.
[13] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
[14] C. Liu, J. Yin, J. Zhou, *Existence of weak solutions for a generalized thin film equation*, Commun. Pure Appl. Anal., 2007, 6: 465–480.
[15] X. Liu, C. Qu, *Existence and blow-up of weak solutions for a sixth-order equation related to thin solid films*, Nonlinear Anal. Real World Appl., 2010, 11: 4214–4222.
[16] J.D. Powers, A.M. Glaeser, *High-temperature healing of cracklike flaws in titanium ion-implanted sapphire*, J. Amer. Ceramic Soc., 1993, 76: 2225–2234.
[17] V. Lods, A. Piétrus, J.M. Rakotoson, *Mathematical study of the equation describing the evolution of the surface of a film*, Asymptotic Analysis, 2003, 33: 67–91.
[18] W.W. Mullins, *Theory of thermal grooving*, J. of Appl. Phys., 1957, 28: 333–339.
[19] F.A. Nichols, W.W. Mullins, *Morphological changes of a surface of revolution due to capillarity–induced surface diffusion*, J. Appl. Phys., 1965, 36: 1826–1835.
[20] B.J. Spencer, S.H. Davis, P.W. Woorhees, *Morphological instability in epitaxially-strained dislocation-free solid films: nonlinear evolution*, Phys. Rev., B, 1993, 47: 9760–9777.
[21] J.E. Rakotoson, J.M. Rakotoson, *Analyse fonctionnelle appliquée aux èquations érivées partielles*, PUF, 1999.
[22] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer–Verlag, New York, 1988.
[23] Y. D'Angelo, G. Joulin, G. Boury, *On model evolution equation for the whole surface of 3D expanding wrinkled premixed flames*, Combustion Theory & Modelling, 2000, 4: 317–338.
[24] L. Rayleigh, *On the capillary phenomena of jets*, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 1879, 29: 71–97.
[25] J.M. Rallison, *The deformation of small viscous drops and bubbles in shear flows*, Annual Review of Fluid Mechanics, 1984, 16: 45–66.
[26] H.A. Stone, *Dynamics of drop deformation and breakup in viscous fluids*, Annual Review of Fluid Mechanics, 1994, 26: 65–102.
[27] M. Xu, S. Zhou, *Existence and uniqueness of weak solutions for a generalized thin film equation*, Nonlinear Anal., 2005, 60: 755–774.
[28] C.H. Zhu, C.Z. Qu, *Classification and reduction of generalized thin film equations*, Commun. Theor. Phys. (Beijing), 2009, 52: 403–410.

(Received April 1, 2016)

*A. Alriyabi*
*41 Rue des Fontanelles 47300 Villeneuve sur Lot*
*France*
*e-mail:* a.alriyabi@gmail.com