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COUPLED SYSTEMS OF FRACTIONAL ∇–DIFFERENCE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the existence of solutions for a coupled system of two-point fractional ∇-difference boundary value problems of the form

$$\begin{pmatrix} \nabla^{\alpha}_{a^+}u(t) \\ \nabla^{\beta}_{a^+}v(t) \end{pmatrix} + \begin{pmatrix} f(t,v(t)) \\ g(t,u(t)) \end{pmatrix} = 0,$$
$$\begin{pmatrix} u(a+1) \\ u(b+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(a+1) \\ v(b+1) \end{pmatrix},$$

where $1 < \alpha, \beta \leqslant 2$, $t \in [a+2, b+1]_{\mathbb{N}} = \{a+2, a+3, ..., b, b+1\}$, $a, b \in \mathbb{Z}$ such that $a \geqslant 0, b \geqslant 3$ and the functions $f, g : [a+2, b+1]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ are continuous. Our analysis relies on the Green functions and the nonlinear alternative of Leray-Schauder and Krasnoselśkii-Zabreiko fixed point theorems. At the end we give some numerical examples to illustrate the main results.

1. Introduction

The theory of fractional calculus basically acts on the differential operators as $D_t^{\alpha} \equiv d^{\alpha}/dt^{\alpha}$ with arbitrary order $\alpha \in \mathbb{R}$, that generalize the integer order integration and differentiation. In recent decades, it has been illustrated that many systems appeared in science and engineering can be simulated by fractional derivatives rather than integer ones [11, 12]. This is why we are interested to the study of the various fractional based approaches related to the both theoretical and computational sciences interacted with mathematics.

Despite the boom of developments in fractional differential equations, the approach of the fractional difference equations have been included to the collection of some elementary analysis of fractional discrete boundary value problems in the early last decade. In this way one can suggest the pioneering works of F. Atici and P. Eloe [2, 3, 4, 5], C. Goodrich [7], C. Goodrich, A. C. Peterson [8], Y. Gholami and K. Ghanbari [9], R. A. C. Ferreira [6].

F. Atici and P. Eloe in [3], studied the two-point fractional Δ-difference boundary value problem

$$\begin{cases} -\Delta^{\nu}y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t = 1, 2, ..., b+1, \\ y(\nu-2) = 0, \; y(\nu+b+1) = 0, \end{cases} \tag{1.1}$$

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where $1 < \nu \leqslant 2$ is a real number and, $b \geqslant 2$ is an integer and Δ^{ν} denotes fractional Δ-difference operator of order $\nu > 0$. They assumed that $f : [\nu, \nu + b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous. In continuation of this work we consider the following coupled system of two-point fractional ∇-difference boundary value problems

$$\begin{aligned} &\begin{pmatrix} \nabla^{\alpha}_{a^+} u(t) \\ \nabla^{\beta}_{a^+} v(t) \end{pmatrix} + \begin{pmatrix} f(t, v(t)) \\ g(t, u(t)) \end{pmatrix} = 0, \\ &\begin{pmatrix} u(a+1) \\ u(b+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(a+1) \\ v(b+1) \end{pmatrix}, \end{aligned} \tag{1.2}$$

where $1 < \alpha, \beta \leqslant 2$, $t \in [a+2, b+1]_{\mathbb{N}} = \{a+2, a+3, ..., b, b+1\}$, $a, b \in \mathbb{Z}$ such that $a \geqslant 0, b \geqslant 3$. Suppose that $f, g : [a+2, b+1]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions. Note that $\nabla^{\alpha}_{a^+}$ denotes the fractional ∇-difference operator of order $\alpha > 0$ that will be defined later.

2. Preliminaries

We devote this section to the general description of ∇-fractional operators that will be divided into the fractional ∇-summations and fractional ∇-differences. Afterward we construct a relevant functional space needed in our work. To this aim, we begin with fractional rising functions that make cornerstone of the kernels of the ∇-fractional operators.

DEFINITION 1. [1],*[*[8], *Chap. 3] Fractional rising function is defined by*

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{..., -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0. \tag{2.1}$$

Note that $\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}$.

DEFINITION 2. [1],*[*[8], *Chap. 3] Fractional left sided* ∇*-sum of order* $\alpha > 0$ *for function* f *is defined by*

$$\nabla^{-\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t - \delta(s))^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_a := \{a, a+1, a+2, ...\} \tag{2.2}$$

where $\delta(s) = s - 1$. *In accordance with fractional* ∇*-summations, fractional* ∇*-difference of order* $\alpha > 0$ *for function* f *is given by* $\nabla^{\alpha}_{a^+} f(t) = \nabla^n \nabla^{\alpha-n}_{a^+} f(t)$, *where* $t \in \mathbb{N}_{a+n}$, $n - 1 < \alpha \leqslant n$ *and* ∇ *denotes the backward difference operator.*

LEMMA 1. *[*[8], *Chap. 3] Assume that* f *is a real-valued function and* $\mu > 0$, $0 \leqslant n - 1 < \nu \leqslant n$. *Then*

(Q_1) $\nabla^{-\mu}_{a^+} \nabla^{-\nu}_{a^+} f(t) = \nabla^{-(\mu+\nu)}_{a^+} f(t) = \nabla^{-\nu}_{a^+} \nabla^{-\mu}_{a^+} f(t)$,

(Q_2) $\nabla_{a^+}^{-\nu}\, \nabla_{a^+}^{\nu} f(t) = f(t) + c_1 (t-a)^{\overline{\nu-1}} + c_2 (t-a)^{\overline{\nu-2}} + \dots + c_n (t-a)^{\overline{\nu-n}}$, $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

(Q_3) $\nabla_{a^+}^{\nu}\, \nabla_{a^+}^{-\nu} f(t) = f(t)$.

(Q_4) $\nabla_{a^+}^{-\nu} (t-a)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\overline{\mu+\nu}}$, $\mu+\nu+1 \notin (-\mathbb{Z}_0)$.

In this paper, we will use the Banach space

$$E = \mathfrak{B} \times \mathfrak{B}, \quad \mathfrak{B} = \left(\left\{ x \middle| \, x : [a+2, b+1]_{\mathbb{N}} \to \mathbb{R} \right\}, \|.\|_{\mathfrak{B}} \right), \tag{2.3}$$

endowed with the norm

$$\|(x,y)\|_E = \|x\|_{\mathfrak{B}} + \|y\|_{\mathfrak{B}}, \quad \|x\|_{\mathfrak{B}} = \sup_{t=a+2,\dots,b+1} |x(t)|. \tag{2.4}$$

To establish solvability of the coupled system (1.2), we use the nonlinear alternative of the Leray-Schauder and Krasnoselśkii-Zabreiko fixed point theorems. One may state these theorems as below respectively.

THEOREM 1. [13] *Let C is a convex subset of a Banach space, U is an open subset of C with $0 \in U$. Then every completely continuous map $T : \overline{U} \to C$ has at least one of the two following properties:*

(E_1) *There exists an $u \in \overline{U}$ such that $Tu = u$.*

(E_2) *There exist an $v \in \partial U$ and $\lambda \in (0,1)$ such that $v = \lambda T v$.*

THEOREM 2. [10] *Let X is a Banach space. Assume that $T : X \to X$ is a completely continuous mapping. If $L : X \to X$ be a linear bounded mapping such that 1 is not an eigenvalue of L and*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0, \tag{2.5}$$

then T has a fixed point in X.

3. Main Results

As stated above, the Green function plays crucial role in this paper. So we begin the main body of our study with unification of the Green function corresponding to the fractional coupled system (1.2) as follows.

LEMMA 2. *The two-point fractional ∇-difference boundary value problem*

$$\begin{cases} \nabla_{a^+}^{\alpha} u(t) + h(t) = 0, & t \in [a+2, b+1]_{\mathbb{N}}, \\ u(a+1) = 0 = u(b+1), \end{cases} \tag{3.1}$$

uniquely solves the following fractional ∇*-sum equation*

$$u(t)=\sum_{s=a+2}^{b+1}\mathscr{G}(t,s)h(s),\quad t=a+2,a+3,...,b,b+1, \tag{3.2}$$

in which

$$\mathscr{G}(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\mathscr{G}_1(t,s); & a+2\leqslant s\leqslant t\leqslant b+1,\\ \mathscr{G}_2(t,s); & a+2\leqslant t\leqslant s\leqslant b+1,\end{cases} \tag{3.3}$$

where

$$\begin{aligned}\mathscr{G}_1(t,s)&=\frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)}(t-a-1)-(t-s+1)^{\overline{\alpha-1}},\\ \mathscr{G}_2(t,s)&=\frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)}(t-a-1).\end{aligned} \tag{3.4}$$

Proof. According to the property (Q_2) in Lemma 1, fractional ∇-difference equation $\nabla^{\alpha}_{a^+}u(t)+h(t)=0$ is equivalent to the fractional ∇-sum equation

$$u(t)=c_1(t-a)^{\overline{\alpha-1}}+c_2(t-a)^{\overline{\alpha-2}}-\nabla^{-\alpha}_{a^+}h(t).$$

Now implementing the boundary conditions $u(a+1)=0$ and $u(b+1)=0$, the coefficients c_1 and c_2 are uniquely determined as below:

$$c_1=\frac{(b-a-1)!\nabla^{-\alpha}_{a^+}h(t)\Big|_{t=b+1}}{\Gamma(\alpha+b-a-1)},\quad c_2=\frac{-(\alpha-1)(b-a-1)!\nabla^{-\alpha}_{a^+}h(t)\Big|_{t=b+1}}{\Gamma(\alpha+b-a-1)}. \tag{3.5}$$

Therefore we have

$$\begin{aligned}u(t)&=\frac{(b-a-1)!(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)}\sum_{s=a+2}^{b+1}(b-s+2)^{\overline{\alpha-1}}h(s)\\ &\quad-\frac{(\alpha-1)(b-a-1)!(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)}\sum_{s=a+2}^{b+1}(b-s+2)^{\overline{\alpha-1}}h(s)\\ &\quad-\frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{t}(t-s+1)^{\overline{\alpha-1}}h(s)\\ &=\Bigg[\frac{(b-a-1)!(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)}\sum_{s=a+2}^{t}(b-s+2)^{\overline{\alpha-1}}\\ &\qquad-\frac{(\alpha-1)(b-a-1)!(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)}\sum_{s=a+2}^{t}(b-s+2)^{\overline{\alpha-1}}\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} (t-s+1)^{\overline{\alpha-1}} \Bigg] h(s) \\
&\quad + \Bigg[\frac{(b-a-1)!(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)} \sum_{s=t+1}^{b+1} (b-s+2)^{\overline{\alpha-1}} \\
&\quad - \frac{(\alpha-1)(b-a-1)!(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)} \sum_{s=t+1}^{b+1} (b-s+2)^{\overline{\alpha-1}} \Bigg] h(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Bigg\{ \frac{(b-a-1)!(t-a)^{\overline{\alpha-1}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} \\
&\quad - \frac{(\alpha-1)(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} - (t-s+1)^{\overline{\alpha-1}} \Bigg\} h(s) \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b+1} \Bigg\{ \frac{(b-a-1)!(t-a)^{\overline{\alpha-1}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} \\
&\quad - \frac{(\alpha-1)(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} \Bigg\} h(s).
\end{aligned}
$$

Finally using the identity

$$(t-a)^{\overline{\alpha-1}} = (\alpha+t-a-2)(t-a)^{\overline{\alpha-2}},$$

we reach the following:

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Bigg\{ \frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} (t-a-1) \\
&\qquad - (t-s+1)^{\overline{\alpha-1}} \Bigg\} h(s) \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b+1} \Bigg\{ \frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)} (t-a-1) \Bigg\} h(s) \\
&= \sum_{s=a+2}^{b+1} \mathscr{G}(t,s) h(s). \qquad (3.6)
\end{aligned}
$$

LEMMA 3. *The Green function $\mathscr{G}(t,s)$ given by (3.3) and (3.4), satisfies:*

$$\sup_{t\in[a+2,b+1]_{\mathbb{N}}} \mathscr{G}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \mathscr{O}_{even}, & a+b: even, \\ \mathscr{O}_{odd}, & a+b: odd, \end{cases} \qquad (3.7)$$

where

$$\mathscr{O}_{even} = \frac{(b-a-1)!}{\left(\frac{b-a}{2}-1\right)!\left(\frac{b-a}{2}\right)!} \cdot \frac{\Gamma\left(\frac{b-a}{2}+\alpha-1\right)\Gamma\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a-1)},$$

$$\mathscr{O}_{odd} = \frac{(b-a-1)!}{\left(\frac{b-a+1}{2}-1\right)!\left(\frac{b-a+1}{2}\right)!} \cdot \frac{\Gamma\left(\frac{b-a+1}{2}+\alpha-1\right)\Gamma\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a-1)}.$$

Proof. As stated above, the Green's function $\mathscr{G}(t,s)$ defined by (3.3) and (3.4) is given as follows

$$\mathscr{G}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \mathscr{G}_1(t,s); & a+2 \leqslant s \leqslant t \leqslant b+1, \\ \mathscr{G}_2(t,s); & a+2 \leqslant t \leqslant s \leqslant b+1, \end{cases} \tag{3.8}$$

where

$$\mathscr{G}_1(t,s) = \frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)}(t-a-1) - (t-s+1)^{\overline{\alpha-1}},$$
$$\mathscr{G}_2(t,s) = \frac{(b-a-1)!(t-a)^{\overline{\alpha-2}}(b-s+2)^{\overline{\alpha-1}}}{\Gamma(\alpha+b-a-1)}(t-a-1).$$

Obviously $\mathscr{G}_2(t,s) \geqslant 0$ for $t = a+2,...,b+1$ and $(t-s+1)^{\overline{\alpha-1}} \geqslant 0$ for $a+2 \leqslant s \leqslant t \leqslant b+1$. So one has

$$\sup_{t=a+2,...,b+1} \mathscr{G}(t,s) = \sup_{t=a+2,...,b+1} \frac{\mathscr{G}_2(t,s)}{\Gamma(\alpha)}, \quad s \in [a+2,b+1]_{\mathbb{N}}.$$

On the other hand, using $\Delta_s \mathscr{G}_2(t,s) \leqslant 0$, it follows that

$$\sup_{t=a+2,...,b+1} \mathscr{G}(t,s) = \frac{\mathscr{G}_2(s,s)}{\Gamma(\alpha)}, \quad s \in [a+2,b+1]_{\mathbb{N}}. \tag{3.9}$$

Consequently by means of the definition of $\mathscr{G}_2(s,s)$, we have for $s \in [a+2,b+1]_{\mathbb{N}}$ that

$$\Delta\mathscr{G}_2(s,s) = \frac{(b-a)!}{\Gamma(\alpha+b-a-1)} \frac{\Gamma(\alpha+s-a-2)\Gamma(\alpha+b-s)}{(s-a-1)!(b-s+1)!}[(a+b+2)-2s].$$

Accordingly, it is clear that $\mathscr{G}_2(s,s)$ is increasing for $s < \frac{a+b}{2}+1$ and is decreasing for $s > \frac{a+b}{2}+1$. Therefore

$$\max_{s\in[a+2,b+1]_{\mathbb{N}}} \mathscr{G}_2(s,s) = \mathscr{G}_2\left(\frac{a+b}{2}+1, \frac{a+b}{2}+1\right) = \frac{(b-a-1)!}{\left(\frac{b-a}{2}-1\right)!\left(\frac{b-a}{2}\right)!} \cdot \frac{\Gamma\left(\frac{b-a}{2}+\alpha-1\right)\Gamma\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a-1)}, \tag{3.10}$$

provided that $a+b$ is even and

$$\max_{s\in[a+2,b+1]_{\mathbb{N}}} \mathscr{G}_2(s,s) = \mathscr{G}_2\left(\frac{a+b+1}{2}+1, \frac{a+b+1}{2}+1\right) = \frac{(b-a-1)!}{\left(\frac{b-a+1}{2}-1\right)!\left(\frac{b-a+1}{2}\right)!} \cdot \frac{\Gamma\left(\frac{b-a+1}{2}+\alpha-1\right)\Gamma\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a-1)}, \tag{3.11}$$

if $a+b$ is odd. This completes the proof.

We now define the operator $\mathfrak{T} : E \to E$ as

$$\mathfrak{T}(u,v)(t) = ((\mathfrak{T}_1 v)(t), (\mathfrak{T}_2 u)(t)), \quad [a+2,b+1]_{\mathbb{N}} \tag{3.12}$$

where

$$(\mathfrak{T}_1)v(t) = \sum_{s=a+2}^{b+1} \mathscr{G}(t,s) f(s,v(s)), \ \ (\mathfrak{T}_2)u(t) = \sum_{s=a+2}^{b+1} \mathscr{H}(t,s) g(s,u(s)), \ \ [a+2,b+1]_{\mathbb{N}}. \tag{3.13}$$

Note that $\mathscr{H}(t,s)$ is also the Green function obtained by replacing α with β in $\mathscr{G}(t,s)$.

REMARK 1. We notice that the operator $\mathfrak{T}$ can be written in the form of the following vector equation

$$\mathfrak{T}(u,v)(t) = \sum_{s=a+2}^{b+1} \begin{pmatrix} \mathscr{G}(t,s) \\ \mathscr{H}(t,s) \end{pmatrix} \begin{pmatrix} f(s,v(s)) & 0 \\ 0 & g(s,u(s)) \end{pmatrix}. \tag{3.14}$$

Thereby, one can deduce that the coupled system of two-point fractional ∇-difference boundary value problems (1.2) solves uniquely the vector equation (3.14).

Achieving to the first part of the main results, we consider the following hypotheses.

HYPTHESES 1. *There exist positive continuous functions ϕ_i, ψ_i, $i = 1,2$ with ψ_i increasing, such that*

(H_1) $|f(t,v)| \leqslant \phi_1(t)\psi_1(|v|)$, $(t,v) \in [a+2,b+1]_{\mathbb{N}} \times \mathbb{R}$;

(H_2) $|g(t,u)| \leqslant \phi_2(t)\psi_2(|u|)$, $(t,u) \in [a+2,b+1]_{\mathbb{N}} \times \mathbb{R}$.

REMARK 2. Finitely discrete nature of the summation operator $\mathfrak{T}$ given by (3.12) and (3.13) together with the Hypotheses 1, implies that $\mathfrak{T}$ is trivially completely continuous.

Now we are ready to state and prove the main results of the paper as follows.

THEOREM 3. *Assume that the hypotheses (H_1) and (H_2) are satisfied. If there exist a positive constant ρ such that*

$$\left(\frac{1}{M}+\frac{1}{N}\right) > \frac{2}{\rho}\left[\psi_1(\rho)\sum_{a+2}^{b+1}|\phi_1(s)| + \psi_2(\rho)\sum_{a+2}^{b+1}|\phi_2(s)|\right], \tag{3.15}$$

where $M \in \{M_{even}, M_{odd}\}$ and $N \in \{N_{even}, N_{odd}\}$, in which

$$M_{even} = \frac{(b-a-1)!}{\left(\frac{b-a}{2}-1\right)!\left(\frac{b-a}{2}\right)!}\cdot\frac{\Gamma\left(\frac{b-a}{2}+\alpha-1\right)\Gamma\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)},$$

$$M_{odd} = \frac{(b-a-1)!}{\left(\frac{b-a+1}{2}-1\right)!\left(\frac{b-a+1}{2}\right)!}\cdot\frac{\Gamma\left(\frac{b-a+1}{2}+\alpha-1\right)\Gamma\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha)\Gamma(\alpha+b-a-1)},$$

$$N_{even} = \frac{(b-a-1)!}{\left(\frac{b-a}{2}-1\right)!\left(\frac{b-a}{2}\right)!}\cdot\frac{\Gamma\left(\frac{b-a}{2}+\beta-1\right)\Gamma\left(\frac{b-a}{2}+\beta\right)}{\Gamma(\beta)\Gamma(\beta+b-a-1)},$$

$$N_{odd} = \frac{(b-a-1)!}{\left(\frac{b-a+1}{2}-1\right)!\left(\frac{b-a+1}{2}\right)!}\cdot\frac{\Gamma\left(\frac{b-a+1}{2}+\beta-1\right)\Gamma\left(\frac{b-a+1}{2}+\beta\right)}{\Gamma(\beta)\Gamma(\beta+b-a-1)},$$

then the coupled system of two-point fractional ∇-difference boundary value problems (1.2) has at least one solution in E.

Proof. Let us consider the following coupled system of fractional λ-parametric ∇-difference boundary value problems

$$\begin{pmatrix}\nabla^{\alpha}_{a^+}u(t)\\ \nabla^{\beta}_{a^+}v(t)\end{pmatrix} + \lambda\begin{pmatrix}f(t,v(t))\\ g(t,u(t))\end{pmatrix} = 0, \quad \begin{pmatrix}u(a+1)\\ u(b+1)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}v(a+1)\\ v(b+1)\end{pmatrix}, \tag{3.16}$$

for $\lambda \in (0,1)$. So solving (3.16) is equivalent to solving the fixed point problem $(u,v) = \lambda\mathfrak{T}(u,v)$ where $\mathfrak{T}$ is given by (3.12)-(3.13). Define

$$\Omega = \left\{(u,v) \in E \mid \|u\|_{\mathfrak{B}}, \|v\|_{\mathfrak{B}} < \rho/2\right\}. \tag{3.17}$$

We have to prove that $(u,v) \neq \lambda\mathfrak{T}(u,v)$ for $(u,v) \in \partial\Omega$ and $\lambda \in (0,1)$. To this aim suppose on the contrary that there exists $(u,v) \in \partial\Omega$ such that $(u,v) = \lambda\mathfrak{T}(u,v) = \lambda(\mathfrak{T}_1 v, \mathfrak{T}_2 u)$. So for $\lambda \in (0,1)$ it follows that

$$\begin{aligned}
\|u\|_{\mathfrak{B}} &= \lambda \sup_{t\in[a+2,b+1]_{\mathbb{N}}} |\mathfrak{T}_1 v| \\
&\leqslant \sup_{t\in[a+2,b+1]_{\mathbb{N}}} \left| \sum_{s=a+2}^{b+1} \mathscr{G}(t,s) f(s,v(s)) \right| \\
&\leqslant M \sum_{s=a+2}^{b+1} |f(s,v(s))| \\
&\leqslant M\psi_1(\rho) \sum_{s=a+2}^{b+1} |\phi_1(s)|.
\end{aligned} \tag{3.18}$$

Therefore it follows that

$$\rho \leqslant 2M\psi_1(\rho) \sum_{s=a+2}^{b+1} |\phi_1(s)|. \tag{3.19}$$

Similarly one has from $v = \lambda\mathfrak{T}_2 u$ that

$$\rho \leqslant 2N\psi_2(\rho) \sum_{s=a+2}^{b+1} |\phi_2(s)|. \tag{3.20}$$

Inequalities (3.19) and (3.20), yield

$$\left(\frac{1}{M} + \frac{1}{N}\right) \leqslant \frac{2}{\rho}\left[\psi_1(\rho)\sum_{a+2}^{b+1}|\phi_1(s)| + \psi_2(\rho)\sum_{a+2}^{b+1}|\phi_2(s)|\right],$$

which contradicts (3.15). This contradiction proves that the second part of Theorem 1, namely (E_2) is not satisfied. Thereby we conclude that there exists an $(u,v) \in \overline{\Omega}$ such that $(u,v) = \mathfrak{T}(u,v)$. Equivalently the coupled system of two-point fractional ∇-difference boundary value problems (1.2) has at least one solution in E. The proof is completed.

HYPTHESES 2. *Assume that the following assumptions are satisfied in what follows.*

(S_1) $\displaystyle\lim_{\|v\|_{\mathfrak{B}}\to\infty} \frac{f(t,v)}{v} = \Theta_1(t)$;

(S_2) $\displaystyle\lim_{\|u\|_{\mathfrak{B}}\to\infty} \frac{g(t,u)}{u} = \Theta_2(t)$.

THEOREM 4. *Let the assumptions (S_1) and (S_2) be satisfied. Suppose*

$$\sum_{s=a+2}^{b+1} \mathscr{G}(t,s) < \|\Theta_1\|_{\mathfrak{B}}^{-1}, \quad \sum_{s=a+2}^{b+1} \mathscr{H}(t,s) < \|\Theta_2\|_{\mathfrak{B}}^{-1}, \quad t \in [a+2, b+1]_{\mathbb{N}}. \tag{3.21}$$

Then the coupled system of two-point fractional ∇-difference boundary value problems (1.2) has at least one solution in E.

Proof. Consider the linear bounded mappings $L_i : E \to E$, $i = 1,2$ given by

$$L_1 v(t) = \sum_{s=a+2}^{b+1} \mathscr{G}(t,s)v(s)\Theta_1(s), \quad L_2 u(t) = \sum_{s=a+2}^{b+1} \mathscr{H}(t,s)u(s)\Theta_2(s). \tag{3.22}$$

Obviously one can derive

$$\|L_1 v\|_{\mathfrak{B}} \leqslant \sum_{s=a+2}^{b+1} \mathscr{G}(t,s)\|v\|_{\mathfrak{B}}\|\Theta_1\|_{\mathfrak{B}} < \|v\|_{\mathfrak{B}}, \tag{3.23}$$

which demonstrates that 1 can not be an eigenvalue of L_1. In the same way L_2 can not admit 1 as its eigenvalue. Therefore if we define $L(u,v) = (L_1 v, L_2 u)$, then $(1,1)$ can not be the eigenvalue of L. Now considering the limit approach of the hypotheses (S_1) and (S_2), for arbitrary $\varepsilon > 0$ we have

$$\begin{aligned} \|\mathfrak{T}_1 v - L_1 v\|_{\mathfrak{B}} &\leqslant \sum_{s=a+2}^{b+1} \mathscr{G}(t,s)\|f(t,v) - v\Theta_1\|_{\mathfrak{B}} \\ &\leqslant \sum_{s=a+2}^{b+1} \mathscr{G}(t,s)\varepsilon\|v\|_{\mathfrak{B}} < (b-a)M\varepsilon\|v\|_{\mathfrak{B}}. \end{aligned} \tag{3.24}$$

Similarly we deduce that

$$\|\mathfrak{T}_2 u - L_2 u\|_{\mathfrak{B}} < (b-a)N\varepsilon\|u\|_{\mathfrak{B}}. \tag{3.25}$$

Note that M and N appeared in (3.24) and (3.25) are defined as in Theorem 3. Thus using inequalities (3.24) and (3.25), we have

$$\begin{aligned} \lim_{(u,v)\to(\infty,\infty)} \frac{\|\mathfrak{T}(u,v) - L(u,v)\|_E}{\|(u,v)\|_E} &\leqslant \lim_{\substack{\|u\|_{\mathfrak{B}}\to\infty \\ \|v\|_{\mathfrak{B}}\to\infty}} \left\{ \frac{\|\mathfrak{T}_1 v - L_1 v\|_{\mathfrak{B}}}{\|v\|_{\mathfrak{B}}} + \frac{\|\mathfrak{T}_2 u - L_2 u\|_E}{\|u\|_{\mathfrak{B}}} \right\} \\ &< \varepsilon(b-a)(M+N), \end{aligned} \tag{3.26}$$

for arbitrary $\varepsilon > 0$. Thereby Theorem 2 ensures that the vector equation $\mathfrak{T}(u,v)$ defined by (3.12)-(3.13) has at least one fixed point in E. Equivalently the coupled system of two-point fractional ∇-difference boundary value problems (1.2) has at least one solution in E. This completes the proof.

4. Numerical Examples

EXAMPLE 1. Consider the following coupled system of two-point fractional ∇-difference boundary value problems

$$\begin{pmatrix} \nabla_{0^+}^{\frac{3}{2}} u(t) \\ \nabla_{0^+}^{\frac{3}{2}} v(t) \end{pmatrix} + \begin{pmatrix} f(t,v(t)) \\ g(t,u(t)) \end{pmatrix} = 0, \quad \begin{pmatrix} u(1) \\ u(9) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(1) \\ v(9) \end{pmatrix}. \tag{4.1}$$

Actually system (4.1) is updated with setting $\alpha = \beta = \frac{3}{2}$ and $a = 0$, $b = 8$. Taking into account that $a+b$ is even, we have $M = M_{even}$ and $N = N_{even}$. Therefore choosing $\rho = 5$ and taking

$$f(t,v) = \underbrace{\exp(-t)}_{\phi_1}\underbrace{\tan^{-1}(v)}_{\psi_1}, \quad g(t,u) = \underbrace{\exp(-t)}_{\phi_2}\underbrace{\ln(1+u)}_{\psi_2}, \tag{4.2}$$

we find that the conditions (H_1) and (H_2) hold. Consequently a direct calculation demonstrates that

$$\left(\frac{1}{M} + \frac{1}{N}\right) \approx 1.022715 > 0.633032 \approx \frac{2}{\rho}\left[\psi_1(\rho)\sum_{a+2}^{b+1}|\phi_1(s)| + \psi_2(\rho)\sum_{a+2}^{b+1}|\phi_2(s)|\right].$$

Therefore Theorem 3 implies that the coupled system (4.1) admits at least one solution in E.

EXAMPLE 2. Let us consider the coupled system (4.1). Suppose that

$$f(t,v) = \underbrace{(t+1)^{-2}}_{\Theta_1} v, \quad g(t,u) = \underbrace{\exp(-t)}_{\Theta_2} u, \quad t \in [2,9]_{\mathbb{N}}. \tag{4.3}$$

Thus hypotheses (S_1) and (S_2) are satisfied. Since $a+b$ is even, we have $M = N = 4.5686585$. On the other hand a simple computation shows that $\|\Theta_1\|_{\mathfrak{B}} = 0.2$ and $\|\Theta_2\|_{\mathfrak{B}} \approx 0.135335$. Thus we have

$$\sum_{s=2}^{9} \mathscr{G}(t,s) \approx 4.5686585 < 5 = \|\Theta_1\|_{\mathfrak{B}}^{-1},$$
$$\sum_{s=2}^{9} \mathscr{H}(t,s) \approx 4.5686585 < 7.389056 \approx \|\Theta_2\|_{\mathfrak{B}}^{-1}.$$

Thereby in accordance with Theorem 4, the coupled system (4.1) has at least one solution in E.

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