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POSITIVE SOLUTIONS FOR INFINITE SEMIPOSITONE/ POSITONE QUASILINEAR ELLIPTIC SYSTEMS WITH SINGULAR AND SUPERLINEAR TERMS

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Abstract. We establish existence and regularity of positive solutions for a class of quasilinear elliptic systems with singular and superlinear terms. The approach is based on sub-supersolution methods for systems of quasilinear singular equations and the Schauder's fixed point theorem.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain with $C^{1,\alpha}$-boundary $\partial\Omega$, $\alpha \in (0,1)$, and let $1 < p,q \leqslant N$. We deal with the following quasilinear singular elliptic problem

$$\begin{cases} -\Delta_p u = \lambda u^{\alpha_1} + v^{\beta_1} & \text{in } \Omega, \\ -\Delta_q v = u^{\alpha_2} + \lambda v^{\beta_2} & \text{in } \Omega, \\ u,v > 0 & \text{in } \Omega, \\ u,v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter. Here Δ_p and Δ_q denote the p-Laplacian and q-Laplacian differential operators defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and $\Delta_q v = div(|\nabla v|^{q-2}\nabla v)$, respectively. We consider system (1) in a singular case assuming that

$$-1 < \alpha_1, \beta_2 < 0. \tag{2}$$

A solution of (1) is understood in the weak sense, which means a pair of functions $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ which are positive a.e. in Ω such that

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, dx = \int_\Omega (\lambda u^{\alpha_1} + v^{\beta_1})\varphi \, dx \\ \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi \, dx = \int_\Omega (u^{\alpha_2} + \lambda v^{\beta_2})\psi \, dx \end{cases}$$

for all $(\varphi,\psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ provided the integrals in the right-hand side of the above equalities exist.

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We explicitly observe that under assumption (2) and depending on the sign of a real number λ, it holds

$$\lim_{s\to 0^+}(\lambda s^{\alpha_1}+s^{\beta_1}) = \lim_{s\to 0^+}(s^{\alpha_2}+\lambda s^{\beta_2}) = \begin{cases} +\infty \text{ if } \lambda > 0 \\ -\infty \text{ if } \lambda < 0. \end{cases}$$

Therefore, system (1) can be referred to as an infinite positone problem if $\lambda > 0$ and as an infinite semipositone problem if $\lambda < 0$.

The principle fact in this work is that the singularity in problem (1) comes out through nonlinearities which are $(p-1)$-superlinear and $(q-1)$-superlinear near $+\infty$. Namely, we assume that

$$\alpha_2 > q-1 \text{ and } \beta_1 > p-1. \tag{3}$$

In this context, system (1) has a cooperative structure, that is, for u (resp. v) fixed the right term in the first (resp. second) equation of (1) is increasing in v (resp. u). Further, according to (3) we have

$$\lim_{s\to+\infty}(\lambda s^{\alpha_1}+s^{\beta_1})/s^{p-1} = \lim_{s\to+\infty}(s^{\alpha_2}+\lambda s^{\beta_2})/s^{q-1} = +\infty.$$

This type of problem is rare in the literature. According to our knowledge, only a positone-type singular system with superlinear terms was examined in [21]. There the authors considered problem (1) depending on two positive parameters in the whole space $\mathbb{R}^N$. The existence of a positive entire solution is shown provided the parameters are sufficiently small. Recall that in [21] an entire solution is defined to be a function in $C^1(\mathbb{R}^N)\times C^1(\mathbb{R}^N)$ satisfying (1) at every point $x\in\mathbb{R}^N$ and vanishing at infinity.

The sublinear condition $\alpha_2 < q-1$ and $\beta_1 < p-1$ for singular systems of type (1) have been thoroughly investigated. For a complete overview on the study of the infinite positone problem (1) we refer to [1, 2, 15, 17], while for the study of the infinite semipositone problem (1), we cite [5, 13, 14]. We also mention [6, 7] focusing on the semilinear case of (1), that is, when $p=q=2$.

Another important class of singular problems considered in the literature is the following

$$\begin{cases} -\Delta_p u = u^{\alpha_1}v^{\beta_1} & \text{in } \Omega, \\ -\Delta_q v = u^{\alpha_2}v^{\beta_2} & \text{in } \Omega, \\ u,v>0 & \text{in } \Omega, \\ u,v=0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Relevant contributions regarding the cooperative case of system (4), that is $\alpha_2,\beta_1>0$, can be found in [8, 9, 18]. With regard to the complementary situation $\alpha_2,\beta_1<0$ which is the so called competitive structure for system (4), we quote the papers [9, 17, 19]. The semilinear case in (1) (i.e. $p=q=2$) was examined in [6, 12, 20] where the linearity of the principal part is essentially used. It is worth pointing out that in the aforementioned works, singular problem (4) was examined only under the sublinear condition $\max\{\alpha_1,\beta_1\}<p-1$ and $\max\{\alpha_2,\beta_2\}<q-1$. The assumptions imposed therein, especially in [19], are not satisfied for our system (1) under hypothesis (3).

Our main concern is the question of existence of a (positive) smooth solution for a singular system a class of elliptic systems where the nonlinearities besides a singular terms have superlinear terms. The main result is formulated in the next theorem.

THEOREM 1. *Assume (2) and (3) hold. Then system (1) has a (positive) solution* (u,v) *in* $(W_0^{1,p}(\Omega)\cap C_0^{1,\gamma}(\overline{\Omega}))\times(W_0^{1,q}(\Omega)\cap C_0^{1,\gamma}(\overline{\Omega}))$ *for some* $\gamma\in(0,1)$.

The proof of Theorem 1 is done in section 3. The main technical difficulty consists in the presence of singular terms in system (1) that can occur under hypothesis (2). This difficulty is heightened by the superlinear character of (1) that arise from (3). Our approach is chiefly based on Theorem 2 proved in Section 2 via Schauder's fixed point theorem (see [22]) and adequate truncations. This is a version of the sub-supersolution method for quasilinear singular elliptic systems with cooperative structure. We mention that in Theorem 2 no sign condition is required on the right-hand side nonlinearities and so it can be used for large classes of quasilinear singular problems. A significant feature of our result lies in the obtaining of the sub- and supersolution. Due to the superlinear character of the nonlinearities in (1), the latter cannot be constructed easily. At this point, the choice of suitable functions with an adjustment of adequate constants is crucial. Here we emphasize that the obtained sub- and supersolution are quite different from functions considered in the aforementioned papers, especially those constructed in [19].

This article is organized as follows. In section 2 we state and prove a general theorem about sub and supersolution method for singular systems. Section 3 contains the proof of Theorem 1.

2. Sub and supersolution theorem

Given $1<p<+\infty$, the spaces $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ are endowed with the usual norms

$$\|u\|_p=\Big(\int_\Omega|u|^p dx\Big)^{1/p}\quad\text{and}\quad\|u\|_{1,p}=\Big(\int_\Omega|\nabla u|^p dx\Big)^{1/p},$$

respectively. In the sequel, corresponding to $1<p<+\infty$, we denote $p'=\frac{p}{p-1}$. We will also use the spaces $C(\overline{\Omega})$ and

$$C_0^{1,\gamma}(\overline{\Omega})=\{u\in C^{1,\gamma}(\overline{\Omega}):u=0\text{ on }\partial\Omega\}$$

with $\gamma\in(0,1)$. We denote by $\lambda_{1,p}$ and $\lambda_{1,q}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ and of $-\Delta_q$ on $W_0^{1,q}(\Omega)$, respectively. Let $\phi_{1,p}$ be the normalized positive eigenfunction of $-\Delta_p$ corresponding to $\lambda_{1,p}$, that is

$$-\Delta_p\phi_{1,p}=\lambda_{1,p}\phi_{1,p}^{p-1}\text{ in }\Omega,\ \ \phi_{1,p}=0\text{ on }\partial\Omega,\ \ \|\phi_1\|_p=1$$

Similarly, let $\phi_{1,q}$ be the normalized positive eigenfunction of $-\Delta_q$ corresponding to $\lambda_{1,q}$, that is

$$-\Delta_q\phi_{1,q}=\lambda_{1,q}\phi_{1,q}^{q-1}\text{ in }\Omega,\ \ \phi_{1,q}=0\text{ on }\partial\Omega,\ \ \|\phi_2\|_q=1.$$

For later use we set

$$R = \max\left\{\max_{\overline{\Omega}}\phi_{1,p}, \max_{\overline{\Omega}}\phi_{1,q}\right\}. \tag{5}$$

We denote by $d(x)$ the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$. It is known that we can find a constant $l > 0$ such that

$$\phi_{1,p}(x), \phi_{1,q}(x) \geqslant l d(x) \text{ for all } x \in \Omega, \tag{6}$$

where $d(x) := dist(x, \partial\Omega)$ (see, e.g., [10, page 121]).

Let us introduce the problem

$$\begin{cases} -\Delta_p u = f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary, $1 < p, q < \infty$ and $f, g : \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ are continuous functions which can exhibit singularities when the variables u and v approach zero. We consider system (7) with cooperative structure assuming that for u (resp. v) fixed the nonlinearity f (resp. g) is increasing in v (resp. u). This makes the sub-supersolution techniques applicable for (7). For systems without cooperative structure, i.e. competitive systems, additional assumptions are required (see [9]).

We recall that a sub-supersolution for (7) is any pair $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in (W_0^{1,p}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,q}(\Omega) \cap L^\infty(\Omega))$ for which there hold $(\overline{u}, \overline{v}) \geqslant (\underline{u}, \underline{v})$ in Ω,

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx - \int_\Omega f(x, \underline{u}, \omega_2)\varphi \, dx + \int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi \, dx - \int_\Omega g(x, \omega_1, \underline{v})\psi \, dx \leqslant 0,$$

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \, dx - \int_\Omega f(x, \overline{u}, \omega_2)\varphi \, dx + \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v} \nabla \psi - \int_\Omega g(x, \omega_1, \overline{v})\psi \, dx \geqslant 0,$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with $\varphi, \psi \geqslant 0$ a.e. in Ω and for all $(\omega_1, \omega_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfying $\underline{u} \leqslant \omega_1 \leqslant \overline{u}$ and $\underline{v} \leqslant \omega_2 \leqslant \overline{v}$ a.e. in Ω (see [4, p. 269]). The main goal in this section is to prove Theorem 2 below, which is a key point in the proof of Theorem 1.

THEOREM 2. *Let $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ be a sub and supersolution pairs of (7) and suppose there exist constants $k_1, k_2 > 0$ and $-1 < \alpha, \beta < 0$ such that*

$$|f(x,u,v)| \leqslant k_1 d(x)^\alpha \text{ and } |g(x,u,v)| \leqslant k_2 d(x)^\beta \text{ in } \Omega \times [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]. \tag{8}$$

Then system (7) has a positive solution (u,v) *in* $(W_0^{1,p}(\Omega)\cap C_0^{1,\gamma}(\overline{\Omega}))\times(W_0^{1,q}(\Omega)\cap C_0^{1,\gamma}(\overline{\Omega}))$ *for certain* $\gamma\in(0,1)$.

Proof. For each $(z_1,z_2)\in C(\overline{\Omega})\times C(\overline{\Omega})$, let $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ be the unique solution of the problem

$$\begin{cases} -\Delta_p u=\widetilde{f}(x,z_1,z_2) & \text{in } \Omega,\\ -\Delta_q v=\widetilde{g}(x,z_1,z_2) & \text{in } \Omega,\\ u,v=0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where

$$\widetilde{f}(x,z_1,z_2)=f(x,\widetilde{z}_1,\widetilde{z}_2) \text{ and } \widetilde{g}(x,z_1,z_2)=g(x,\widetilde{z}_1,\widetilde{z}_2) \tag{10}$$

with

$$\widetilde{z}_1=\min(\max(z_1,\underline{u}),\overline{u}) \text{ and } \widetilde{z}_2=\min(\max(z_2,\underline{v}),\overline{v}). \tag{11}$$

On account of (11) it follows that $\underline{u}\leqslant\widetilde{z}_1\leqslant\overline{u}$ and $\underline{v}\leqslant\widetilde{z}_2\leqslant\overline{v}$. Then, bearing in mind (8) we have

$$\left|\widetilde{f}(x,z_1,z_2)\right|\leqslant k_1 d(x)^{\alpha} \text{ and } |\widetilde{g}(x,z_1,z_2)|\leqslant k_2 d(x)^{\beta} \text{ for a.e. } x\in\Omega. \tag{12}$$

We point out that the estimates (12) enable us to deduce that

$$\widetilde{f}(x,z_1,z_2)\in W^{-1,p'}(\Omega) \text{ and } \widetilde{g}(x,z_1,z_2)\in W^{-1,q'}(\Omega).$$

This is a consequence of Hardy-Sobolev inequality (see, e.g., [1, Lemma 2.3]), that can be applied since the exponents $\alpha,\beta\in(-1,0)$. Then the unique solvability of (u,v) in (9) is readily derived from Minty-Browder Theorem (see, e.g., [3]).

Let us introduce the operator

$$\begin{array}{rcl} \mathscr{T}: C(\overline{\Omega})\times C(\overline{\Omega}) & \to & C(\overline{\Omega})\times C(\overline{\Omega}) \\ (z_1,z_2) & \mapsto & (u,v). \end{array}$$

We note from (9) that any fixed point of $\mathscr{T}$ coincides with the weak solution of (7). Consequently, to achieve the desired conclusion it suffices to prove that $\mathscr{T}$ has a fixed point. To this end we apply Schauder's fixed point theorem. Using (12) with $\alpha,\beta\in(-1,0)$, there exists $\gamma\in(0,1)$ such that

$$(u,v)\in C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega}) \quad\text{and}\quad \|u\|_{C_0^{1,\gamma}(\overline{\Omega})},\ \|u\|_{C_0^{1,\gamma}(\overline{\Omega})}\leqslant C,$$

where $C>0$ is independent of u and v (see [11, Lemma 3.1]). Then the compactness of the embedding $C_0^{1,\gamma}(\overline{\Omega})\subset C(\overline{\Omega})$ implies that $\mathscr{T}(C(\overline{\Omega})\times C(\overline{\Omega}))$ is a relatively compact subset of $C(\overline{\Omega})\times C(\overline{\Omega})$.

Next, we show that $\mathscr{T}$ is continuous with respect to the topology of $C(\overline{\Omega})\times C(\overline{\Omega})$. Let $(z_{1,n},z_{2,n})\to(z_1,z_2)$ in $C(\overline{\Omega})\times C(\overline{\Omega})$ for all n. Denote $(u_n,v_n)=\mathscr{T}(z_{1,n},z_{2,n})$, which reads as

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi=\int_\Omega \widetilde{f}(x,z_{1,n},z_{2,n})\varphi\,dx \tag{13}$$

and

$$\int_\Omega |\nabla v_n|^{q-2}\nabla v_n \nabla \psi = \int_\Omega \widetilde{g}(x,z_{1,n},z_{2,n})\psi\, dx \tag{14}$$

for all $(\varphi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Inserting $(\varphi,\psi)=(u_n,v_n)$ in (13) and (14), using (8) we get

$$\|u_n\|_{1,p} = \int_\Omega \widetilde{f}(x,z_{1,n},z_{2,n})u_n\, dx \leqslant \int_\Omega d^\alpha u_n dx \tag{15}$$

and

$$\|v_n\|_{1,q} = \int_\Omega \widetilde{g}(x,z_{1,n},z_{2,n})v_n\, dx \leqslant \int_\Omega d^\beta v_n dx. \tag{16}$$

Since $-1<\alpha,\beta<0$, by virtue of the Hardy-Sobolev inequality (see, e.g., [1]), the last integrals in (15) and (16) are finite which in turn imply that $\{u_n\}$ and $\{v_n\}$ are bounded in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively. So, passing to relabeled subsequences, we can write the weak convergence in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$

$$(u_n,v_n)\rightharpoonup (u,v) \tag{17}$$

for some $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Setting $\varphi=u_n-u$ in (13) and $\psi=v_n-v$ in (14), we find that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla (u_n-u) = \int_\Omega \widetilde{f}(x,z_{1,n},z_{2,n})(u_n-u)\, dx$$

and

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla (v_n-v) = \int_\Omega \widetilde{g}(x,z_{1,n},z_{2,n})(v_n-v)\, dx.$$

Lebesgue's dominated convergence theorem ensures that

$$\lim_{n\to\infty}\left\langle -\Delta_p u_n, u_n-u\right\rangle = \lim_{n\to\infty}\left\langle -\Delta_q v_n, v_n-v\right\rangle = 0.$$

The S_+-property of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ and of $-\Delta_q$ on $W_0^{1,q}(\Omega)$ (see, e.g. [16, Proposition 3.5]), along with (17), implies

$$u_n\to u \text{ in } W_0^{1,p}(\Omega) \text{ and } v_n\to v \text{ in } W_0^{1,q}(\Omega).$$

Then, through (13), (14) and the invariance of $C(\overline{\Omega})\times C(\overline{\Omega})$ by $\mathscr{T}$, we infer that $(u,v)=\mathscr{T}(z_1,z_2)$. On the other hand, from (13) and (14) we know that the sequence $\{(u_n,v_n)\}$ is bounded in $C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ for certain $\gamma\in(0,1)$. Since the embedding $C_0^{1,\gamma}(\overline{\Omega})\subset C(\overline{\Omega})$ is compact, along a relabeled subsequence there holds $(u_n,v_n)\to(u,v)$ in $C(\overline{\Omega})\times C(\overline{\Omega})$. We conclude that $\mathscr{T}$ is continuous.

We are thus in a position to apply Schauder's fixed point theorem to the map $\mathscr{T}$, which establishes the existence of $(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega})$ satisfying $(u,v)=\mathscr{T}(u,v)$.

Let us justify that

$$\underline{u}\leqslant u\leqslant \overline{u} \text{ and } \underline{v}\leqslant v\leqslant \overline{v} \text{ in } \Omega.$$

Put $\zeta=(\underline{u}-u)^+$ and suppose $\zeta\neq 0$. Then, bearing in mind that system (7) is cooperative, from (11), (9) and (10), we infer that

$$\int_{\{u<\underline{u}\}} |\nabla u|^{p-2}\nabla u\nabla\zeta dx = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\zeta\, dx$$

$$= \int_{\{u<\underline{u}\}} \widetilde{f}(x,u,v)\zeta \, dx = \int_{\{u<\underline{u}\}} f(x,\widetilde{u},\widetilde{v})\zeta dx$$
$$= \int_{\{u<\underline{u}\}} f(x,\underline{u},\widetilde{v})\zeta \, dx \geqslant \int_{\{u<\underline{u}\}} |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\zeta \, dx.$$

This implies that

$$\int_{\{u<\underline{u}\}} (|\nabla u|^{p-2}\nabla u - |\nabla \underline{u}|^{p-2}\nabla \underline{u})\nabla\zeta \, dx \leqslant 0,$$

a contradiction. Hence $u \geqslant \underline{u}$ in Ω. A quite similar argument provides that $v \geqslant \underline{v}$ in Ω. In the same way, we prove that $u \leqslant \overline{u}$ and $v \leqslant \overline{v}$ in Ω.

Finally, thanks to [11, Lemma 3.1] one has $(u,v) \in C_0^{1,\gamma}(\overline{\Omega}) \times C_0^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. This completes the proof. ■

3. Proof of the main result

This section is devoted to the proof of Theorem 1. It relies on sub-supersolution techniques shown by Theorem 2.

Let y_1 and y_2 be the unique solutions of the problems

$$\begin{cases} -\Delta_p y_1 = y_1^{\alpha_1} & \text{in } \Omega \\ y_1 > 0 & \text{in } \Omega \\ y_1 = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_q y_2 = y_2^{\beta_2} & \text{in } \Omega \\ y_2 > 0 & \text{in } \Omega \\ y_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

respectively, which are known to satisfy

$$c_1\phi_{1,p}(x) \leqslant y_1(x) \leqslant c_2\phi_{1,p}(x) \ \text{ and } \ c_3\phi_{1,q}(x) \leqslant y_2(x) \leqslant c_4\phi_{1,q}(x), \tag{19}$$

with constants $c_2 \geqslant c_1 > 0$ and $c_4 \geqslant c_3 > 0$ (see Lemma 3.1, Theorem B.1 and the proof of Lemma A.7 in [10]). For $\delta > 0$ sufficiently small we denote

$$\Omega_\delta = \{x \in \Omega : dist\,(x, \partial\Omega) < \delta\}$$

and $\mu = \mu(\delta) > 0$ a constant such that

$$\phi_{1,p}(x), \phi_{1,q}(x) \geqslant \mu \text{ in } \Omega\backslash\Omega_\delta. \tag{20}$$

Let $\underline{u}$ and $\underline{v}$ satisfy

$$-\Delta_p \underline{u}(x) = C \begin{cases} y_1^{\alpha_1}(x) & \text{if } x \in \Omega\backslash\overline{\Omega}_\delta \\ -y_1^{\alpha_1}(x) & \text{if } x \in \Omega_\delta \end{cases}, \ \underline{u} = 0 \text{ on } \partial\Omega \tag{21}$$

and

$$-\Delta_q \underline{v}(x) = C \begin{cases} y_2^{\beta_2}(x) & \text{if } x \in \Omega\backslash\overline{\Omega}_\delta \\ -y_2^{\beta_2}(x) & \text{if } x \in \Omega_\delta \end{cases}, \ \underline{v} = 0 \text{ on } \partial\Omega, \tag{22}$$

with a constant $C > 1$ to be chosen later on. The Hardy-Sobolev inequality (see e.g. [1]) guarantees that the right hand side of (21) and (22) are in $W^{-1,p'}(\Omega)$ and $W^{-1,q'}(\Omega)$,

respectively. This allows to apply the Minty-Browder theorem (see [3, Theorem V.15]) to deduce the existence of unique solutions $\underline{u}$ and $\underline{v}$ for problems (21) and (22), respectively. Moreover, (18), (19), (21), (22) and the monotonicity of the operators $-\Delta_p$ and $-\Delta_q$ together with [11, Corollary 3.1] imply that

$$\begin{cases} \frac{c_1}{2}C^{\frac{1}{p-1}}\phi_{1,p}(x) \leqslant \underline{u}(x) \leqslant c_2 C^{\frac{1}{p-1}}\phi_{1,p}(x), \\ \frac{c_3}{2}C^{\frac{1}{q-1}}\phi_{1,q}(x) \leqslant \underline{v}(x) \leqslant c_4 C^{\frac{1}{q-1}}\phi_{1,q}(x) \text{ in } \Omega. \end{cases} \tag{23}$$

For $\lambda \geqslant 0$, the positivity of $\underline{u}, \underline{v}, y_1, y_2$ and C enable us to have

$$-Cy_1^{\alpha_1} - \lambda \underline{u}^{\alpha_1} \leqslant 0 \leqslant \underline{v}^{\beta_1} \text{ in } \Omega_\delta \tag{24}$$

and

$$-Cy_2^{\beta_2} - \lambda \underline{v}^{\beta_2} \leqslant 0 \leqslant \underline{u}^{\alpha_2} \text{ in } \Omega_\delta. \tag{25}$$

For $\lambda < 0$, (19), (23) and (2) imply

$$-Cy_1^{\alpha_1} - \lambda \underline{u}^{\alpha_1} \leqslant (-Cc_2^{\alpha_1} - \lambda(\tfrac{c_1}{2}C^{\frac{1}{p-1}})^{\alpha_1})\phi_{1,p}^{\alpha_1} \leqslant 0 \leqslant \underline{v}^{\beta_1} \text{ in } \Omega_\delta \tag{26}$$

and

$$-Cy_2^{\beta_2} - \lambda \underline{v}^{\beta_2} \leqslant \left(-C^{q-1}c_4^{\beta_2} - \lambda(\tfrac{c_3}{2}C^{\frac{1}{q-1}})^{\beta_2}\right)\phi_{1,q}^{\beta_2} \leqslant 0 \leqslant \underline{u}^{\alpha_2} \text{ in } \Omega_\delta, \tag{27}$$

provided that C is sufficiently large. Now we deal with the corresponding estimates on $\Omega \backslash \overline{\Omega}_\delta$. If $\lambda \geqslant 0$, we get from (19), (23), (20), (6) and (2) that

$$\begin{aligned} (Cy_1^{\alpha_1} - \lambda \underline{u}^{\alpha_1})\underline{v}^{-\beta_1} &\leqslant Cy_1^{\alpha_1}\underline{v}^{-\beta_1} \leqslant C^{1-\frac{\beta_1}{q-1}}c_1^{\alpha_1}(\frac{c_3}{2}l)^{-\beta_1}\phi_{1,p}^{\alpha_1-\beta_1} \\ &\leqslant C^{1-\frac{\beta_1}{q-1}}c_1^{\alpha_1}(\frac{c_3}{2}l)^{-\beta_1}\mu^{\alpha_1-\beta_1} \leqslant 1 \text{ in } \Omega\backslash\overline{\Omega}_\delta \end{aligned} \tag{28}$$

and

$$\begin{aligned} (Cy_2^{\beta_2} - \lambda \underline{v}^{\beta_2})\underline{u}^{-\alpha_2} &\leqslant Cy_2^{\beta_2}\underline{u}^{-\alpha_2} \leqslant C^{1-\frac{\alpha_2}{p-1}}c_3^{\beta_2}(\frac{c1}{2}l)^{-\alpha_2}\phi_{1,q}^{\beta_2-\alpha_2} \\ &\leqslant C^{1-\frac{\alpha_2}{p-1}}c_3^{\beta_2}(\frac{c1}{2}l)^{-\alpha_2}\mu^{\beta_2-\alpha_2} \leqslant 1 \text{ in } \Omega\backslash\overline{\Omega}_\delta, \end{aligned} \tag{29}$$

provided that C is sufficiently large. For $\lambda < 0$, (19), (23), (20), (6) and (2) imply

$$\begin{aligned} (Cy_1^{\alpha_1} - \lambda \underline{u}^{\alpha_1})\underline{v}^{-\beta_1} &\leqslant \left(Cc_1^{\alpha_1} - \lambda C^{\frac{\alpha_1}{p-1}}(\frac{c_1}{2})^{\alpha_1}\right)C^{-\frac{\beta_1}{q-1}}(\frac{c_3}{2}l)^{-\beta_1}\phi_{1,p}^{\alpha_1-\beta_1} \\ &= C^{1-\frac{\beta_1}{q-1}}c_1^{\alpha_1}\left(1 - \lambda C^{\frac{\alpha_1}{p-1}-1}2^{-\alpha_1}\right)(\frac{c_3}{2}l)^{-\beta_1}\phi_{1,p}^{\alpha_1-\beta_1} \\ &\leqslant C^{1-\frac{\beta_1}{q-1}}c_1^{\alpha_1}\left(1 - \lambda 2^{-\alpha_1}\right)(\frac{c_3}{2}l)^{-\beta_1}\mu^{\alpha_1-\beta_1} \leqslant 1 \text{ in } \Omega\backslash\overline{\Omega}_\delta \end{aligned}$$

and

$$(Cy_2^{\beta_2} - \lambda \underline{v}^{\beta_2})\underline{u}^{-\alpha_2} \leqslant \left(Cc_3^{\beta_2} - \lambda C^{\frac{\beta_2}{q-1}}(\frac{c_3}{2})^{\beta_2}\right)C^{-\frac{\alpha_2}{p-1}}(\frac{c_1}{2}l)^{-\alpha_2}\phi_{1,q}^{\beta_2-\alpha_2}$$

$$= C^{1-\frac{\alpha_2}{p-1}} c_3^{\beta_2} \left(1 - \lambda C^{\frac{\beta_2}{q-1}-1} 2^{-\beta_2}\right) (\frac{c_1}{2} l)^{-\alpha_2} \phi_{1,q}^{\beta_2-\alpha_2}$$
$$\leqslant C^{1-\frac{\alpha_2}{p-1}} c_3^{\beta_2} \left(1 - \lambda 2^{-\beta_2}\right) (\frac{c_1}{2} l)^{-\alpha_2} \mu^{\beta_2-\alpha_2} \leqslant 1 \text{ in } \Omega\backslash\overline{\Omega}_\delta,$$

provided that C is sufficiently large. This is equivalent to

$$C y_1^{\alpha_1} \leqslant \lambda \underline{u}^{\alpha_1} + \underline{v}^{\beta_1} \text{ in } \Omega\backslash\overline{\Omega}_\delta \tag{30}$$

and

$$C y_2^{\beta_2} \leqslant \underline{u}^{\alpha_2} + \lambda \underline{v}^{\beta_2} \text{ in } \Omega\backslash\overline{\Omega}_\delta, \tag{31}$$

for all $\lambda \in \mathbb{R}$.

Due to the definition of $\underline{u}$ and $\underline{v}$ (see (21) and (22)) we actually have

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx = \int_{\Omega\backslash\overline{\Omega}_\delta} C y_1^{\alpha_1} \varphi \, dx - \int_{\Omega_\delta} C y_1^{\alpha_1} \varphi \, dx \tag{32}$$

and

$$\int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi \, dx = \int_{\Omega\backslash\overline{\Omega}_\delta} C y_2^{\beta_2} \psi \, dx - \int_{\Omega_\delta} C y_2^{\beta_2} \psi \, dx, \tag{33}$$

where $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with $\varphi, \psi \geqslant 0$. Then combining (24)-(27), (30), (31) with (32), (33), it is readily seen that

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, dx \leqslant \int_\Omega (\lambda \underline{u}^{\alpha_1} + \underline{v}^{\beta_1}) \varphi \, dx$$

and

$$\int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi \, dx \leqslant \int_\Omega (\underline{u}^{\alpha_2} + \lambda \underline{v}^{\beta_2}) \psi \, dx,$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with $\varphi, \psi \geqslant 0$, showing that $(\underline{u}, \underline{v})$ is a subsolution of problem (1).

Next, we construct a supersolution part for problem (1). To this end, let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,\alpha}$ boundary $\partial\widetilde{\Omega}$, $\alpha \in (0,1)$, such that $\overline{\Omega} \subset \widetilde{\Omega}$. We denote by $\widetilde{\lambda}_{1,p}$ and $\widetilde{\lambda}_{1,q}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\widetilde{\Omega})$ and of $-\Delta_q$ on $W_0^{1,q}(\widetilde{\Omega})$, respectively. Let $\widetilde{\phi}_{1,p}$ be the normalized positive eigenfunction of $-\Delta_p$ corresponding to $\widetilde{\lambda}_{1,p}$, that is

$$-\Delta_p \widetilde{\phi}_{1,p} = \widetilde{\lambda}_{1,p} \widetilde{\phi}_{1,p}^{p-1} \text{ in } \widetilde{\Omega}, \quad \widetilde{\phi}_{1,p} = 0 \text{ on } \partial\widetilde{\Omega}.$$

Similarly, let $\widetilde{\phi}_{1,q}$ be the normalized positive eigenfunction of $-\Delta_q$ corresponding to $\widetilde{\lambda}_{1,q}$, that is

$$-\Delta_q \widetilde{\phi}_{1,q} = \widetilde{\lambda}_{1,q} \widetilde{\phi}_{1,q}^{q-1} \text{ in } \Omega, \quad \widetilde{\phi}_{1,q} = 0 \text{ on } \partial\widetilde{\Omega}.$$

By the definition of $\widetilde{\Omega}$ and the strong maximum principle, there exists a constant $\rho > 0$ sufficiently small such that

$$\widetilde{\phi}_{1,p}(x), \widetilde{\phi}_{1,q}(x) > \rho \text{ in } \overline{\Omega}. \tag{34}$$

Without lost of generality we assume that

$$R = \max\left\{\max_{\overline{\Omega}}\widetilde{\phi}_{1,p}, \max_{\overline{\Omega}}\widetilde{\phi}_{1,q}\right\}. \tag{35}$$

Let $\xi_1, \xi_2 \in C^1\left(\overline{\widetilde{\Omega}}\right)$ be the solutions of the homogeneous Dirichlet problems:

$$\begin{cases} -\Delta_p \xi_1 = C^{\delta(p-1)}\xi_1^{\theta_1} \text{ in } \widetilde{\Omega} \\ \xi_1 = 0 \text{ on } \widetilde{\Omega} \end{cases}, \quad \begin{cases} -\Delta_q \xi_2 = C^{\delta(q-1)}\xi_2^{\theta_2} \text{ in } \widetilde{\Omega} \\ \xi_2 = 0 \text{ on } \widetilde{\Omega}, \end{cases} \tag{36}$$

with constants δ, θ_1 and θ_2 satisfying

$$\theta_1 \in (\alpha_1, 0), \quad \theta_2 \in (\beta_2, 0) \text{ and } \delta < \min\{\tfrac{1}{\theta_1}, \tfrac{1}{\theta_2}\} < 0. \tag{37}$$

Functions ξ_1 and ξ_2 verifying

$$C^\delta c_0 \widetilde{\phi}_{1,p} \leqslant \xi_1 \leqslant C^\delta c \widetilde{\phi}_{1,p} \text{ and } C^\delta c_0' \widetilde{\phi}_{1,q} \leqslant \xi_2 \leqslant C^\delta c' \widetilde{\phi}_{1,q}, \tag{38}$$

for some positive constants c_0, c_0', c and c' (see [10]). Set

$$(\overline{u}, \overline{v}) = C^{-\delta}(\xi_1, \xi_2). \tag{39}$$

Then we have $(\overline{u}, \overline{v}) \geqslant (\underline{u}, \underline{v})$ in $\overline{\Omega}$. Indeed, on the one hand, through (21), (22) and (36), one has

$$-\Delta_p \overline{u} \geqslant -\Delta_p \underline{u} \text{ and } -\Delta_p \overline{v} \geqslant -\Delta_p \underline{v} \text{ in } \Omega_\delta.$$

On the other hand, on the basis of (21), (22), (36), (38), (19), (20), (35), (37) and for C large enough, we achieve

$$\begin{aligned} -\Delta_p \overline{u} = C^{-\delta(p-1)}C^{\delta(p-1)}\xi_1^{\theta_1} &= \xi_1^{\theta_1} \geqslant C^{\delta\theta_1}(c\widetilde{\phi}_{1,p})^{\theta_1} \\ &\geqslant C^{\delta\theta_1}(cR)^{\theta_1} \geqslant C(c_1\mu)^{\alpha_1} \\ &\geqslant C(c_1\phi_{1,p})^{\alpha_1} = Cy_1^{\alpha_1} = -\Delta_p \underline{u} \text{ in } \Omega\backslash\overline{\Omega}_\delta, \end{aligned}$$

and

$$\begin{aligned} -\Delta_q \overline{v} = C^{-\delta(q-1)}C^{\delta(q-1)}\xi_2^{\theta_2} &= \xi_2^{\theta_2} \geqslant C^{\delta\theta_2}(c'\phi_{1,q})^{\theta_2} \\ &\geqslant C^{\delta\theta_2}(c'R)^{\theta_2} \geqslant C(c_3\mu)^{\beta_2} \\ &\geqslant C(c_3\phi_{1,q})^{\beta_2} \geqslant Cy_2^{\beta_2} = -\Delta_q \underline{v} \text{ in } \Omega\backslash\overline{\Omega}_\delta. \end{aligned}$$

Then the monotonicity of the operators $-\Delta_p$ and $-\Delta_q$ leads to the conclusion.

Now, taking into account (37), (38), (35), (34) and (2), for all $\lambda \in \mathbb{R}$, we derive that in $\overline{\Omega}$ one has

$$\begin{cases} \xi_1^{\theta_1} \geqslant C^{\delta\theta_1}(c\phi_{1,p})^{\theta_1} \geqslant C^{\delta\theta_1}(cR)^{\theta_1} \geqslant \lambda(c_0\rho)^{\alpha_1} + (c'R)^{\beta_1} \\ \lambda(c_0\widetilde{\phi}_{1,p})^{\alpha_1} + (c'\widetilde{\phi}_{1,q})^{\beta_1} \geqslant \lambda(C^{-\delta}\xi_1)^{\alpha_1} + (C^{-\delta}\xi_2)^{\beta_1} = \lambda\overline{u}^{\alpha_1} + \overline{v}^{\beta_1} \text{ in } \overline{\Omega} \end{cases} \tag{40}$$

and

$$\begin{cases} \xi_2^{\theta_2} \geqslant C^{\delta\theta_2}(c'\phi_{1,q})^{\theta_2} \geqslant C^{\delta\theta_2}(c'R)^{\theta_2} \geqslant \lambda(cR)^{\alpha_2} + (c_0'\rho)^{\beta_1} \\ \lambda(c\widetilde{\phi}_{1,p})^{\alpha_1} + (c_0'\widetilde{\phi}_{1,q})^{\beta_1} \geqslant \lambda(C^{-\delta}\xi_1)^{\alpha_1} + (C^{-\delta}\xi_2)^{\beta_1} = \lambda\overline{u}^{\alpha_1} + \overline{v}^{\beta_1} \ \text{ in } \overline{\Omega} \end{cases} \tag{41}$$

provided that C is sufficiently large. Consequently, it turns out from (36), (40) and (41) that

$$\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi\,dx = \int_\Omega \xi_1^{\theta_1}\varphi \geqslant \int_\Omega(\lambda\overline{u}^{\alpha_1} + \overline{v}^{\beta_1})\varphi\,dx$$

and

$$\int_\Omega |\nabla\overline{v}|^{q-2}\nabla\overline{v}\nabla\psi\,dx = \int_\Omega \xi_2^{\theta_2}\psi \geqslant \int_\Omega \left(\overline{u}^{\alpha_2} + \lambda\overline{v}^{\beta_2}\right)\psi\,dx,$$

for all $(\varphi,\psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. This proves that the pair $(\overline{u},\overline{v})$ is a supersolution for problem (1).

Finally, owing to Theorem 2 problem (1) has a positive solution $(u,v) \in C_0^{1,\gamma}(\overline{\Omega}) \times C_0^{1,\gamma}(\overline{\Omega})$, for certain $\gamma \in (0,1)$, within $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. This completes the proof.

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