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EXISTENCE OF SOLUTIONS TO NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we provide sufficient conditions for the existence of solutions to nonlinear boundary value problems. We do so by applying a general abstract strategy for solving nonlinear equations with a linear component. We apply this to general systems by first isolating a linear periodic system and using the general theory of periodic solutions to find conditions on the additional nonlinear components to guarantee solutions.

1. Introduction

In this paper we study general nonlinear systems of differential equations with boundary conditions. The goal is to provide sufficient conditions under which solutions exist. We do so by applying an abstract strategy previously used by the authors. We first consider the linear portion of the problem, together with periodic boundary conditions, and then analyze the requirements on the remaining nonlinearities to guarantee a solution.

In Section 2, we layout the basic strategy for proving the existence of solutions to general nonlinearly perturbed equations. This strategy had been used previously in the context of Sturm-Liouville problems [20, 19, 18]. This approach provides a type of global inverse function theorem, which can be seen as extending results from [6, 5].

For a textbook study of multiple solutions to boundary value problems, see [11]. [22] studied similar boundary value problems to the present paper, but for multipoint boundary conditions. [21] provided conditions for when nonlinear boundary value problems, such as those considered here, can be reduced to finite-dimensional alternative problems. [17] used Galerkin's method to study nonlinear boundary value problems. For nonlinear boundary value problems with linear, Stieltjes boundary conditions, see [16]. [15] studied nonlinear two-point boundary value problems, and [14] covered the analagous case for partial differential equations. For the case of second order differential equations with three point boundary conditions, see [4]. For second order equations with boundary conditions that can be represented as continuous linear functionals, see [24]. Elliptic boundary value problems were considered by [2]. Boundary value problems with nonlocal, integral boundary conditions were studied in [3, 23, 13, 10]

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Related applications of fixed point theorems in the setting of fractional differential equations were dealt with in [1, 7]. [8] studied nonlinear Hammerstein integral equations with applications to elliptical partial differential equations.

Section 2 gives the general abstract theory that will be applied in Section 4. Section 3 describes the necessary background and defines the spaces on which we work. Section 4 applies the general theory to the case of nonlinear systems of differential equations. Section 5 shows how the previous results on Sturm-Liouville problems can be seen in the present context, and Section 6 provides an example of the application of the main theorem of the present paper to a more general problem. Finally, in Section 7, we discuss alternative choices of normed spaces and how these choices would affect the main results.

2. General strategy

In this section, we present a general strategy for proving the solvability of nonlinear equations when there is a linear component that is well-understood. This formalizes the abstract results from [20].

Our first lemma describes the form of the inverse for a linear operator with two components under special conditions. For a linear operator, L, let the kernel be denoted by $Ker(L) = L^{-1}\left(\{0\}\right)$.

LEMMA 1. *Let D, X, and Y be vector spaces. Assume $L : D \to Y$ and $B : D \to Z$ are linear operators, and define $\mathscr{L} = \begin{pmatrix} L \\ B \end{pmatrix}$. If $L \restriction_{Ker(B)}$ and $B \restriction_{Ker(L)}$ are both bijections, then*

$$\mathscr{L}^{-1}(y,z) \equiv \begin{pmatrix} L \\ B \end{pmatrix}^{-1}(y,z) = \left(L \restriction_{Ker(B)}\right)^{-1}(y) + \left(B \restriction_{Ker(L)}\right)^{-1}(z). \tag{1}$$

Proof. It is clear that $Ker(\mathscr{L}) = \{0\}$, which shows that $\mathscr{L}$ is injective. Applying L and B to the right hand side of (1) shows that it is surjective, and that this is the inverse of $\mathscr{L}$. $\square$

The general strategy pursued herewithin can be summarized in the following theorem:

THEOREM 1. *Let X be a Banach space, Y and Z be normed linear spaces, and $D \subseteq X$ be a subspace. Assume $L : D \to Y$ and $B : D \to Z$ are linear operators. Assume $F_1 : X \to Y$ and $F_2 : X \to Z$ are Lipschitz continuous with constants K_1 and K_2, respectively. Assume $L \restriction_{Ker(B)}$ and $B \restriction_{Ker(L)}$ are both bijections, their inverses are bounded linear operators, and*

$$K^* \equiv K_1 \left\| \left(L \restriction_{Ker(B)}\right)^{-1} \right\| + K_2 \left\| \left(B \restriction_{Ker(L)}\right)^{-1} \right\| < 1.$$

If $\mathscr{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, *then*

$$\mathscr{L} - \mathscr{F} \equiv \begin{pmatrix} L \\ B \end{pmatrix} - \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

is invertible, and, in particular, $\left(\mathscr{L} - \mathscr{F}\right)^{-1}$ *is Lipschitz with constant*

$$K = \left(1 - K^*\right)^{-1} \max\left\{ \left\| \left(L \restriction_{Ker(B)}\right)^{-1} \right\|, \left\| \left(B \restriction_{Ker(L)}\right)^{-1} \right\| \right\}.$$

If $\mathscr{L}^{-1}$ *is compact, then so is* $\left(\mathscr{L} - \mathscr{F}\right)^{-1}$.

Furthermore, if $\mathscr{G} : X \to Y \times Z$ *is a compact operator that satisfies:* $\exists M \in \mathbb{N}$, *such that for* $\|x\| \leqslant M$, $\|\mathscr{G}(x)\| \leqslant K^{-1}\left(M - \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\|\right)$, *then the equation,*

$$\mathscr{L}(x) - \mathscr{F}(x) = \mathscr{G}(x)$$

has a solution in D. *If* $\mathscr{L}^{-1}$ *is compact, then* $\mathscr{G}$ *need only be continuous.*

Proof. $\mathscr{L} \equiv \begin{pmatrix} L \\ B \end{pmatrix}$ is invertible, and its inverse is given by Lemma 1. From this, it is clear that

$$\left\|\mathscr{L}^{-1}\right\| \leqslant \max\left\{ \left\| \left(L \restriction_{Ker(B)}\right)^{-1} \right\|, \left\| \left(B \restriction_{Ker(L)}\right)^{-1} \right\| \right\}.$$

We also have that

$$\mathscr{L}^{-1}\mathscr{F}(x) = \left(L \restriction_{Ker(B)}\right)^{-1} F_1(x) + \left(B \restriction_{Ker(L)}\right)^{-1} F_2(x).$$

It is clear from this that $\mathscr{L}^{-1}\mathscr{F}$ is Lipschitz with constant K^*. Since, by assumption, $K^* < 1$, this satisfies the conditions of [20, Lemma 1].

Define $H \equiv \left(\mathscr{L} - \mathscr{F}\right)^{-1} \circ \mathscr{G}$. If both operators are continuous, and either is compact, then H is compact. Let $B = \{z \in D \| \|z\| \leqslant M\}$. Then,

$$\begin{aligned} \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}\mathscr{G}(x)\right\| &\leqslant \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}\mathscr{G}(x) - \left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\| + \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\| \\ &\leqslant KK^{-1}\left(M - \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\|\right) + \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\| \\ &\leqslant M. \end{aligned}$$

Therefore, $H(B) \subseteq B$, and, since B is clearly closed, bounded, and convex, by Schauder fixed point theorem, H has at least one fixed point. □

REMARK 1. If the operator, $\mathscr{G}$, satisfies a sublinear growth condition, that is

$$\|\mathscr{G}(x)\| \leqslant a + b\|x\|^{\varepsilon},$$

for some $a, b \in \mathbb{R}$ and $\varepsilon < 1$, then the growth condition in the previous theorem is automatically satisfied.

The second part of the previous theorem can be extended in the case that the growth condition is made stricter. The proof of the following can be found in [20, Corollary 2].

COROLLARY 1. *Assume the conditions of Theorem 1, but assume that for some* $\delta > 0$, $\exists M \in \mathbb{N}$, *such that for* $\|x\| \leqslant M$, $\|\mathscr{G}(x)\| \leqslant K^{-1}\big(M - \big\|\big(\mathscr{L} - \mathscr{F}\big)^{-1}(0)\big\|\big) - \delta$. *Consider* $\mathscr{G}_\varepsilon \equiv \mathscr{G} + \varepsilon\mathscr{H}$, *where* $\mathscr{H} : X \to Y$ *is continuous, such that*

$$\sup_{\|x\| \leqslant M} \|\mathscr{H}(x)\| = H < \infty.$$

Then for every $\varepsilon \leqslant K\delta/H$, *there exists at least one point,* $x_0 \in D$, *such that* $\mathscr{L}(x_0) + \mathscr{F}(x_0) = \mathscr{G}_\varepsilon(x_0)$.

3. Preliminaries

We start by defining the spaces that we will consider. First, let $C[0,1]$ be the set of continuous functions from $[0,1]$ into $\mathbb{R}^n$, and let $C^1[0,1]$ be those that are continuously differentiable. Let

$$C(1) = \big\{f : [0,1] \to \mathbb{R}^n | f(0) = f(1)\big\}. \tag{2}$$

$$C^1(1) = \big\{f : [0,1] \to \mathbb{R}^n | f(0) = f(1)\big\}. \tag{3}$$

Let $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$, where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^n$. Let $\mathscr{C} = \big(C[0,1], \|\cdot\|_\infty\big)$. We will also consider the following norm on $C^1[0,1]$:

$$\|x\|_{C^1} = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |x'(t)|.$$

Then let

$$\mathscr{C}_1^1 = \big(C^1[0,1], \|\cdot\|_\infty\big), \text{ and } \mathscr{C}^1 = \big(C^1[0,1], \|\cdot\|_{C^1}\big).$$

Note that the $\|\cdot\|_\infty$-completion of $C^1(1)$ is $C(1)$, and the $\|\cdot\|_\infty$-completion of $C^1[0,1]$ is $C[0,1]$.

4. Nonlinear systems

In this paper we consider systems of ordinary differential equations on $[0,1]$ of the form

$$\dot{x}(t) - f(x)(t) = g(x)(t) \tag{4}$$

$$x(0) - x(1) - \eta(x) = \phi(x), \tag{5}$$

where η and ϕ are operators from $C[0,1]$ into $\mathbb{R}^n$ and f and g are operators from $C[0,1]$ to $C[0,1]$. We look for solutions to (4) in $C^1[0,1]$. We first shift the differential operator by a constant matrix, so that

$$\dot{x} - Ax - (f(x) - Ax) = g(x) \tag{6}$$

$$x(0)-x(1)-\eta(x)=\phi(x), \tag{7}$$

where A is a constant $n\times n$ matrix. Let $|\lambda_1|\leqslant\ldots\leqslant|\lambda_J|$ be the eigenvalues of A. We assume that for $j=1,\ldots,J$, $\lambda_j\neq 2\pi ik$ for any $k\in\mathbb{Z}$.

Let $L,L_A:\mathscr{C}_1^1\subset\mathscr{C}\to\mathscr{C}$ be defined by $Lz(t)=\dot{z}(t)$ and $L_Az(t)=\dot{z}(t)-Az(t)$. Now, let $B:\mathscr{C}_1^1\subset\mathscr{C}\to\mathbb{R}^n$ be defined by $Bz=z(0)-z(1)$, and consider the operators $\mathscr{L},\mathscr{L}_A:\mathscr{C}_1^1\subset\mathscr{C}\to\mathscr{C}\times\mathbb{R}^n$, where

$$\mathscr{L}\equiv\begin{bmatrix}L\\B\end{bmatrix}\text{ and }\mathscr{L}_A\equiv\begin{bmatrix}L_A\\B\end{bmatrix}. \tag{8}$$

Then we have the following:

LEMMA 2. *Let $h\in C[0,1]$ have the Fourier series $h(t)\overset{L^2}{=}\sum_{k=-\infty}^{\infty}\alpha_k e^{2\pi ikt}$ for $j=1,\ldots,n$. Then, $L_A\restriction_{Ker(B)}$: $Ker(B)\subset\mathscr{C}_1^1\to\mathscr{C}$ is an invertible map, whose inverse is given by*

$$\left(L_A\restriction_{Ker(B)}^{-1}\right)h\,(t)=\sum_{k=-\infty}^{\infty}\left(2\pi ikI-A\right)^{-1}\alpha_k e^{2\pi ikt}, \tag{9}$$

where the convergence is uniform.

Proof. If $h\in C[0,1]$, $h=L_Az$, with Fourier series, $h(t)\sim\sum_k\alpha_k e^{2\pi ikt}$. All solutions to $h=L_Az$ are given by the variations of constants formula. With the restriction that $z(0)=z(1)$, there is a unique solution for any $h\in C[0,1]$ obtained by solving for the initial condition. Thus, $L_Az=h$ has a solution in $C^1(1)$ given by

$$z(t)=e^{At}\left(I-e^A\right)^{-1}e^A\int_0^1 e^{-As}h(s)ds+e^{At}\int_0^t e^{-As}h(s)ds, \tag{10}$$

which has a uniformly convergent Fourier series since it is a member of $C^1(1)$. Let z be in the domain of L_A with Fourier series, $\sum_k\beta_k e^{2\pi ikt}$. It is well known that the Fourier series of an absolutely continuous function can be differentiated termwise [9, 2.3.4]. Also, multiplication by a constant matrix can clearly be moved inside the infinite sum. Therefore, the Fourier series (in the L^2 sense) of L_Az is given by

$$L_Az(t)=\sum_k(2\pi ikI-A)\beta_k e^{2\pi ikt}.$$

Equating coefficients, we see the given form for the inverse. □

The following lemma provides estimates that will help to bound the operator norm of this inverse.

LEMMA 3.

$$\left\|\left(L_A\restriction_{Ker(B)}\right)^{-1}h\right\|_\infty\leqslant\left(\sum_k\left\|\left(2\pi ikI-A\right)^{-1}\right\|^2\right)^{1/2}\|h\|_\infty\equiv a_1\|h\|_\infty \tag{11}$$

Proof. To show that $\sum_k \left\| \left(2\pi ikI - A\right)^{-1} \right\|^2$ is summable, note that

$$\left\| \left(2\pi ikI - A\right)^{-1} \right\|^2 = \max_j \sigma_j \left(\left(2\pi ikI - A\right)^{-1} \right)^2,$$

where σ_j is the jth singular value. Then we have

$$\max_j \sigma_j \left(\left(2\pi ikI - A\right)^{-1} \right) = \min_j \sigma_j \left(2\pi ikI - A\right)^{-1} \leqslant \left(\min_j \sigma_i(A) - 2\pi k \right)^{-1}.$$

The inequality comes from the fact [12, Theorem 3.3.16] that

$$\sigma_j(2\pi ikI - A) \geqslant \sigma_j(A) - 2\pi k.$$

Now, for the inequality,

$$\begin{aligned} \left| \sum_k (2\pi ikI - A)^{-1} \alpha_k e^{2\pi ikt} \right| &\leqslant \sum_k \left\| \left(2\pi ikI - A\right)^{-1} \right\| \left|\alpha_k\right| \\ &\leqslant \sum_k \left\| \left(2\pi ikI - A\right)^{-1} \right\|^2 \sum_k |\alpha_k|^2 \\ &\leqslant \sum_k \left\| \left(2\pi ikI - A\right)^{-1} \right\|^2 \|h\|_\infty^2. \end{aligned}$$

LEMMA 4. *It holds:*

$$\left(\left(B \restriction_{Ker(L)}\right)^{-1} v \right)(t) = e^{At} \left(I - e^A\right)^{-1} v. \tag{12}$$

Proof. The kernel of L_A is $\left\{ e^{At} w | w \in \mathbb{R}^n \right\}$. On this set, $Bx = v$ if and only if $x(t) = e^{At} \left(I - e^A\right)^{-1} v$. $\square$

The following immediately follows from the form of the inverse.

LEMMA 5. *It holds:*

$$\left\| \left(\left(B \restriction_{Ker(L)}\right)^{-1} v \right) \right\|_\infty \leqslant \sup_{t \in [0,1]} \left\| e^{At} \left(I - e^A\right)^{-1} \right\| \|v\| \equiv a_2 \|v\|. \tag{13}$$

We now consider the left-hand side nonlinearities. Since we have shifted the linear portion by a constant matrix, we shift the nonlinear operator correspondingly. Define $\mathscr{F}$, $\mathscr{F}_A$, and $\mathscr{G}$ as

$$\mathscr{F} \equiv \begin{bmatrix} f \\ \eta \end{bmatrix}, \mathscr{F}_A \equiv \begin{bmatrix} f - A \\ \eta \end{bmatrix}, \text{ and } \mathscr{G} \equiv \begin{bmatrix} g \\ \phi \end{bmatrix}. \tag{14}$$

The original problem, (4), can now be written as

$$\mathscr{L} - \mathscr{F} = \mathscr{L}_A - \mathscr{F}_A = \mathscr{G}. \tag{15}$$

The following provides conditions on $\mathscr{F}_A$ so that $\mathscr{L} - \mathscr{F}$ is invertible; it is a direct consequence of Theorem 1.

THEOREM 2. *Assume that $f-A$ and η are Lipschitz continuous, with constants K_1 and K_2, respectively.*

Let $\mathscr{F}_A : \mathscr{C} \to \mathscr{C} \times \mathbb{R}^2$ and $L_A : \mathscr{C}_1^1 \subset \mathscr{C} \to \mathscr{C}$, as above. If

$$K^* = a_1 K_1 + a_2 K_2 < 1,$$

Then for any $h \in C[0,1]$ and $v \in \mathbb{R}^n$, there exists a solution, $x_0 \in C^1[0,1]$, such that $\mathscr{L}(x_0) - \mathscr{F}(x_0) = (h, v)$.

To give an idea of the types of problems covered by the previous theorem, we give examples of Lipschitz functions that could appear in the boundary conditions.

EXAMPLE 1. Let $\{t_i\}_{i=1}^n \subseteq [0,1]$ and $h : \mathbb{R} \to \mathbb{R}^n$ Lipschitz.

1. $\eta : \mathscr{C} \to \mathbb{R}^2$, $\eta(x) = \sum_{j=1}^n h(x(t_j))$,
2. $\eta : \mathscr{C} \to \mathbb{R}^2$, $\eta(x) = \int_0^1 h(x(t))dt$.

The importance of this example is to notice that both multipoint evaluations and global boundary conditions can be considered in this framework. The following is a direct consequence of the last portion of Theorem 1.

THEOREM 3. *Assume the conditions of Theorem 2. Let*

$$K = (1 - K^*)^{-1} \max\{a_1, a_2\}. \tag{16}$$

If $\mathscr{G} : \mathscr{C} \to \mathscr{C} \times \mathbb{R}^2$ is such that there exists an $M \in \mathbb{N}$ such that for $\|x\| \leqslant M$, $\|\mathscr{G}(x)\| \leqslant K^{-1}\left(M - \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\|\right)$, then there exists at least one point, $x_0 \in C^1[0,1]$ such that

$$\mathscr{L}(x_0) - \mathscr{F}(x_0) = \mathscr{G}(x_0).$$

Correspondingly, the following is a direct consequence of Corollary 1. It allows for further small perturbations.

THEOREM 4. *Assume the conditions of Theorem 2. Let K be defined in (16), $\delta > 0$, and $\mathscr{H} : \mathscr{C} \to \mathscr{C} \times \mathbb{R}^2$ be continuous such that $\sup_{\|x\| \leqslant M} \|\mathscr{H}(x)\| = H < \infty$. If $\mathscr{G} : \mathscr{C} \to \mathscr{C} \times \mathbb{R}^2$ is such that there exists an $M \in \mathbb{N}$ such that for $\|x\| \leqslant M$,*

$$\|\mathscr{G}(x)\| \leqslant K^{-1}\left(M - \left\|\left(\mathscr{L} - \mathscr{F}\right)^{-1}(0)\right\|\right) - \delta,$$

then, if $\varepsilon < K\delta/H$ there exists at least one point, $x_0 \in \mathscr{C}^1$ such that

$$\mathscr{L}(x_0) - \mathscr{F}(x_0) = \mathscr{G}(x_0) + \varepsilon \mathscr{H}(x_0).$$

5. Comparison to previous results

The results presented in this paper allow us to establish sufficient conditions for the solvability of systems of differential equations that do not fall within the framework of [20] and [19]. Conversely, the examples of Sturm-Liouville problems that appear in [20] and [19] can be studied within the framework of the present results, and this is what we present in this section. It is important to note, however, that the sufficient conditions from those previous papers neither imply, nor are consequences of, the sufficient conditions in the present paper. Results in the present paper allow us to consider more general problems, but do not take advantage of special properties that exist in more specific contexts. In the previous paper, differential equations on $[0,1]$ that were considered were of the form

$$(p(t)x'(t))' + q(t)x(t) + \psi(x)(t) = G(x)(t), \tag{17}$$

subject to boundary conditions of the form

$$\alpha x(0) + \beta x'(0) + \eta_1(x) = \phi_1(x) \tag{18a}$$
$$\gamma x(1) + \delta x'(1) + \eta_2(x) = \phi_2(x). \tag{18b}$$

Clearly this can be rewritten as the following two dimensional problem:

$$x_1'(t) - x_2(t) = 0 \tag{19}$$

$$x_2'(t) + \frac{1}{p(t)}\big(p'(t)x_2(t) + q(t)x_1(t) + \psi(x)(t)\big) = \frac{1}{p(t)}G(x)(t) \tag{20}$$

subject to boundary conditions of the form

$$x_1(0) - x_1(1) + (\alpha - 1)x_1(0) + \beta x_2(0) + \eta_1(x) = \phi_1(x) \tag{21a}$$
$$x_2(0) - x_2(1) + \gamma x_1(1) + (\delta - 1)x_2(1) + \eta_2(x) = \phi_2(x). \tag{21b}$$

Thus, a choice can be made as to which theorems to apply. The main practical consideration would typically come down to whether the eigenvalues for the linear Sturm-Liouville problem are easily calculated compared with the Lipschitz constants and growth conditions for the transformed problem.

Again, it should be noted that by applying the present results to these types of problems ignores the special structure of Sturm-Liouville problems.

6. Example

In this section we establish the solvability of a system of integro-differential equations that could not be established using the results of [20] and [19]. Since the theorems herewithin can cover more than 2-dimensional problems, we provide a more concrete example of a 3-dimensional system to which our main theorems apply.

COROLLARY 2. *Let* $\{s_{j,i}\}\bigcup\{t_{j,i}\}\bigcup\{u_{j,i}\}\subseteq[0,1]$. *Consider the multipoint boundary value problem on the interval* $[0,1]$

$$\begin{aligned} x'(t)+x(t)+f_1\big(x(t),y(t),z(t)\big) &= \int_0^1 \psi\big(y(s),t\big)ds \\ y'(t)+y(t)+f_2\big(x(t),y(t),z(t)\big) &= g_1\big(x(t),y(t),z(t)\big) \\ z'(t)+z(t)+f_3\big(x(t),y(t),z(t)\big) &= g_2\big(x(t),y(t),z(t)\big) \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned} x(0)-x(1)+\sum_{i=1}^{N} h_{1,i}(x(s_{1,i}),y(t_{1,i}),z(u_{1,i})) &= \phi_1(x,y,z) \\ y(0)-y(1)+\sum_{i=1}^{N} h_{2,i}(x(s_{2,i}),y(t_{2,i}),z(u_{2,i})) &= \phi_2(x,y,z) \\ z(0)-z(1)+\sum_{i=1}^{N} h_{3,i}(x(s_{3,i}),y(t_{3,i}),z(u_{3,i})) &= \phi_3(x,y,z) \end{aligned}$$

where $\{h_{j,i}\}:\mathbb{R}^3\to\mathbb{R}$ *are Lipschitz continuous functions with constants* $\{k_{j,i}\}$. *Let* $\psi:\mathbb{R}^n\to\mathbb{R}$ *be bounded, and* $\{f_i\}:\mathbb{R}^3\to\mathbb{R}$ *are Lipschitz continuous with constants* $\{c_i\}$. *Let* $C_1=\sum_{j=1}^{3}c_i$ *and* $C_2=\sum_{j=1}^{3}\sum_{i=1}^{N}k_{j,i}$. *Assume* $\exists\, a_i,b_i\in\mathbb{R},\zeta\in[0,1)$, *such that* $|g_i(x,y,z)|\leqslant a_i+b_i|(x,y,z)|^{\zeta}$ *for all* $t\in\mathbb{R}$ *and* $i=1,2$. *Let* $\phi_i:\mathscr{C}\to\mathbb{R}^3$ *be continuous and additionally assume that* $\exists\, A_i,B_i\in\mathbb{R},\xi\in[0,1)$, *such that* $|\phi_i(x,y,z)|\leqslant A_i+B_i\|(x,y,z)\|_{\infty}^{\xi}$ *for all* $(x,y,z)\in\mathscr{C}$ *and* $i=1,2,3$. *Let* $K_1=\big(.5\coth(.5)\big)^{1/2}$, $K_2=e/(1-e)$, *and assume that,* $C_1K_1+C_2K_2<1$. *Then there exists at least one solution to the above problem.*

This example particularly shows the flexibility of Theorem 3. It can handle relatively large nonlinearities along with multipoint boundary conditions, and even in the context of integro-differential problems.

7. Discussion

In the language of Theorem 1, we have been working in the following spaces:

$$D=\mathscr{C}_1^1, X=\mathscr{C}, Y=\mathscr{C}, Z=\mathbb{R}^n.$$

In this case, D is not complete, but can be seen as a subset of its completion, X. One possible modification would be to consider the following spaces:

$$D=\mathscr{C}^1, X=\mathscr{C}^1, Y=\mathscr{C}^1, Z=\mathbb{R}^n.$$

This has the effect of allowing more flexibility in the nonlinearities that can be considered since they would only need to be defined on a smaller set. For example, functions of the derivative could be considered. This comes at the expense of larger constants

to consider when formulating an analogue to Theorem 2. This type of trade-off is discussed in detail in [20].

Two other sets of spaces should be mentioned as well, and also correspond closely with the choices in [20]. Let $\|x\|_2 = \left(\int_{[0,1]} |x|^2\right)^{1/2}$, and consider

$$\|x\|_{1,2} = \left(\int_{[0,1]} |x|^2\right)^{1/2} + \left(\int_{[0,1]} |x'|^2\right)^{1/2}.$$

We use the notation, x', to denote the weak derivative. Let H^1 denote the Sobolev space of square integrable functions whose weak derivatives are also members of L^2. Then let

$$\mathscr{H}_2^1 = \left(H^1, \|\cdot\|_2\right) \text{ and } \mathscr{H}^1 = \left(H^1, \|\cdot\|_{1,2}\right).$$

Analagously to the previous paragraph, the following pair of choices can be considered:

$$D = \mathscr{H}_2^1, X = \mathscr{L}^2, Y = \mathscr{L}^2, Z = \mathbb{R}^n$$
$$D = \mathscr{H}^1, X = \mathscr{H}^1, Y = \mathscr{H}^1, Z = \mathbb{R}^n.$$

Again, there is a trade-off between the types of nonlinearities that can be covered by the corresponding version of the major theorems and the constants that become part of the sufficient conditions for the existence of solutions.

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