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ANTI–PERIODIC SOLUTIONS OF ABEL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DISCONTINUITIES

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Abstract. Given $T>0$, the Abel-like equation $\theta' = f_0 + \sum_{j\in\mathbb{N}} f_j\theta^j$ is generalized to the case where θ and θ' are real functions on $[0,T]$ subject to given state dependent discontinuities. Each f_j is a real function of bounded variation for which $f_j(0) = (-1)^{j+1} f_j(T)$. Under appropriate conditions, this equation is shown to admit a solution of bounded variation on $[0,T]$ which is T-anti-periodic in the sense that $\theta(0) = -\theta(T)$. The contraction principle yields a bound for the rate of uniform convergence to the solution of a sequence of iterates.

1. Introduction

For given $T>0$, let i) $NBV(2T)$ designate the family of $2T$-periodic functions $f:\mathbb{R}\to\mathbb{R}$ of bounded variation on $[0,2T]$ and normalized in the sense that

$$f(t) = \big(f(t+) + f(t-)\big)/2 \tag{1}$$

for all $t\in\mathbb{R}$ and ii) NBV denote the real linear space of all functions in $NBV(2T)$ restricted to $[0,T]$. For $f\in NBV$, $v(f)$ will designate its total variation over the interval $[0,T]$. For a start, consider the Abel-like equation

$$\theta' = f_0 + \sum_{j\in\mathbb{N}} f_j\theta^j \tag{2}$$

on $[0,T]$ with T-anti-periodic boundary condition

$$\theta(0) = -\theta(T) \tag{3}$$

accompanied by the following assumption:

A1. f_j lies in NBV and satisfies

$$f_j(0) = (-1)^{j+1} f_j(T)$$

for $j\in\{0,1,2,\dots\}$.

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For $r \geqslant 0$, let B_r denote the set of functions given by

$$B_r = \{f \in NBV : v(f) \leqslant r, f(0) = -f(T)\} \tag{4}$$

and $p_\star : [0,\infty) \to [0,\infty]$ be the convex function given by the Maclaurin series

$$p_\star(r) = T \sum_{j=0}^{\infty} \|f_j\|_1 r^j \tag{5}$$

where

$$\|f_j\|_1 = \frac{1}{T} \int_0^T |f_j(t)|\, dt. \tag{6}$$

Our general result (see Theorem 2), in the absence of state dependent discontinuities in θ and θ', reduces to the following.

THEOREM 1. *Let (2) on $[0,T]$ be subject to both (3) and A1 and let B_r and $p_\star$ be given by (4) and (5) respectively. If $\|f_0\|_1 \neq 0$ and if there exists $r_0 > 0$ for which $p_\star(r_0) \leqslant r_0$ and $p'_\star(r_0) < 1$ simultaneously then, in $B_{p_\star(r_0)}$, there exists a unique nontrivial solution θ of (2). Furthermore, θ is absolutely continuous on $[0,T]$ and $\theta' \in NBV$.*

Note that the closed balls B_r and

$$B_r(2T) = \{f \in NBV(2T) : v(f) \leqslant r, f(t) = -f(t+T)\ \forall t \in \mathbb{R}\} \tag{7}$$

are equivalent. Furthermore, each f_j in A1 can be extended from $[0,T]$ to $[0,2T]$ by way of

$$f_j(t+T) = (-1)^{j+1} f_j(t) \tag{8}$$

and then to all of $\mathbb{R}$ by 2T-periodicity. Conserving the same notation f_j for all these extended functions, there is no loss of generality in studying (2) and

$$\theta(t) = -\theta(t+T) \tag{9}$$

simultaneously on $\mathbb{R}$ rather than (2) and (3) on $[0,T]$, as we shall see. Clearly A1 is equivalent to

A′1. f_j lies in $NBV(2T)$ and satisfies (8) for all $t \in \mathbb{R}$ and all $j \in \{0,1,2,\ldots\}$.

Consequently, Theorem 1 yields the following result on the existence of periodic solutions of Abel-like equations.

COROLLARY 1. *Let (2) on $\mathbb{R}$ be subject to both (9) and A′1 and let $p_\star$ be given by (5). If $\|f_0\|_1 \neq 0$ and if there exists $r_0 > 0$ for which $p_\star(r_0) \leqslant r_0$ and $p'_\star(r_0) < 1$ simultaneously then, in $B_{p_\star(r_0)}(2T)$ given by (7), there exists a unique nontrivial solution θ of (2). Furthermore, θ is absolutely continuous on bounded intervals and $\theta' \in NBV(2T)$.*

The corollary can be viewed as a result on the number of limit cycles for Abel's equation (2) on $\mathbb{R}$. It constitutes one more small contribution to Hilbert's 16th problem. In the next section, our more general Theorem 2 allows the presence of jumps in θ' of state dependent amplitude $a_k(\theta)$ at state dependent instant $\tau_k(\theta) \in [0,T)$ for all $k \in \mathbb{N}$ and jumps in θ of state dependent amplitude $b_l(\theta)$ at state independent instant $\sigma_l \in [0,T)$ for all $l \in \mathbb{N}$. These amplitudes are characterized by

$$a_k(\theta) = \begin{cases} \theta'(\tau_k(\theta)+) - \theta'(\tau_k(\theta)-), & 0 < \tau_k(\theta) < T \\ \theta'(0+) + \theta'(T-), & \tau_k(\theta) = 0 \end{cases} \tag{10}$$

and

$$b_l(\theta) = \begin{cases} \theta(\sigma_l+) - \theta(\sigma_l-), & 0 < \sigma_l < T \\ \theta(0+) + \theta(T-), & \sigma_l = 0 \end{cases} \tag{11}$$

respectively. Formulas (10) for $\tau_k(\theta) = 0$ and (11) for $\sigma_l = 0$ will be evident once we extend θ that satisfies (3) to a $2T$-periodic function that satisfies (9) on $\mathbb{R}$ since we will then have $\theta(0-) = \theta(2T-) = -\theta(T-)$ and $\theta'(0-) = \theta'(2T-) = -\theta'(T-)$. We make the following assumptions regarding the instant and amplitude of the discontinuities:

A2. $a_k : NBV \to [-\alpha_k, \alpha_k]$ for some $\alpha_k > 0$ and there exists $a'_k \geqslant 0$ such that

$$|a_k(x) - a_k(y)| \leqslant a'_k v(x-y)$$

for all $k \in \mathbb{N}$ and all $x, y \in NBV$. We write

$$A = \sum_{k \in \mathbb{N}} \alpha_k, \quad A' = \sum_{k \imath \mathbb{N}} a'_k$$

which we assume finite.

A3. $\tau_k : NBV \to [0,T)$ and there exists $\tau'_k \geqslant 0$ such that

$$|\tau_k(x) - \tau_k(y)| \leqslant \tau'_k v(x-y)$$

for all $k \in \mathbb{N}$ and all $x, y \in NBV$. We write

$$C = \sum_{k \in \mathbb{N}} \alpha_k \tau'_k$$

which we assume finite.

A4. $b_l : NBV \to [-\beta_l, \beta_l]$ for some $\beta_l > 0$ and there exists $b'_l \geqslant 0$ such that

$$|b_l(x) - b_l(y)| \leqslant b'_l v(x-y)$$

for all $l \in \mathbb{N}$ and all $x, y \in NBV$. We write

$$B = \sum_{l \in \mathbb{N}} \beta_l, \quad B' = \sum_{l \in \mathbb{N}} b'_l$$

which we assume finite.

A5. $\sigma_l \in [0,T)$ for all $l \in \mathbb{N}$.

Given such discontinuities, we replace (2) with the more general equation

$$\theta' = f_0 + \sum_{j\in\mathbb{N}} f_j\theta^j + \sum_{k\in\mathbb{N}} a_k(\theta)J_{\tau_k\theta)} + \sum_{l\in\mathbb{N}} b_l(\theta)J'_{\sigma_l} \tag{12}$$

restricted to $[0,T]$ where, for arbitrary $t_0 \in [0,T)$, $J_{t_0} : \mathbb{R} \to \mathbb{R}$ denotes the $2T$-periodic function characterized by

$$J_{t_0}(t) = \begin{cases} -1/2, & t_0 < t < t_0 + T \\ 1/2, & t_0 + T < t < t_0 + 2T \\ 0, & t = t_0,\ \ t_0 + T,\ \ t_0 + 2T. \end{cases} \tag{13}$$

As we shall see, each term $a_k\theta)J_{\tau_k(\theta)}$ corresponds to a jump in θ' of amplitude $a_k(\theta)$ at state dependent instant $\tau_k(\theta)$ and each generalized derivative $b_l(\theta)J'_{\sigma_l}$ is associated with a jump in θ of state dependent amplitude $b_l(\theta)$ at state independent instant $\sigma_l \in [0,T)$. (See for example [16] and [37] for the definition of a generalized periodic function and its generalized derivative.) The conditions $A < \infty$ in A2 and $B < \infty$ in A4 imply that the sums $\sum_{k\in\mathbb{N}} a_k(\theta)J_{\tau_k(\theta)}$ and $\sum_{l\in\mathbb{N}} b_l(\theta)J_{\sigma_l}$ are of bounded variation on $[0,T]$. For (3) to hold, we impose

$$\sum_{k\in\mathbb{N}} a_k(\theta) = \sum_{l\in\mathbb{N}} b_l(\theta) = 0 \tag{14}$$

as an added requirement.

The concept of a solution of a differential equation satisfying T-anti-periodic boundary condition (3) was introduced by Okochi [28] and subsequently proved useful for a variety of nonlinear equations with or without state discontinuities (see [1], [2], [13]–[15], [17]–[20], [23]–[27], [29], [33]–[36]). So far, there are no results in the literature on the existence of anti-periodic solutions of Abel's equation, even in the absence of state discontinuities. Results for this equation are limited to the existence of limit cycles (see [4]–[12]) with some attention given to the center problem, which consists in finding conditions guaranteeing that all solutions are periodic of given period (see [21], [30] and the references therein). The equation $\theta' = \theta$ without state discontinuities is an Abel equation satisfying A1. Up to multiplication by a constant, its unique nontrivial solution on $[0,T]$ is $\theta(t) = e^t$, which is not T-anti-periodic for any $T > 0$. On the other hand, $\theta' = \cos\omega t + \theta$ ($\omega = \pi/T$) without state discontinuities is also an Abel equation that satisfies A1. Its particular solution $\theta(t) = (-\cos\omega t + \omega\sin\omega t)/(1+\omega^2)$ ($0 \leqslant t \leqslant T$) is clearly T-anti-periodic for any $T > 0$. In this paper, our aim is to obtain conditions that guarantee the existence of a solution of (12) on $[0,T]$ subject to (3), A1–A5 and (14).

2. Main result

Before we state our main result, we introduce the following notions and notation. The family AC of real absolutely continuous functions on $[0,T]$ is an important

subspace of NBV [3, p. 269] which in turn is a subset of the space L^1 of Lebesgue integrable functions on $[0,T]$. Any $\theta \in NBV$ admits a derivative $\theta' \in L^1$ and $v(\theta) \geqslant \int_0^T |\theta'|$ [31, p. 104] with equality when $\theta \in AC$ [3, p. 273]. For X either NBV or AC, let X_{Tap} denote the subspace of all functions in X T-anti-periodic in the sense of (3). Any $\theta \in X_{Tap}$ can be extended from $[0,T]$ to $[0,2T]$ by way of (9) and then to all $\mathbb{R}$ by $2T$-periodicity, in which case the extended function satisfies (9) for all $t \in \mathbb{R}$. We write $X_{Tap}(2T)$ to denote the real linear space of such $2T$-periodic extended functions and note that X_{Tap} can be identified with $X_{Tap}(2T)$. NBV_{Tap} with norm v and $NBV_{Tap}(2T)$ with total variation norm over any interval of length T are equivalent Banach spaces.

Given $t_0 \in [0,T)$, the $2T$-periodic function $J_{t_0} : \mathbb{R} \to \mathbb{R}$ characterized by (13) satisfies (1) and (9). It is also of bounded variation on $[0,2T]$, so making it an element of $NBV_{Tap}(2T)$. Its restriction to $[0,T]$ clearly satisfies (3). On the right hand side of (2) we add jumps to θ' (but not θ) of state dependent amplitude $a_k(\theta)$ at state dependent instant $\tau_k(\theta) \in [0,T)$ for all $k \in \mathbb{N}$ by introducing the term $\sum_{k\in\mathbb{N}} a_k(\theta) S_{\tau_k(\theta)}$ where

$$S_{t_0} = \frac{1}{2} + J_{t_0}$$

for any $t_0 \in [0,T)$. For condition (3) to be maintained, we assume $\sum_{k\in\mathbb{N}} a_k(\theta) = 0$ and so

$$\sum_{k\in\mathbb{N}} a_k(\theta) S_{\tau_k(\theta)} = \sum_{k\in\mathbb{N}} a_k(\theta) J_{\tau_k(\theta)}.$$

Similarly, if we add $\sum_{l\in\mathbb{N}} b_l(\theta) S'_{\sigma_l}$ to the right hand side of (2), then each generalized derivative $b_l(\theta) S'_{\sigma_l}$ is associated with a jump in θ of state dependent amplitude $b_l(\theta)$ at state independent instant σ_l. Again for condition (3) to be maintained, we assume $\sum_{l\in\mathbb{N}} b_l(\theta) = 0$ and so

$$\sum_{l\in\mathbb{N}} b_l(\theta) S'_{\sigma_l} = \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l}$$

in the sense of generalized functions. Thus we obtain (12) by adding discontinuities that satisfy (14) to the right hand side of (2). Subject to A1–A5 and (14), studying (12) and (3) simultaneously on $[0,T]$ is equivalent to studying (12) and (9) simultaneously on $\mathbb{R}$. This fact will be used to prove our results.

For arbitrary $t \in [0,2T]$, let ρ_t denote the radius of convergence of the Maclaurin series $x \mapsto \sum_{j\in\mathbb{N}} f_j(t) x^j$ and define

$$\rho = \inf\{\rho_t : 0 \leqslant t \leqslant 2T\}. \tag{15}$$

In what follows, we assume $\rho \neq 0$. The Maclaurin series $r \mapsto \sum_{j=0}^{\infty} \|f_j\|_1 r^j$ converges for all $r \in (0,\rho)$. Thus, for all $r \in (0,\rho)$ we have convergence of the Maclaurin series in the definitions

$$p_0(r) = p_\star(t) + TA/2 = T\left(\|f_0\|_1 + \frac{A}{2} + \sum_{j\in\mathbb{N}} \|f_j\|_1 r^j\right), \tag{16}$$

$$p(r) = p_0(r) + B = T\left(\|f_0\|_1 + \frac{A}{2} + \sum_{j\in\mathbb{N}} \|f_j\|_1 r^j\right) + B \tag{17}$$

and

$$q(r) = p'(r) + T(A' + 2C) + B'. \tag{18}$$

Denote by ε_{t_0} the real $2T$-periodic function

$$\varepsilon_{t_0}(t) = -\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{in\omega(t-t_0)}}{n^2\omega^2}, \quad \omega = \pi/T, \quad i = +\sqrt{-1} \tag{19}$$

with generalized derivative

$$\varepsilon'_{t_0}(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{in\omega(t-t_0)}}{in\omega}.$$

By comparing Fourier coefficients we obtain, for all $t \in [t_0, t_0 + 2T]$,

$$\varepsilon_{t_0}(t) = \frac{6T(t-t_0) - 3(t-t_0)^2 - 2T^2}{6}$$

and

$$\varepsilon'_{t_0}(t) = \begin{cases} T + t_0 - t, & t_0 < t < t_0 + 2T \\ 0, & t = t_0,\ t_0 + 2T \end{cases}$$

respectively [22, p. 53]. Clearly ε_{t_0} is absolutely continuous and ε'_{t_0} is of bounded variation on $[0, 2T]$. Furthermore, in the sense of generalized functions, $1 + \varepsilon''_{t_0}(t)$ is precisely the Dirac delta function

$$\delta_{t_0}(t) = \sum_{n\in\mathbb{Z}} e^{in\omega(t-t_o)}$$

associated with an impulse that gives rise to a jump in ε'_{t_0} of amplitude $2T$ at instants $t_0 + 2kT$ for all $k \in \mathbb{Z}$ [37, p. 333]. It is easy to show that

$$J_{t_0}(t) = \frac{\varepsilon'_{t_0+T}(t) - \varepsilon'_{t_0}(t)}{2T} \tag{20}$$

for all $t \in \mathbb{R}$. The generalized derivative J'_{t_0} becomes the generalized function

$$J'_{t_0} = \frac{\varepsilon''_{t_0+T} - \varepsilon''_{t_0}}{2T} = \frac{\delta_{t_0+T} - \delta_{t_0}}{2T}.$$

Let F_0 and F be the operators on $NBV_{Tap}(2T)$ given explicitly by

$$F_0(\theta) = f_0 * \varepsilon'_0 + \sum_{j\in\mathbb{N}} G_j(\theta) + \sum_{k\in\mathbb{N}} a_k(\theta) H_k(\theta) \tag{21}$$

and

$$F(\theta) = F_0(\theta) + \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l} \tag{22}$$

where G_j and H_k are defined on $NBV_{Tap}(2T)$ by

$$G_j(\theta) = (f_j\theta^j) * \varepsilon_0' \tag{23}$$

and

$$H_k(\theta) = J_{\tau_k(\theta)} * \varepsilon_0' = \frac{\varepsilon_{\tau_k(\theta)+T} - \varepsilon_{\tau_k(\theta)}}{2T} \tag{24}$$

with convolutions in the sense of generalized $2T$-periodic functions. Clearly F_0 and F can be viewed as operators on either NBV_{Tap} or $NBV_{Tap}(2T)$ since the two spaces are equivalent in the sense that an element of NBV_{Tap} is the restriction to $[0,T]$ of a unique element of $NBV_{Tap}(2T)$. We can now state our main result.

THEOREM 2. *Suppose (12) with discontinuities on $[0,T]$ is subject to (3), A1–A5 and (14). Let $p(r)$ and $q(r)$ be given by (17) and (18), respctively, and B_r be defined by (4) for all $r \in (0,\rho)$ where $\rho > 0$ is given by (15). If there exists $r_0 \in (0,\rho)$ for which $p(r_0) \leqslant r_0$ and $q(r_0) < 1$ simultaneously then, in the ball $B_{p(r_0)}$, there exists a unique solution θ of (12) on $[0,T]$. Furthermore, we have $\theta - \sum_{l\in\mathbb{N}} b_l(\theta)J_{\sigma_l} = F_0(\theta) \in B_{p_0(r_0)} \cap AC_{Tap}$ and $\theta' - \sum_{l\in\mathbb{N}} b_l(\theta)J'_{\sigma_l} \in NBV_{Tap}$ where J_{σ_l} is given by (13) and $F_0(\theta)$ by (21).*

REMARK 1. In Theorem 2, the solution θ is nontrivial (i.e. $\theta \neq 0$) provided

$$f_0 + \sum_{k\in\mathbb{N}} a_k(0)J_{\tau_k(0)} + \sum_{l\in\mathbb{N}} b_l(0)J'_{\sigma_l} \neq 0 \tag{25}$$

as is the case when $\|f_0\|_1 \neq 0$. For this reason θ in Theorem 1 is nontrivial. To prove Theorem 2 we will show that F is a total variation norm contraction on B_{r_0}. Thus, under the conditions of the theorem, for any $\theta_0 \in B_{r_0}$, the iterates $\{F^n(\theta_0)\}_{n=1}^{\infty}$ converge uniformly to the solution θ with uniform norm $\left\|\theta - F^n(\theta_0)\right\|_\infty$ bounded by (see (37))

$$\left\|\theta - F^n(\theta_0)\right\|_\infty \leqslant v\big(\theta - F^n(\theta_0)\big) \leqslant \frac{2\lambda^n r_0}{1-\lambda}$$

for $\lambda = q(r_0)$.

We prove Theorem 2 in section 3. For $\theta' = \theta$ we have $p(r) = p_\star(r) = Tr$. The graph of $p(r)$ for $r > 0$ fails to cross that of the identity and so the theorem is not applicable here. In fact, the equation admits no nontrivial T-anti-periodic solution for any $T > 0$, as mentioned in section 1. On the other hand, for $\theta' = \cos\omega t + \theta$ we have $p(r) = 2/\omega + Tr$ and so the graph of p intersects that of the identity at $r_0 = 2/\omega(1-T)$ whenever $0 < T < 1$, in which case $q(r_0) = p'(r_0) = T < 1$. Thus the theorem guarantees the existence of a differentiable T-anti-periodic solution whenever $0 < T < 1$. In reality, there exists such a solution for all $T > 0$, as mentioned in section 1. Hence, Theorem 2 only provides sufficient conditions for the existence of anti-periodic solutions. It can yield information difficult to obtain otherwise, as the following nontrivial examples show.

EXAMPLE 1. Consider

$$\theta'(t) = (2t-T) + \sum_{j\in\mathbb{N}} (2t-T)^{j+1}\theta^j(t) + \sum_{k\in\mathbb{N}} a_k(\theta) J_{\tau_k(\theta)}(t) + \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l}(t) \tag{26}$$

for

$$a_k(\theta) = \frac{\sin\theta(T/k)}{k^2}, \quad \tau_k(\theta) = \frac{T}{2}\cos^2\theta(T/k), \quad b_l(\theta) = \frac{\sin\theta(T/l)}{100l^2} \tag{27}$$

and $\sigma_l \in [0,T)$. Choosing $\alpha_k = a'_k = 1/k^2$, $\beta_l = b'_l = 1/100l^2$ and $\tau'_k = T$ we get $A = A' = \zeta(2) = \pi^2/6$, $B = B' = \zeta(2)/100 = \pi^2/600$ and $C = T\pi^2/6$. Furthermore $f_j(t) = (2t-T)^{j+1}$ and so

$$\|f_j\|_1 = \frac{T^{j+1}}{j+2}$$

for all $j \in \{0,1,2,\ldots\}$ and $\rho = 1/T$. We obtain by (17) and (18) the continuous monotone increasing functions

$$\begin{aligned} p(r) &= \frac{T}{2}(T+A) + T\sum_{j\in\mathbb{N}} \frac{T^{j+1}}{j+2} r^j + B \\ &= \frac{1}{2}\Big(T^2 + T\frac{\pi^2}{6}\Big) - \frac{1}{r^2}\left[\ln(1-Tr) + Tr + \frac{(Tr)^2}{2}\right] + \frac{\pi^2}{600} \\ &= T\pi^2/12 - \big[\ln(1-Tr) + Tr\big]/r^2 + \pi^2/600 \end{aligned}$$

and

$$\begin{aligned} q(r) &= p'(r) + \frac{T\pi^2}{6}(1+4T) + \frac{\pi^2}{600} \\ &= 2\big[\ln(1-Tr) + Tr\big]/r^3 + T^2/(r-Tr^2) + T\pi^2(1+2T)/6 + \pi^2/600 \end{aligned}$$

for $0 < r < 1/T$. For example, if we take $T = 0.1$ and $r_0 = 1$, then $0 < r_0 < 1/T$, $p(r_0) = 0.30406 < r_0$ and $q(r_0) = 0.21423 < 1$. Hence, by Theorem 2, (26) with $T = 0.1$ admits, in the ball $B_{p(r_0)}$, a unique nontrivial solution θ. Furthermore, this solution is such that $\theta - \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l} \in AC_{Tap}$, $\theta' - \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l} \in NBV_{Tap}$ and $v(\theta) \leqslant 0.30406$.

EXAMPLE 2. Consider, for ω given in (19),

$$\theta'(t) = \cos(\omega t) + \sum_{j\in\mathbb{N}} \cos^{j+1}(\omega t)\theta^j(t) + \sum_{k\in\mathbb{N}} a_k(\theta) J_{\tau_k(\theta)}(t) \tag{28}$$

where

$$a_k(\theta) = \frac{\sin\theta(T/k))}{k^2}, \quad \tau_k(\theta) = \frac{T}{2}\cos^2\theta(T/k), \quad b_l(\theta) = 0$$

and $\sigma_l \in [0,T)$ for all $j,k,l \in \mathbb{N}$. Choosing $\alpha_k = a'_k = 1/k^2$, $\beta_l = b'_l = 0$ and $\tau'_k = T$ we get $A = A' = \zeta(2) = \pi^2/6$, $B = B' = 0$ and $C = T\pi^2/6$. We have $f_j(t) = \cos^{j+1}(\omega t)$ and so

$$\|f_0\|_1 = \frac{2}{\pi}$$

and, by Wallis' integrals,

$$\|f_j\|_1 = \frac{2}{\pi} W_{j+1}$$

where

$$W_n(t) = \begin{cases} 4^p (p!)^2/(2p+1)!, & n = 2p+1 \\ \pi(2p)!/2(p!)^2 4^p, & n = 2p \end{cases}$$

for all $n \in \mathbb{N}$. It is easy to see that $\rho_t = 1/|\cos(\omega t)|$ and so $\rho = 1$. Hence (17) and (18) become

$$p(r) = T\left(\frac{2}{\pi} + \frac{A}{2}\right) + \frac{2T}{\pi} \sum_{j\in\mathbb{N}} W_{j+1} r^j$$

and

$$q(r) = p'(r) + \frac{T\pi^2}{6}(1+2T) = \frac{2T}{\pi} \sum_{j\in\mathbb{N}} j W_{j+1} r^{j-1} + \frac{T\pi^2}{6}(1+2T)$$

for $0 < r < 1$. For arbitrary $T > 0$ small enough we can find $r_0 \in (0,1)$ such that $p(r_0) \leqslant r_0$ and $q(r_0) < 1$ simultaneously. Thus by Theorem 2 there exists for all $T > 0$ sufficiently small a unique nontrivial solution $\theta \in AC_{Tap}$ of (28) on $[0,T]$ for which $\theta' \in NBV_{Tap}$ and $v(\theta) \leqslant p(r_0)$.

If (12) is of the form

$$\theta' = f_0 + f_1\theta^j + f_2\theta^2 + f_3\theta^3 + \sum_{k\in\mathbb{N}} a_k(\theta) J_{\tau_k(\theta)} + \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l} \tag{29}$$

for $f_3 \neq 0$ then $\rho = \infty$ and so, for all $r > 0$, (17) and (18) become

$$p(r) = T\Big(A/2 + \|f_0\|_1 + \|f_1\|_1 r + \|f_2\|_1 r^2 + \|f_3\|_1 r^3\Big) + B \tag{30}$$

and

$$q(r) = T\Big(A' + 2C + \|f_1\|_1 + 2\|f_2\|_1 r + 3\|f_3\|_1 r^2\Big) + B' \tag{31}$$

respectively. Suppose that the graph of $p : (0,\infty) \to (0,\infty)$ intersects that of the identity $\iota(r) = r$ at some point $r_0 > 0$, which we take to be the least of all such points of intersection. But, the graph of the monotone increasing function p intersects that of the identity $\iota(r) = r$ at $r_0 > 0$ if and only if there exists a point $\bar{r} \geqslant r_0$ for which $p'(\bar{r}) = 1$ and $p(\bar{r}) \leqslant \bar{r}$ simultaneously. By way of the quadratic formula, there exists $\bar{r}$ such that $p'(\bar{r}) = 1$ if and only if $T\|f_1\|_1 < 1$, in which case

$$\bar{r} = \left(-T\|f_2\|_1 + \sqrt{T^2\|f_2\|_1^2 + 3T(1 - T\|f_1\|_1)\|f_3\|_1}\right)/3T\|f_3\|_1. \tag{32}$$

Thus, if $T\|f_1\|_1 < 1$ and if $p(\bar{r}) \leqslant \bar{r}$ and $q(\bar{r}) < 1$ for $\bar{r}$ given by (32) then (29) fulfills the conditions of Theorem 2 and so admits, in $B_{p(\bar{r})}$, a unique nontrivial solution θ for which $\theta - \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l} = F_0(\theta) \in B_{p_0(r_0)} \cap AC_{Tap}$ and $\theta' - \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l} \in NBV_{Tap}$.

EXAMPLE 3. Consider, for ω given in (19),

$$\theta' = \cos\omega t + \theta + J_0(t)\theta^2 + \big(\sin 2\omega t\big)\theta^3 + \sum_{k=1}^{m} a_k(\theta) J_{\tau_k(\theta)}(\theta) \tag{33}$$

where a_k and τ_k are again given by (27). Here we have $f_0(t) = \cos\omega t$, $f_1(t) = 1$, $f_2(t) = J_0(t)$ and $f_3(t) = \sin 2\omega t$ and so $\|f_0\|_1 = \|f_3\|_1 = 2/\pi$ and $\|f_1\|_1 = 2\|f_2\|_1 = 1$. Again $B = B' = 0$ and choosing $\alpha_k = a'_k = 1/k^2$ and $\tau'_k = T$ we get $A = A' = \zeta(2) = \pi^2/6$ and $C = T\pi^2/6$. Equations (30) and (31) now become

$$p(r) = T\left(\frac{2}{\pi} + \frac{\pi^2}{12} + r + \frac{1}{2}r^2 + \frac{2}{\pi}r^3\right)$$

and

$$q(r) = T\left(1 + r + \frac{6}{\pi}r^2 + \frac{\pi^2}{6}(1+2T)\right).$$

Thus, if $T < 1$ (i.e. $T < 1/\|f_1\|_1$), then $p'(\bar{r}) = 1$ for

$$\bar{r} = \frac{-T\pi + \sqrt{T^2\pi^2 + 24T\pi(1-T)}}{12T} = \frac{-\pi + \sqrt{\pi^2 + 24\pi(T^{-1}-1)}}{12}$$

by (32). Furthermore, $p(\bar{r}) \leqslant \bar{r}$ and $q(\bar{r}) < 1$ for all $T > 0$ small enough. Hence, (33) is in the conditions of Theorem 2 for all $T \in (0,1)$ small enough. In other words, for all $T \in (0,1)$ small enough, (33) admits a unique nontrivial solution $\theta \in AC_{Tap}$ for which $\theta' \in NBV_{Tap}$ and $v(\theta) \leqslant \bar{r}$.

Suppose now that (12) is of the form

$$\theta' = f_0 + f_1\theta^j + f_2\theta^2 + \sum_{k\in\mathbb{N}} a_k(\theta) J_{\tau_k(\theta)} + \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l} \tag{34}$$

for $f_0 \neq 0$ and $f_2 \neq 0$. For all $r > 0$, (17) and (18) become

$$p(r) = T\Big(A/2 + \|f_0\|_1 + \|f_1\|_1 r + \|f_2\|_1 r^2\Big) + B$$

and

$$q(r) = p'(r) + T(A' + 2C) + B' = T\Big(A' + 2C + \|f_1\|_1 + 2\|f_2\|_1 r\Big) + B'$$

respectively. Let the graph of the monotone increasing function $p : (0,\infty) \to (0,\infty)$ intersect that of the identity at some point $r_0 > 0$, which we take to be the least of all such points of intersection. Such a point r_0 exists if and only if there exists $\bar{r} > 0$ for which $p'(\bar{r}) = 1$ and $p(\bar{r}) \leqslant \bar{r}$ (in which case $r_0 \leqslant \bar{r}$). The condition $p'(\bar{r}) = 1$ yields

$$\bar{r} = \Big(1 - T\|f_1\|_1\Big)/2T\|f_2\|_1 \tag{35}$$

and so $\bar{r} > 0$ exists if and only if $0 < T\|f_1\|_1 < 1$. Thus, when $0 < T\|f_1\|_1 < 1$, if $p(\bar{r}) \leqslant \bar{r}$ and $q(\bar{r}) < 1$ for $\bar{r}$ given by (35) then (34) fulfills the conditions of Theorem 2 and so admits, in $B_{p(\bar{r})}$, a unique nontrivial solution θ for which $\theta - \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l} = F_0(\theta) \in B_{p_0(r_0)} \cap AC_{Tap}$ and $\theta' - \sum_{l\in\mathbb{N}} b_l(\theta) J'_{\sigma_l} \in NBV_{Tap}$.

EXAMPLE 4. Consider, for ω given in (19),

$$\theta' = \cos\omega t + \theta + J_0(t)\theta^2 + \sum_{k=1}^{m} a_k(\theta) J_{\tau_k(\theta)}(\theta) \tag{36}$$

where a_k and τ_k are given by (27). Then $f_0(t) = \cos\omega t$, $f_1(t) = 1$ and $f_2(t) = J_0(t)$ and so $\|f_0\|_1 = 2/\pi$ and $\|f_1\|_1 = 2\|f_2\|_1 = 1$. Again we have $B = B' = 0$. Choosing $\alpha_k = a'_k = 1/k^2$ and $\tau'_k = T$ we get $A = A' = \zeta(2) = \pi^2/6$ and $C = T\pi^2/6$. So we have

$$p(r) = T\left(\frac{2}{\pi} + \frac{\pi^2}{12} + r + \frac{1}{2}r^2\right)$$

and

$$q(r) = T\left(1 + r + \frac{\pi^2}{6}(1+2T)\right).$$

Reasoning as in Example 3, it follows that for all $T \in (0,1)$ small enough, (36) admits a unique nontrivial solution $\theta \in AC_{Tap}$ for which $\theta' \in NBV_{Tap}$ and $v(\theta) \leqslant p(\bar{r})$ where

$$\bar{r} = \big(1-T\big)/T$$

by (35).

In the absence of jumps in θ and θ' (i.e. $A/2 = \big(A' + 4C'\big) = B = B' = 0$), (12) becomes (2), p reduces to $p_\star$ given by (5) and $q = p'_\star$. By Theorem 2 we obtain Theorem 1 and Corollary 1. The condition $q(r_0) < 1$ becomes superfluous when $p_\star(r) - r$ has more than one (and so exactly two) strictly positive distinct roots. In fact, if we take for r_0 the smaller of the two, then the graph of $p_\star$ intersects that of the identity $\iota(r) = r$ at r_0 with slope $p'_\star(r_0) < 1$ (i.e. $q(r_0) < 1$). Furthermore, for F given by (22) and $\theta_0 \in B_{r_0}$, the iterates $\{F^n(\theta_0)\}_{n=1}^{\infty}$ converge uniformly to θ with uniform norm $\big\|\theta - F^n(\theta_0)\big\|_\infty$ bounded by

$$\big\|\theta - F^n(\theta_0)\big\|_\infty \leqslant \frac{2\lambda^n r_0}{1-\lambda}$$

where $\lambda = p'_\star(r_0)$.

3. Proof of Theorem 2

Let $L^1(2T)$ denote, as usual, the Banach space of almost everywhere $2T$-periodic Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$ with norm (6) and essential supremum $\|f\|_\infty$. We write $L^1_{Tap}(2T)$ for the subspace of all $\theta \in L^1(2T)$ that satisfy (9) almost everywhere. If $x \in L^1_{Tap}(2T)$ and $y \in L^1(2T)$ then $x * y$, which is known to lie in $L^1(2T)$ [22, pp. 4–5], satisfies (9) since

$$\begin{aligned}(x*y)(t) &= \frac{1}{2T}\int_0^{2T} x(t-s)y(s)\,ds \\ &= -\frac{1}{2T}\int_0^{2T} x(t+T-s)y(s)\,ds \\ &= -(x*y)(t+T).\end{aligned}$$

This proves the following.

LEMMA 1. *If* $x \in L^1_{Tap}(2T)$ *and* $y \in L^1(2T)$ *then* $x * y \in L^1_{Tap}(2T)$.

Given $x \in NBV_{Tap}(2T)$ and $t \in [0,T]$, there exists $t' \in [0,T]$ such that $|x(t)| \leqslant |x(t) - x(t')|$ and so we have

$$|x(t)| \leqslant v(x) \tag{37}$$

for all $t \in [0,T]$. By (9) we obtain (37) for all $t \in \mathbb{R}$.

PROPOSITION 1. *If* $x \in NBV_{Tap}(2T)$ *then* $x * \varepsilon_0' \in AC_{Tap}(2T)$ *and*

$$v\big(x * \varepsilon_0'\big) = T\|x\|_1. \tag{38}$$

Proof. By Lemma 1, $x * \varepsilon_0'$ is T-anti-periodic. Given $0 \leqslant t_1 \leqslant t_2 \leqslant 2T$ we have

$$|(x * \varepsilon_0')(t_2) - (x * \varepsilon_0')(t_1)| \leqslant \|x\varepsilon_{t_2}' - x\varepsilon_{t_1}'\|_1 \leqslant v(x)\|\varepsilon_{t_2}' - \varepsilon_{t_1}'\|_1$$

where

$$\begin{aligned}\|\varepsilon_{t_2}' - \varepsilon_{t_1}'\|_1 &= \frac{1}{2T}\int_{[t_1,t_2]} |\varepsilon_{t_2}' - \varepsilon_{t_1}'| + \frac{1}{2T}\int_{[0,2T]\setminus[t_1,t_2]} |\varepsilon_{t_2}' - \varepsilon_{t_1}'| \\ &\leqslant \frac{1}{2T}\left(\int_{[t_1,t_2]} 2T\right) + \frac{1}{2T}\left(\int_{[0,2T]\setminus[t_1,t_2]} |t_2 - t_1|\right)\end{aligned}$$

and so

$$\|\varepsilon_{t_2}' - \varepsilon_{t_1}'\|_1 \leqslant 2|t_2 - t_1|. \tag{39}$$

By (39) we obtain

$$|\big(x * \varepsilon_0'\big)(t_2) - \big(x * \varepsilon_0'\big)(t_1)| \leqslant 2v(x)|t_2 - t_1|$$

and so $x * \varepsilon_0'$ is absolutely continuous on any interval of length $2T$. Thus $x * \varepsilon_0' \in AC_{Tap}(2T)$ and so $(x * \varepsilon_0')'$ exists in $L^1_{Tap}(2T)$. Since $\big(x * \varepsilon_0'\big)' = x$ Lebesgue almost everywhere [22, p. 13] we get

$$v\big(x * \varepsilon_0'\big) = \int_0^T \big|\big(x * \varepsilon_0'\big)'\big| = \int_0^T |x| = \frac{1}{2}\int_0^{2T} |x| = T\|x\|_1$$

which proves (38). □

The following corollary is a consequence of (23), (24) and Proposition 1.

COROLLARY 2. *The operators* G_j *and* H_k *map* $NBV_{Tap}(2T)$ *into* $AC_{Tap}(2T)$ *and so we have* $F_0 : B_r(2T) \to AC_{Tap}(2T)$ *for all* $r \in (0,\rho)$.

By (22) and A2 we have, for all $\theta \in NBV_{Tap}(2T)$,

$$v\big(F_0(\theta)\big) \leqslant \sum_{j\in\mathbb{N}} v\big(G_j(\theta)\big) + \sum_{k\in\mathbb{N}} \alpha_k v\big(H_k(\theta)\big)$$

where, by (37) and (38),

$$v\big(G_j(\theta)\big) = T\|f_j\theta^j\|_1 \leqslant T\|f_j\|_1 v\big(\theta\big)^j$$

and by (24)

$$v\big(H_k(\theta)\big) = T\|J_{\tau_k(\theta)}\|_1 = \frac{T}{2}. \tag{40}$$

Thus, for all $r \in (0,\rho)$ and all $\theta \in B_r(2T)$, we have

$$v\big(F_0(\theta)\big) \leqslant p_0(r)$$

for p_0 given by (16) and so the operator F_0 given by (21) is such that

$$F_0 : B_r(2T) \to B_{p_0(r)}(2T) \cap AC_{Tap}(2T)$$

by Corollary 2. We also have

$$v\big(b_l(\theta)J_{\sigma_l}\big) \leqslant \beta_l v\big(J_{\sigma_l}\big) = \beta_l. \tag{41}$$

Thus

$$\begin{aligned} v\big(F(\theta)\big) &\leqslant v\big(F_0(\theta)\big) + v\Big(\sum_{l\in\mathbb{N}} b_l(\theta)J_{\sigma_l}\Big) \\ &\leqslant p_0(r) + B = p(r) \end{aligned}$$

for all $\theta \in NBV_{Tap}(2T)$ such that $v(\theta) < \rho$, which proves the following corollary.

COROLLARY 3. *If $\theta \in B_r(2T)$ for $r \in (0,\rho)$ then*

$$v\big(F(\theta)\big) \leqslant p(r)$$

where p is given by (17).

For arbitrary $\theta_1, \theta_2 \in B_r(2T)$ we have

$$v\big(F_0(\theta_2) - F_0(\theta_1)\big) \leqslant \sum_{j\in\mathbb{N}} v\big(G_j(\theta_2) - G_j(\theta_1)\big) + \sum_{k\in\mathbb{N}} \alpha_k v\big(H_k(\theta_2) - H_k(\theta_1)\big).$$

The identity $x^j - y^j = \big(\sum_{k=0}^{j-1} x^{j-1-k}y^k\big)(x-y)$ along with (37) and (38) yields

$$v\big(G_j(\theta_2) - G_j(\theta_1)\big) = T\|f_j\theta_2^j - f_j\theta_1^j\|_1 \leqslant T\|f_j \sum_{k=0}^{j-1} \theta_2^{j-1-k}\theta_1^k\|_1 v\big(\theta_2 - \theta_1\big)$$

and so

$$v\big(G_j(\theta_2)-G_j(\theta_1)\big)\leqslant T\|f_j\|_1 jr^{j-1}v\big(\theta_2-\theta_1\big).$$

Using

$$a_k(\theta_2)H_k(\theta_2)-a_k(\theta_1)H_k(\theta_1)=\big(a_k(\theta_2)-a_k(\theta_1)\big)H_k(\theta_2)+a_k(\theta_1)\big(H_k(\theta_2)-H_k(\theta_1)\big)$$

we obtain

$$\begin{aligned}&v\big(a_k(\theta_2)H_k(\theta_2)-a_k(\theta_1)H_k(\theta_1)\big)\\&\qquad\leqslant\big|a_k(\theta_2)-a_k(\theta_1)\big|v\big(H_k(\theta_2)\big)+\big|a_k(\theta_1)\big|v\big(H_k(\theta_2)-H_k(\theta_1)\big).\end{aligned}$$

By (20), (38) and (39) we have

$$v\big(H_k(\theta_2)-H_k(\theta_1)\big)=T\|J_{\tau_k(\theta_2)}-J_{\tau_k(\theta_1)}\|_1\leqslant 2T\big|\tau_k(\theta_2)-\tau_k(\theta_1)\big|$$

from which follows that

$$v\big(H_k(\theta_2)-H_k(\theta_1)\big)\leqslant 2T\tau_k' v(\theta_2-\theta_1)$$

by A3. Hence

$$v\big(a_k(\theta_2)H_k(\theta_2)-a_k(\theta_1)H_k(\theta_1)\big)\leqslant T(A'+2C) \tag{42}$$

by A2 and so

$$v\big(F_0(\theta_2)-F_0(\theta_1)\big)\leqslant q_0(r)v\big(\theta_2-\theta_1\big)$$

for $q_0=q-B'$.

By A4 and $v\big(J_{\sigma_l}\big)=1$ we have

$$v\big(b_l(\theta_2)J_{\sigma_l}-b_l(\theta_1)J_{\sigma_l}\big)\leqslant\big|b_l(\theta_2)-b_l(\theta_1)\big|v\big(J_{\sigma_l}\big)\leqslant b_l' v\big(\theta_2-\theta_1\big)$$

and so, if $0<r<\rho$, then

$$v\big(F(\theta)\big)\leqslant v\big(F_0(\theta)\big)+\sum_{l\in\mathbb{N}}v\big(b_l(\theta)J_{\sigma_l}\big)\leqslant p_0(r)+B=p(r)$$

and

$$\begin{aligned}v\big(F(\theta_2)-F(\theta_1)\big)&\leqslant v\big(F_0(\theta_2)-F_0(\theta_1)\big)+B'v\big(\theta_2-\theta_1\big)\\&\leqslant\big(q_0(r)+B'\big)v\big(\theta_2-\theta_1\big)\\&=q(r)v\big(\theta_2-\theta_1\big)\end{aligned}$$

for all $\theta\in B_r(2T)$. Hence, if $0<r<\rho$, then $F:B_r(2T)\to B_{p(r)}(2T)$ and F is a contraction on $B_r(2T)$ when $q(r)<1$. By Corollary 2, we have $F(\theta)-\sum_{l\in\mathbb{N}}b_l(\theta)J_{\sigma_l}=F_0(\theta)\in AC_{Tap}(2T)$ for all $\theta\in B_r(2T)$. Theorem 2 now follows by the contraction principle.

4. A refinement

By (38) we obtain

$$v\Big(\big(f_0+\sum_{j\in\mathbb{N}} f_j\theta^j\big)*\varepsilon_0'\Big)=T\|f_0+\sum_{j\in\mathbb{N}} f_j\theta^j\|_1 \tag{43}$$

for all $\theta\in NBV_{Tap}(2T)$ such that $v(\theta)<\rho$. For any $r\in(0,\rho)$, the function

$$\varphi_r(t)=\sup\Big\{\big|f_0(t)+\sum_{j\in\mathbb{N}} f_j(t)x^j\big|:-r\leqslant x\leqslant r, x\in\mathbb{Q}\Big\}$$

is Lebesgue integrable [32] and so we can define the monotone increasing function

$$\chi(r)=T\|\varphi_r\|_1=\frac{1}{2}\int_0^{2T}\varphi_r(t)\,dt.$$

With respect to this notation (43) yields

$$v\Big(\big(f_0+\sum_{j\in\mathbb{N}} f_j\theta^j\big)*\varepsilon_0'\Big)\leqslant\chi(r)$$

for all $\theta\in B_r(2T)$. This, in conjunction with (40) and (41), gives

$$v\big(F_0(\theta)\big)\leqslant P_0(r),\quad \theta\in B_r(2T) \tag{44}$$

and

$$v\big(F(\theta)\big)\leqslant P(r),\quad \theta\in B_r(2T) \tag{45}$$

for

$$P_0(r)=\chi(r)+\frac{TA}{2} \tag{46}$$

and

$$P(r)=P_0(r)+B=\chi(r)+\frac{TA}{2}+B. \tag{47}$$

By Corollary 2, (44) and (45) we have $F_0:B_r(2T)\to B_{P_0(r)}(2T)\cap AC_{Tap}(2T)$ and $F:B_r(2T)\to B_{P(r)}(2T)$.

Similarly, for any $r\in(0,\rho)$, the function

$$\psi_r(t)=\sup\Big\{\big|\frac{d}{dz}\sum_{j\in\mathbb{N}} f_j(t)z^j\big|:-r\leqslant z\leqslant r, z\in\mathbb{Q}\Big\}$$

is Lebesgue integrable and so we can introduce the function

$$\phi(r)=T\|\psi_r\|_1=\frac{1}{2}\int_0^{2T}\psi_r(t)\,dt$$

which is monotone increasing in $r>0$. By (38) we have

$$v\Big(\big(\sum_{j\in\mathbb{N}} f_j(\theta_2^j-\theta_1^j)\big)*\varepsilon_0'\Big)=T\Big\|\sum_{j\in\mathbb{N}} f_j(\theta_2^j-\theta_1^j)\Big\|_1$$

for all $\theta_1, \theta_2 \in B_r(2T)$. For any $t \in \mathbb{R}$, the mean value theorem yields

$$\sum_{j\in\mathbb{N}} f_j(t)\big(\theta_2^j(t) - \theta_1^j(t)\big) = \Big(\sum_{j\in\mathbb{N}} f_j(t) j\theta_t^{j-1}\Big)\big(\theta_2(t) - \theta_1(t)\big)$$

and so

$$\Big|\sum_{j\in\mathbb{N}} f_j(t)\big(\theta_2^j(t) - \theta_1^j(t)\big)\Big| \leqslant \Big|\sum_{j\in\mathbb{N}} f_j(t) j\theta_t^{j-1}\Big| v\big(\theta_2 - \theta_1\big)$$

for some $\theta_t \in \mathbb{R}$ between $\theta_1(t)$ and $\theta_2(t)$ (and so between $-r$ and r). From this follows that

$$v\Big(\big(\sum_{j\in\mathbb{N}} f_j(\theta_2^j - \theta_1^j)\big) * \varepsilon_0'\Big) \leqslant \phi(r) v\big(\theta_2 - \theta_1\big)$$

which, in conjunction with (42), yields

$$v\big(F(\theta_2) - F(\theta_1)\big) \leqslant Q(r) v\big(\theta_2 - \theta_1\big); \quad \theta_1, \theta_2 \in B_r(2T)$$

for

$$Q(r) = \phi(r) + T\big(A' + 2C\big) + B'. \tag{48}$$

The contraction principle now yields the following result.

THEOREM 3. *Given the simultaneous equations (12) and (3) on $[0,T]$ subject to A1–A5, let $P_0(r)$, $P(r)$ and $Q(r)$ be defined by (46), (47) and (48) and B_r by (4) for all $r \in (0,\rho)$ where ρ is given by (15). If there exists $R_0 \in (0,\rho)$ for which $P(R_0) \leqslant R_0$ and $Q(R_0) < 1$ simultaneously then there exists, in $B_{P(R_0)}$, a unique solution θ of (12) on $[0,T]$. Furthermore, $\theta - \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l} = F_0(\theta) \in B_{P_0(R_0)} \cap AC_{Tap}$ and $\theta' - \sum_{l\in\mathbb{N}} b_l(\theta) J_{\sigma_l}' \in NBV_{Tap}$ where J_{σ_l} is given by (13) and $F_0(\theta)$ by (21).*

In the context of Theorem 3, θ is nontrivial if and only if (25) is satisfied. Furthermore, for F on NBV_{Tap} given by (22) and $\theta_0 \in B_{R_0}$, the iterates $\big\{F^n(\theta_0)\big\}_{n=1}^{\infty}$ converge uniformly to θ with uniform norm $\big\|\theta - F^n(\theta_0)\big\|_\infty$ bounded by

$$\big\|\theta - F^n(\theta_0)\big\|_\infty \leqslant \frac{2\lambda^n R_0}{1-\lambda}$$

for $\lambda = Q(R_0)$.

We have

$$\varphi_r(t) \leqslant |f_0(t)| + \sum_{j\in\mathbb{N}} |f_j(t)| r^j$$

and so

$$\chi(r) \leqslant \|f_0\|_1 + \sum_{j\in\mathbb{N}} \|f_j\|_1 r^j.$$

From this we obtain the two inequalities

$$P_0(r) \leqslant p_0(r), \quad P(r) \leqslant p(r) \tag{49}$$

for $0 < r < \rho$. Similarly we have

$$Q(r) \leqslant q(r) \tag{50}$$

again for $0 < r < \rho$. The following example shows that, in some cases, Theorem 2 and Theorem 3 yield the same result in the sense that $P = p$ and $Q = q$.

EXAMPLE 5. In the context of Example 1, (26) can be conveniently written as

$$\theta'(t) = \frac{2t - T}{1 - (2t - T)\theta(t)} + \sum_{k \in \mathbb{N}} a_k(\theta) J_{\tau_k(\theta)}(t) + \sum_{l \in \mathbb{N}} b_l(\theta) J'_{\sigma_l}(t)$$

provided $(2t - T)\theta(t) < 1$. Hence $\rho_t = 1/|2t - T|$ and so $\rho = 1/T$ from which follows that

$$\varphi_r(t) = \sup\left\{ \left| \frac{2t - T}{1 - (2t - T)x} \right| : -r \leqslant x \leqslant r, x \in \mathbb{Q} \right\} = \frac{|T - 2t|}{1 - (T - 2t)r}$$

and

$$\psi_r(t) = \sup\left\{ \left| \frac{d}{dz}\left(\frac{2t - T}{1 - (2t - T)z} \right) \right| : -r \leqslant z \leqslant r, z \in \mathbb{Q} \right\} = \frac{(2t - T)^2}{(1 - (T - 2t)r)^2}$$

for all $t \in [0, 2T]$ and all $r \in (0, 1/T)$. Thus we have

$$\chi(r) = 2\int_0^{T/2} \frac{T - 2t}{1 - (T - 2t)r} dt = -\big[\ln(1 - Tr) + Tr\big]/r^2$$

and

$$\phi(r) = 2\int_0^{T/2} \frac{(T - 2t)^2}{(1 - (T - 2t)r)^2} dt = 2\big[\ln(1 - Tr) + Tr\big]/r^3 + T^2/(r - Tr^2)$$

and so

$$P(r) = T\pi^2/12 - \big[\ln(1 - Tr) + Tr\big]/r^2 + \pi^2/600$$

and

$$Q(r) = 2\big[\ln(1 - Tr) + Tr\big]/r^3 + T^2/(r - Tr^2) + T\pi^2(1 + 2T)/6 + \pi^2/600$$

for all $r < 1/T$. Hence $P = p$ and $Q = q$ from which follows that there is no gain in using the refinements of this section for this example.

The next example shows that Theorem 3 can yield stronger results than Theorem 2 in the sense that $P \neq p$ and/or $Q \neq q$.

EXAMPLE 6. For all $t \in [0, T]$, consider the equation

$$\theta'(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (2t - T)^{2n+2}}{(2n+1)!} \theta^{2n+1}(t) + \sum_{k \in \mathbb{N}} a_k(\theta) J_{\tau_k(\theta)}(t) + \sum_{l \in \mathbb{N}} b_l(\theta) J'_{\sigma_l}(t) \tag{51}$$

with

$$a_k(\theta) = \frac{\sin\theta(T/k))}{k^2}, \quad \tau_k(\theta) = \frac{T}{2}\cos^2\theta(T/k), \quad b_l(\theta) = \frac{\sin\theta(T/l))}{100l^2}$$

and $\sigma_l \in [0,T)$. Choosing $\alpha_k = a'_k = 1/k^2$, $\beta_l = b'_l = 1/100l^2$ and $\tau'_k = T$ we get $A = A' = \zeta(2) = \pi^2/6$, $B = B' = \zeta(2)/100 = \pi^2/600$ and $C = T\pi^2/6$. We also have $\rho_t = \infty$ for all $t \in [0,T]$ and so $\rho = \infty$. Here each f_j can be extended to a T-periodic function on $\mathbb{R}$ such that

$$f_j(t) = \begin{cases} (-1)^n(2t-T)^{2n+2}/(2n+1)!, & j = 2n+1,\ 0 \leqslant t \leqslant T \\ 0, & j = 2n,\ 0 \leqslant t \leqslant T \end{cases}$$

and so

$$\|f_j\|_1 = \begin{cases} T^{2n+2}/(2n+1)!(2n+3), & j = 2n+1 \\ 0, & j = 2n. \end{cases}$$

Thus

$$p_0(r) = \sum_{n=0}^{n=\infty} \frac{T^{2n+2}}{(2n+1)!(2n+3)} r^{2n+1} + \frac{T\pi^2}{12}, \tag{52}$$

$$p(r) = \sum_{n=0}^{n=\infty} \frac{T^{2n+2}}{(2n+1)!(2n+3)} r^{2n+1} + \frac{T\pi^2}{12} + \frac{\pi^2}{600} \tag{53}$$

and

$$q(r) = \sum_{n=0}^{n=\infty} \frac{T^{2n+2}}{(2n)!(2n+3)} r^{2n} + \frac{T\pi^2(1+2T)}{6} + \frac{\pi^2}{600} \tag{54}$$

for all $r > 0$.

Clearly

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2t-T)^{2n+2}}{(2n+1)!} x^{2n+1} = (2t-T)\sin\big((2t-T)x\big)$$

for all $t \in [0,T]$ and so we have for all $t \in [0,T/2]$ and $t_m = \max\big\{0, (T/2 - \pi/4r)\big\}$

$$\begin{aligned} \varphi_r(t) &= \sup\Big\{\big|(2t-T)\sin\big((2t-T)x\big)\big| : -r \leqslant x \leqslant r, x \in \mathbb{Q}\Big\} \\ &= \begin{cases} (T-2t)\sin\big((T-2t)r\big), & t_m < t \leqslant T/2 \\ T-2t, & 0 \leqslant t \leqslant t_m \end{cases} \end{aligned}$$

and

$$\begin{aligned} \psi_r(t) &= \sup\Big\{\Big|\frac{d}{dz}(2t-T)\sin\big((2t-T)z\big)\Big| : -r \leqslant z \leqslant r, z \in \mathbb{Q}\Big\} \\ &= \sup\Big\{(2t-T)^2\big|\cos\big((2t-T)z\big)\big| : -r \leqslant z \leqslant r, z \in \mathbb{Q}\Big\} \\ &= (2t-T)^2 \end{aligned}$$

for all $r > 0$. From these we obtain

$$\begin{aligned}\chi(r) &= 2\int_0^{t_m}(T-2t)\,dt + 2\int_{t_m}^{T/2}(T-2t)\sin\big((T-2t)r\big)\,dt\\ &= 2t_m(T-t_m) - 2\frac{d}{dr}\int_{t_m}^{T/2}\cos\big((T-2t)r\big)\,dt\\ &= 2t_m(T-t_m) - \frac{1}{r}(T-2t_m)\cos\big((T-2t_m)r\big) + \frac{1}{r^2}\sin\big((T-2t_m)r\big)\\ &= \begin{cases}(\sin Tr - Tr\cos Tr)/r^2, & t_m = 0\\ T^2/2 - \pi^2/8r^2 + 1/r^2, & 0 < t_m \leqslant T/2\end{cases}\end{aligned}$$

and

$$\phi(r) = \frac{T^3}{3}$$

for all $r \in (0,\infty)$ and so

$$P_0(r) = T\pi^2/12 + \begin{cases}(\sin Tr - Tr\cos Tr)/r^2, & 0 < r < \pi/2T\\ T^2/2 - \pi^2/8r^2 + 1/r^2, & r \geqslant \pi/2T\end{cases}$$

by (46),

$$P(r) = T\pi^2/12 + \frac{\pi^2}{600} + \begin{cases}(\sin Tr - Tr\cos Tr)/r^2, & 0 < r < \pi/2T\\ T^2/2 - \pi^2/8r^2 + 1/r^2, & r \geqslant \pi/2T\end{cases}$$

by (47) and

$$Q(r) = \frac{T^3}{3} + \frac{T\pi^2(1+2T)}{6} + \frac{\pi^2}{600}$$

by (48). Regardless of $T > 0$, there exists a point $R_0 > 0$ where the graph of P intersects that of the identity. It now follows by Theorem 3 and (25) that (51) admits, in $B_{P(R_0)}$, a unique nontrivial T-anti-periodic solution for all T such that

$$T^3/3 + T\pi^2(1+2T)/6 + \pi^2/600 < 1.$$

By (49), (52) and (53) we have $P_0(r) < p_0(r)$ and $P(r) < p(r)$. Thus, if the graph of p intersects that of the identity at some point $r_0 > 0$, then $R_0 < r_0$ and so $B_{P_0(R_0)} \subsetneq B_{p_0(r_0)}$ and $B_{P(R_0)} \subsetneq B_{p(r_0)}$. Furthermore, by (50) and (54) we obtain $Q(r) < q(r)$ for all $r > 0$. Hence, for this example, the refinements of this section lead to sharper results.

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