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EXISTENCE AND MULTIPLICITY RESULTS FOR THE FRACTIONAL p–LAPLACIAN EQUATION WITH HARDY–SOBOLEV EXPONENTS

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Abstract. In this paper, we investigate the following fractional p-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + \dfrac{|u|^{p^*_{s,\alpha}-2}u}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain containing the origin in $\mathbb{R}^N$ with Lipschitz boundary, $p \in (1,\infty)$, $s \in (0,1)$, $0 \leqslant \alpha < ps < N$ and $p^*_{s,\alpha} = (N-\alpha)p/(N-ps)$ is the fractional Hardy-Sobolev exponent. We prove the existence, multiplicity and bifurcation results for the above problem. Our results extend some results in the literature for the fractional p-Laplacian problem involving critical Sobolev exponent and the p-Laplacian problem involving Hardy-Sobolev exponents.

1. Introduction and main results

Let Ω be a bounded domain containing the origin in $\mathbb{R}^N$ with Lipschitz boundary, we consider the following fractional p-Laplacian equation

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + \dfrac{|u|^{p^*_{s,\alpha}-2}}{|x|^\alpha}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p \in (1,\infty)$, $s \in (0,1)$, $0 \leqslant \alpha < ps < N$, $\lambda > 0$ is a parameter, and $p^*_{s,\alpha} = (N-\alpha)p/(N-ps)$ is the fractional Hardy-Sobolev exponent. For $p \in (1,\infty)$, $s \in (0,1)$ and $N > ps$, the fractional p-Laplacian operator $(-\Delta)_p^s$ is the nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0^+} \int_{B_\varepsilon(x)^c} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \qquad x \in \mathbb{R}^N.$$

This definition is consistent, up to a normalization constant depending on N and s. We would like to point out that, in the last decades, great attention has been attracted to

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the study of problems involving fractional Laplacian operators. We refer the readers to [2, 8, 9, 14, 15, 18, 22, 23, 27] and the references therein.

When $s=1$, problem (1.1) reduces to the following p-Laplacian problem involving critical Hardy-Sobolev exponents

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \dfrac{|u|^{p^*(\alpha)}}{|x|^\alpha} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ containing the origin, $1<p<N$, $\lambda>0$ is a parameter, $0<\alpha<p$ and $p^*(\alpha)=(N-\alpha)p/(N-p)$ is the critical Hardy-Sobolev exponent. In this setting, denoting λ_1 the first eigenvalue of the eigenvalues problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

in $W_0^{1,p}(\Omega)$. Perera-Zou [21] proved that:

- $N \geqslant p^2$ and $0<\lambda<\lambda_1$, then problem (1.2) has a positive ground state solution;
- $N \geqslant p^2$ and $\lambda>\lambda_1$ is not an eigenvalue of problem (1.3), then problem (1.2) has a nontrivial solution;
- $(N-p^2)(N-\alpha)>(p-\alpha)p$ and $\lambda \geqslant \lambda_1$, then problem (1.2) has a nontrivial solution.

We refer the readers to [12] for more details.

When $\alpha=0$, problem (1.1) reduces to the fractional p-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + |u|^{p_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda>0$ and $p_s^*=Np/(N-ps)$ is the fractional critical Sobolev exponent. In [17], by using an increasing and unbounded sequence of variational eigenvalues $0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$ of $(-\Delta)_p^s$, Mosconi, Perera, Squassina and Yang proved that the above problem has a nontrivial weak solution in the following cases:

- $N=sp^2$ and $\lambda<\lambda_1$;
- $N>sp^2$ and λ is not one of the eigenvalues λ_k;
- $N^2/(N+s)>sp^2$;
- $(N^3+s^3p^3)/N(N+s)>sp^2$ and $\partial\Omega\in C^{1,1}$.

In fact, Much work on Brezis-Nirenberg problem has been done after the celebrated paper by Brezis and Nirenberg [4], see [1, 7, 11, 12, 24, 25, 26, 28, 29, 30] and the references therein for the local case, and [24, 25, 28, 29] for the nonlocal case.

Now let us recall the weak formulation of problem (1.1). Let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}$$

be the Galiardo seminorm of a measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = ([u]_{s,p}^p + |u|_p^p)^{1/p},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We work in the closed linear subspaces

$$X_p^s(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\},$$

equivalently renormed by setting $\|\cdot\| = [\,\cdot\,]_{s,p}$, which is a uniformly convex Banach space. A function $u \in X_p^s(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy \\ &= \lambda \int_\Omega |u|^{p-2} uv dx + \int_\Omega \frac{|u|^{p^*_{s,\alpha}-2}}{|x|^\alpha} uv dx, \qquad \forall v \in X_p^s(\Omega). \end{aligned} \tag{1.4}$$

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$I(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{p}|u|^p - \frac{1}{p^*_{s,\alpha}} \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha} dx, \qquad u \in X_p^s(\Omega).$$

Let

$$\mu_\alpha = \inf_{u \in X_p^s(\Omega)\backslash\{0\}} \frac{\|u\|^p}{\left(\int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha} dx \right)^{p/p^*_{s,\alpha}}}, \tag{1.5}$$

which is positive by the fractional Hardy-Sobolev inequality and independent of Ω. Our first major difficulty is that an explicit formula for a minimizer for μ_α is not available. A natural conjecture is whether the family of minimizers consists of constant multiplies, translations and dilations of the function

$$U(x) = \frac{1}{\left(1 + |x|^{\frac{p-\alpha/s}{p-1}}\right)^{\frac{N-sp}{p-\alpha/s}}}, \qquad x \in \mathbb{R}^N.$$

This conjecture has been proved in [12] when $s = 1$, $p > 1$, and $\alpha \in [0, p)$ through Bliss inequality; and in [6] if $s \in (0,1)$, $p = 2$, $\alpha = 0$. However, up to now, the explicit

form of optimizers is not known for general $p \neq 2$ and $s \in (0,1)$. We will overcome this difficulty by working with certain asymptotic estimates for minimizers recently obtained in Marano and Mosconi [16].

The second difficulty is the lack of a direct sum decomposition suitable for applying the classical linking theorem. We will get around this difficulty by applying some more general critical point theorems (see [20, 31]), which will be described in Section 2.

The Dirichlet spectrum of $(-\Delta)_p^s$ in Ω consists of those $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

has a nontrivial solution. Although a complete description of the specturm is not known when $p \neq 2$, we can define an increasing and unbounded sequence of eigenvalues via a suitable minimax scheme. We will use the following scheme based on the cohomological index as in Iannizzotto et al. [13] (see also Perera [19]). The eigenvalue of problem (1.6) coincide with critical values of the functional

$$\Psi(u) = \left(\int_\Omega |u|^p dx \right)^{-1} = \frac{1}{|u|_p^p}$$

on the unit sphere $\mathscr{M} = \{u \in X_p^s(\Omega) : \|u\| = 1\}$. Set

$$\Psi^a = \{u \in \mathscr{M} : \Psi(u) \leqslant a\}, \ \ \Psi_a = \{u \in \mathscr{M} : \Psi(u) \geqslant a\}, \ \ a \in \mathbb{R}.$$

Let $\mathscr{F}$ denote the class of symmetric subsets of $\mathscr{M}$ and $i(M)$ stands for the $\mathbb{Z}_2$-cohomological index of $M \in \mathscr{F}$ which will be introduced in Section 2. Set

$$\lambda_k := \inf_{M \in \mathscr{F}, i(M) \geqslant k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.6) and

$$\lambda_k < \lambda_{k+1} \Rightarrow i(\Psi^{\lambda_k}) = i(\mathscr{M} \backslash \Psi_{\lambda_{k+1}}) = k. \tag{1.7}$$

Set

$$V_\alpha(\Omega) = \int_\Omega |x|^{\frac{(N-sp)\alpha}{sp-\alpha}} dx,$$

and note that

$$\int_\Omega |u|^p dx \leqslant V_\alpha(\Omega)^{(sp-\alpha)/(N-\alpha)} \left(\int_\Omega \frac{|u|^{p_{s,\alpha}^*}}{|x|^\alpha} dx \right)^{p/p_{s,\alpha}^*}, \qquad \forall u \in X_p^s(\Omega) \tag{1.8}$$

by the Hölder inequality.

Our main results are as follows:

THEOREM 1.1. *(Nonlocal Brezis-Nirenberg problem) Let* $p \in (1,\infty)$, $s \in (0,1)$, $0 \leqslant \alpha < ps < N$ *and* $\lambda > 0$. *Then problem (1.1) has a nontrivial weak solution in the following cases:*

(i) $N = sp^2$ *and* $0 < \lambda < \lambda_1$;

(ii) $N > sp^2$ *and* λ *is not one of the eigenvalues* λ_k;

(iii) $(N - sp^2)(N - \alpha) > sp(sp - \alpha)$;

(iv) $N(N - sp^2)(N - \alpha) > (N - sp)sp(sp - \alpha)$ *and* $\partial\Omega \in C^{1,1}$.

REMARK 1.1. In the nonsingular case $\alpha = 0$, Theorem 1.1 reduces to Mosconi et al. [17, Theorem 1.3]. In the case $s = 1$, Theorem 1.1 reduces to Perera-Zou [21, Theorem 1.1, 1.2 and 1.3]. Therefore, Theorem 1.1 extends the main results of [17, 21].

THEOREM 1.2. *Let* $p \in (1,\infty)$, $s \in (0,1)$, $0 \leqslant \alpha < ps < N$.

(a) *If*

$$\lambda_1 - \frac{\mu_\alpha}{V_\alpha(\Omega)^{(sp-\alpha)/(N-\alpha)}} < \lambda < \lambda_1,$$

then problem (1.1) has a pair of nontrivial solutions $\pm u^\lambda$ *such that* $u^\lambda \to 0$ *as* $\lambda \nearrow \lambda_1$.

(b) *If* $\lambda_k \leqslant \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ *for some* $k, m \in \mathbb{N}$ *and*

$$\lambda > \lambda_{k+1} - \frac{\mu_\alpha}{V_\alpha(\Omega)^{(sp-\alpha)/(N-\alpha)}}, \tag{1.9}$$

then problem (1.1) has m *distinct pairs of nontrivial solutions* $\pm u_j^\lambda$, $j = 1, \cdots, m$, *such that* $u_j^\lambda \to 0$ *as* $\lambda \nearrow \lambda_{k+1}$.

Now we note that $\lambda_1 \geqslant \dfrac{\mu_\alpha}{V_\alpha(\Omega)^{(sp-\alpha)/(N-\alpha)}}$. Indeed, let φ_1 be an eigenfunction associated with λ_1, then

$$\begin{aligned}
\lambda_1 &= \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|\varphi_1(x) - \varphi_1(y)|^p}{|x-y|^{N+sp}} dxdy}{\displaystyle\int_\Omega \varphi_1^p dx} \\
&\geqslant \frac{\mu_\alpha \Big(\displaystyle\int_\Omega \frac{|\varphi|^{p^*_{s,\alpha}}}{|x|^\alpha} \Big)^{p/p^*_{s,\alpha}}}{\displaystyle\int_\Omega \varphi_1^p dx} \\
&\geqslant \frac{\mu_\alpha}{V_\alpha(\Omega)^{(sp-\alpha)/(N-\alpha)}}
\end{aligned}$$

by (1.5) and (1.8).

REMARK 1.2. Since $V_0(\Omega)$ is the volume of Ω, in the nonsingular case $\alpha = 0$, Theorem 1.2 reduces to Perera et al. Theorem 1.1 in [20]. In the case $s = 1$, Theorem 1.2 reduces to Perera-Zou [21, Theorem 1.6 and 1.7]. Theorem 1.2 improves the main results of [20, 21].

The remainder of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we prove Theorem 1.1 and in Section 4, we prove Theorem 1.2.

2. Preminaries

2.1. Abstract critical point theorems

Here we will use a different sequence of eigenvalues introduced in Perera [13] that is based on a cohomological index. The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [10] is defined as follows. Let W be a Banach space and let $\mathscr{A}$ denote the class of symmetric subsets of $W\backslash\{0\}$. For $A \in \mathscr{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}P^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}P^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\{m \geqslant 1 : f^*(\omega^{m-1}) \neq 0\} & A \neq \emptyset, \\ 0 & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geqslant 1$ is the inclusion $\mathbb{R}P^{m-1} \subset \mathbb{R}P^\infty$, which induces isomorphisms on H^q for $q \leqslant m-1$, so $i(S^{m-1}) = m$.

We will prove Theorem 1.1 using the following abstract critical point theorem proved in Yang and Perera [31].

THEOREM 2.1. *Let I be a C^1-functional defined on a Banach space W, A_0 and B_0 be disjoint nonempty closed symmetric subsets of the unit sphere $S_1 = \{u \in W : \|u\| = 1\}$ such that*

$$i(A_0) = i(S_1\backslash B_0) < \infty.$$

Assume that there exist $R > r > 0$ and $v \in S_1\backslash A_0$ such that

$$\sup I(A) \leqslant \inf I(B), \qquad \sup I(X) < \infty,$$

where

$$\begin{aligned} A &= \{tu : u \in A_0, 0 \leqslant t \leqslant R\} \cup \{R\pi((1-t)u + tv) : u \in A_0, 0 \leqslant t \leqslant 1\}, \\ B &= \{ru : u \in B_0\}, \\ X &= \{tu : u \in A, \|u\| = R, 0 \leqslant t \leqslant 1\}, \end{aligned}$$

and $\pi : W\backslash\{0\} \to S_1, u \mapsto u/\|u\|$ *is the radial projection onto* S_1. *Let*

$$\Gamma = \{\gamma \in C(X,W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\},$$

and set

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in\Gamma(X)} I(u).$$

Then

$$\inf I(B) \leqslant c \leqslant \sup I(X),$$

in particular, c *is finite. If, in addition,* I *satisfies the* $(PS)_c$ *condition, then* c *is a critical value of* I.

Now let I be an even C^1-functional defined on a Banach space W, and let $\mathscr{A}^*$ denote the class of symmetric subsets of W. Let $r > 0$, $S_r = \{u \in W : \|u\| = r\}$, and Γ denote the group of odd homeomorphisms of W that are the identity outside $I^{-1}(0,b)$ for $0 < b \leqslant +\infty$. The pseudo-index of $M \in \mathscr{A}^*$ related to S_r and Γ is defined by

$$i^*(M) = \min_{\gamma\in\Gamma} i(\gamma(M) \cap S_r)$$

(see Benci [3]). We will prove Theorem 1.2 using the following critical point theorem.

THEOREM 2.2. *([20]) Let* A_0 *and* B_0 *be symmetric subsets of* S_1 *such that* A_0 *is compact,* B_0 *is closed, and*

$$i(A_0) \geqslant k+m, \qquad i(S_1\backslash B_0) \leqslant k$$

for some integers $k \geqslant 0$ *and* $m \geqslant 1$. *Assume that there exists* $R > r$ *such that*

$$\sup I(A) \leqslant 0 < \inf I(B), \qquad \sup I(X) < b,$$

where $A = \{Ru : u \in A_0\}$, $B = \{ru : u \in B_0\}$ *and* $X = \{tu : u \in A,\ 0 \leqslant t \leqslant 1\}$. *For* $j = k+1, \cdots, k+m$, *let*

$$\mathscr{A}_j^* = \{M \in \mathscr{A}^* : M \text{ is compact and } i^*(M) \geqslant j\}$$

and set

$$c_j^* := \inf_{M\in\mathscr{A}_j^*} \max_{u\in M} I(u).$$

Then

$$\inf I(B) \leqslant c_{k+1}^* \leqslant \cdots \leqslant c_{k+m}^* \leqslant \sup I(X),$$

in particular, $0 < c_j^* < b$. *If, in addition,* I *satisfies the* $(PS)_c$ *condition for all* $c \in (0,b)$, *then each* c_j^* *is a critical value of* I *and there are* m *distinct pairs of associated critical points.*

2.2. Minimizers for the fractional Hardy-Sobolev inequality

We have the following proposition from Marano and Mosconi [16] regarding the minimization problem (1.5).

PROPOSITION 1. *Let $1 < p < \infty$, $0 < s < 1$, and $0 \leqslant \alpha < ps < N$. Let μ_α be as in (1.5). Then*

(1) *There exists a minimizer for μ_α;*

(2) *Every minimizer U is of constant sign, radially monotone.*

We can see that for every minimizer U, there exists $\lambda_U > 0$ such that

$$\int_{\mathbb{R}^{2N}} \frac{(U(x)-U(y))^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}} dxdy = \lambda_U \int_{\mathbb{R}^N} \frac{U^{p^*_{s,\alpha}-1} v}{|x|^\alpha} dx, \qquad \forall v \in X^s_p(\mathbb{R}^N).$$

In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for μ_α. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)^s_p U = \frac{U^{p^*_{s,\alpha}-1}}{|x|^\alpha}. \tag{2.1}$$

Testing this equation with U and using (1.5), we obtain

$$\begin{aligned} \|U\|^p &= \int_{\mathbb{R}^N} \frac{U^{p^*_{s,\alpha}}}{|x|^\alpha} dx \\ &= \mu_s^{(N-\alpha)/(sp-\alpha)}. \end{aligned} \tag{2.2}$$

For any $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{\frac{N-sp}{p}}} U\left(\frac{|x|}{\varepsilon}\right) \tag{2.3}$$

is also a minimizer for μ_α satisfying (2.1) and (2.2), so after a rescaling we may assume that $U(0) = 1$. Henceforth, U will denote such a normalized (with respect to constant multiples and rescaling) minimizer and U_ε will denote the associated family of minimizers given by (2.3). In the absence of an explicit formula for U, we will use the following asymptotic estimates.

LEMMA 2. *There exist constants c_1, $c_2 > 0$ and $\theta > 1$ such that for all $r \geqslant 1$,*

$$\frac{c_1}{r^{(N-sp)/(p-1)}} \leqslant U(r) \leqslant \frac{c_2}{r^{(N-sp)/(p-1)}} \tag{2.4}$$

and

$$\frac{U(\theta r)}{U(r)} \leqslant \frac{1}{2}. \tag{2.5}$$

Proof. The inequalities in (2.4) were proved in Marano and Mosconi [16]. They imply that

$$\frac{U(\theta r)}{U(r)} \leqslant \frac{c_2}{c_1}\frac{1}{\theta^{(N-sp)/(p-1)}},$$

and (2.5) follows for sufficiently large θ. $\square$

2.3. Regularity estimates

Now let θ be as in Lemma 2, let $\eta \in C^\infty(\mathbb{R}^N,[0,1])$ be such that

$$\eta(x) = \begin{cases} 0, \text{ if } |x| \leqslant 2\theta, \\ 1, \text{ if } |x| \geqslant 3\theta, \end{cases}$$

and let $\eta_\delta(x) = \eta\left(\frac{x}{\delta}\right)$ for $\delta > 0$.

LEMMA 3. *(* [17, Lemma 2.6]*) Assume that* $0 \in \Omega$. *Then there exists a constant* $C = C(N,\Omega,p,s) > 0$ *such that for any* $v \in X_p^s(\Omega)$ *such that* $(-\Delta)_p^s v \in L^\infty(\Omega)$ *and* $\delta > 0$ *such that* $B_{5\theta\delta} \subset \Omega$,

$$\|v\eta_\delta\|^p \leqslant \|v\|^p + C\left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \delta^{N-sp}.$$

2.4. Auxiliary estimates

We now construct some auxiliary functions and estimate their norms. In what follows θ is the universal constant in Lemma 2 that depends only on N, p, s and α. For any $\varepsilon,\delta > 0$, let

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

let

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta}^p(t - U_\varepsilon(\theta\delta)), & \text{if } U_\varepsilon(\theta\delta) \leqslant t \leqslant U_\varepsilon(\delta), \\ t + U_\varepsilon(\delta)(m_{\varepsilon,\delta}^{p-1} - 1), & \text{if } t \geqslant U_\varepsilon(\delta) \end{cases}$$

and let

$$\begin{aligned} G_{\varepsilon,\delta}(t) &= \int_0^t g'_{\varepsilon,\delta}(\tau)^{1/p} d\tau \\ &= \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant U_\varepsilon(\theta\delta), \\ m_{\varepsilon,\delta}(t - U_\varepsilon(\theta\delta)), & \text{if } U_\varepsilon(\theta\delta) \leqslant t \leqslant U_\varepsilon(\delta), \\ t, & \text{if } t \geqslant U_\varepsilon(\delta). \end{cases} \end{aligned} \tag{2.6}$$

The functions $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric nonincreasing function

$$u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r)),$$

which satisfies

$$u_{\varepsilon,\delta}(r) = \begin{cases} U_\varepsilon(r), & \text{if } r \leqslant \delta, \\ 0, & \text{if } r \geqslant \theta\delta. \end{cases} \tag{2.7}$$

We have the following estimates for $u_{\varepsilon,\delta}$.

LEMMA 4. *There exists a constant $C = C(N,p,s,\alpha) > 0$ such that for any $\varepsilon \leqslant \delta/2$,*

$$\|u_{\varepsilon,\delta}\|^p \leqslant \mu_\alpha^{(N-\alpha)/(sp-\alpha)} + C\left(\frac{\varepsilon}{\delta}\right)^{(N-sp)/(p-1)}, \tag{2.8}$$

$$|u_{\varepsilon,\delta}|_p^p \geqslant \begin{cases} \frac{1}{C}\varepsilon^{sp}\log\left(\frac{\delta}{\varepsilon}\right), & \text{if } N = sp^2, \\ \frac{1}{C}\varepsilon^{sp}, & \text{if } N > sp^2, \end{cases} \tag{2.9}$$

$$\int_\Omega \frac{u_{\varepsilon,\delta}^{p^*_{s,\alpha}}}{|x|^\alpha}dx \geqslant \mu_\alpha^{(N-\alpha)/(sp-\alpha)} - C\left(\frac{\varepsilon}{\delta}\right)^{(N-\alpha)/(p-1)}. \tag{2.10}$$

Proof. Using Brasco and Parini [5, Lemma A.2] and testing the equation

$$(-\Delta)_p^s U_\varepsilon = \frac{U_\varepsilon^{p^*_{s,\alpha}-1}}{|x|^\alpha}$$

with $g_{\varepsilon,\delta}(U_\varepsilon) \in X_p^s(\Omega)$ implies that

$$\begin{aligned} \|G_{\varepsilon,\delta}(U_\varepsilon)\|^p &\leqslant \int_{\mathbb{R}^{2N}} \frac{(U_\varepsilon(x)-U_\varepsilon(y))^{p-1}(g_{\varepsilon,\delta}(U_\varepsilon(x)) - g_{\varepsilon,\delta}(U_\varepsilon(y)))}{|x-y|^{N+sp}}dxdy \\ &= \int_{\mathbb{R}^N} \frac{U_\varepsilon^{p^*_{s,\alpha}-1}}{|x|^\alpha} g_{\varepsilon,\delta}(U_\varepsilon(x))dx \\ &= \int_{\mathbb{R}^N} \frac{U_\varepsilon^{p^*_{s,\alpha}}}{|x|^\alpha}dx + \int_{\mathbb{R}^N} \frac{U_\varepsilon^{p^*_{s,\alpha}-1}}{|x|^\alpha}(g_{\varepsilon,\delta}(U_\varepsilon(x)) - U_\varepsilon(x))dx. \end{aligned}$$

We have $\int_{\mathbb{R}^N} \frac{U_\varepsilon^{p^*_{s,\alpha}}}{|x|^\alpha}dx = \mu_\alpha^{(N-\alpha)/(sp-\alpha)}$ by (2.2),

$$g_{\varepsilon,\delta}(t) - t \leqslant U_\varepsilon(\delta)m_{\varepsilon,\delta}^{p-1}$$

$$= \frac{1}{\varepsilon^{(N-sp)/p}} U\left(\frac{\delta}{\varepsilon}\right)\left[1 - U\left(\frac{\theta\delta}{\varepsilon}\right)/U\left(\frac{\delta}{\varepsilon}\right)\right]^{-(p-1)}$$
$$\leqslant 2^{p-1} c_2 \frac{\varepsilon^{(N-sp)/p(p-1)}}{\delta^{(N-sp)/(p-1)}}, \qquad \forall t \geqslant 0$$

by (2.4) and (2.5),

$$\int_\Omega \frac{U_\varepsilon^{p^*_{s,\alpha}-1}}{|x|^\alpha} dx = \varepsilon^{(N-sp)/p} \int_\Omega \frac{U(x)^{p^*_{s,\alpha}-1}}{|x|^\alpha} dx$$

and the last integral is finite by (2.4) again, so (2.8) follows. Using (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(x)^p dx &\geqslant \int_{B_\delta(0)} u_{\varepsilon,\delta}(x)^p dx \\ &= \int_{B_\delta(0)} U_\varepsilon(x)^p dx \\ &= \varepsilon^{sp} \int_{B_{\delta/\varepsilon}(0)} U(x)^p dx\end{aligned}$$

and the last integral is greater than or equal to

$$\int_1^{\delta/\varepsilon} U(r)^p r^{N-1} dr \geqslant c_1^p \int_1^{\delta/\varepsilon} r^{-(N-sp^2)/(p-1)-1} dr$$

by (2.4). A direct evaluation of the integral on the right gives (2.9) since $\delta/\varepsilon \geqslant 2$. Using (2.7)and (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u_{\varepsilon,\delta}(x)^{p^*_{s,\alpha}}}{|x|^\alpha} dx &\geqslant \int_{B_\delta(0)} \frac{u_{\varepsilon,\delta}(x)^{p^*_{s,\alpha}}}{|x|^\alpha} dx \\ &= \int_{B_\delta(0)} \frac{U_\varepsilon(x)^{p^*_{s,\alpha}}}{|x|^\alpha} dx \\ &= \mu_\alpha^{(N-sp)/(sp-\alpha)} - \int_{B_{\delta/\varepsilon}(0)^c} \frac{U(x)^{p^*_{s,\alpha}}}{|x|^\alpha} dx.\end{aligned}$$

By (2.4), the last integral is less than or equal to

$$c_2^{p^*_{s,\alpha}} \int_{\delta/\varepsilon}^\infty r^{-(N-\alpha)/(p-1)-1} dr = \frac{(p-1)c_2^{p^*_{s,\alpha}}}{(N-\alpha)} \left(\frac{\varepsilon}{\delta}\right)^{(N-\alpha)/(p-1)},$$

so (2.10) follows. □

By Lemma 4, we have the following estimate for

$$\mu_{\varepsilon,\delta}(\lambda) := \frac{\|u_{\varepsilon,\delta}\|^p - \lambda |u_{\varepsilon,\delta}|_p^p}{\left(\int_\Omega \frac{u_{\varepsilon,\delta}(x)^{p^*_{s,\alpha}}}{|x|^\alpha} dx\right)^{p/p^*_{s,\alpha}}},$$

there exists a constant $C = C(N,p,s,\alpha) > 0$ such that for any $\varepsilon \leqslant \delta/2$,

$$\mu_{\varepsilon,\delta}(\lambda) \leqslant \begin{cases} \mu_\alpha - \dfrac{\lambda}{C}\varepsilon^{sp}\log\left(\dfrac{\delta}{\varepsilon}\right) + C\left(\dfrac{\varepsilon}{\delta}\right)^{sp}, & \text{if } N = sp^2, \\ \mu_\alpha - \dfrac{\lambda}{C}\varepsilon^{sp} + C\left(\dfrac{\varepsilon}{\delta}\right)^{(N-sp)/(p-1)}, & \text{if } N > sp^2. \end{cases} \tag{2.11}$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For $0 < \lambda < \lambda_1$, mountain pass theorem and (2.11) will give us a positive critical level of I below the threshold level for compactness given in Proposition 5. For $\lambda \geqslant \lambda_1$, we will use the abstract linking theorem, Theorem 2.1.

PROPOSITION 5. *For any $\lambda > 0$, I satisfies the $(PS)_c$ condition for all*

$$c < \frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)}.$$

LEMMA 6. *([20]) If (u_j) is bounded in $X_p^s(\Omega)$, and $u_j \to u$ a.e. in Ω, then*

$$\|u_j\|^p = \|u_j - u\|^p + \|u\|^p + o(1) \qquad \text{as } j \to \infty.$$

By a similar way, we can prove the following lemma, here we omit the details.

LEMMA 7. *If (u_j) is bounded in $X_p^s(\Omega)$, and $u_j \to u$ a.e. in Ω, then*

$$\int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx = \int_\Omega \frac{|u_j - u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx + \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx + o(1) \qquad \text{as } j \to \infty. \tag{3.1}$$

Proof of Proposition 5. Let $c < \dfrac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)}$ and (u_j) be a $(PS)_c$ sequence. First we show that (u_j) is bounded in $X_p^s(\Omega)$. We have

$$\begin{aligned} I(u_j) &= \frac{1}{p}\|u_j\|^p - \frac{\lambda}{p}|u_j|_p^p - \frac{1}{p^*_{s,\alpha}}\int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx \\ &= c + o(1), \end{aligned} \tag{3.2}$$

$$\begin{aligned} I'(u_j)v &= \int_{\mathbb{R}^{2N}} \frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy \\ &\quad -\lambda\int_\Omega |u_j|^{p-2}u_j v dx - \int_\Omega \frac{|u_j|^{p^*_{s,\alpha}-2}}{|x|^\alpha}u_j v dx \\ &= o(\|v\|), \qquad \forall v \in X_p^s(\Omega), \end{aligned} \tag{3.3}$$

as $j \to \infty$. Then we have

$$\frac{ps-\alpha}{(N-\alpha)p}\int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx = I(u_j) - \frac{1}{p}I'(u_j)u_j$$
$$= o(\|u_j\|) + O(1),$$

which together with (3.2) and (1.8) shows that (u_j) is bounded in $X^s_p(\Omega)$. So a renamed subsequence of (u_j) converges to some u weakly in $X^s_p(\Omega)$, strongly in $L^r(\Omega)$ for all $r \in [1, p^*_s)$ and a.e. in Ω. Denoting by $p' = p/(p-1)$ the Hölder conjugate of p,

$$|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))/|x-y|^{(N+sp)/p'}$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ and converges to

$$|u(x) - u(y)|^{p-2}(u(x) - u(y))/|x-y|^{(N+ps)/p'}$$

a.e. in $\mathbb{R}^{2N}$ and

$$(v(x) - v(y))/|x-y|^{(N+ps)/p} \in L^p(\mathbb{R}^{2N}),$$

so the first integral in (3.3) converges to

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}dxdy$$

for a further subsequence. Moreover,

$$\int_\Omega |u_j|^{p-2}u_j v dx \to \int_\Omega |u|^{p-2}uvdx$$

and

$$\int_\Omega \frac{|u_j|^{p^*_{s,\alpha}-2}}{|x|^\alpha}u_j v dx \to \int_\Omega \frac{|u|^{p^*_{s,\alpha}-2}}{|x|^\alpha}uvdx$$

since $\frac{|u_j|^{p^*_{s,\alpha}-2}}{|x|^\alpha}u_j \rightharpoonup \frac{|u|^{p^*_{s,\alpha}-2}}{|x|^\alpha}u$ in $L^{(p^*_{s,\alpha})'}(\Omega)$. So passing to the limit in (3.3) implies that $u \in X^s_p(\Omega)$ is a weak solution of (1.1), i.e. (1.4) holds.

Setting $\widetilde{u}_j = u_j - u$, we will show that $\widetilde{u}_j \to 0$ in $X^s_p(\Omega)$. We have

$$\|\widetilde{u}_j\|^p = \|u_j\|^p - \|u\|^p + o(1) \tag{3.4}$$

by Lemma 6 and

$$\int_\Omega \frac{|\widetilde{u}_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx = \int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx - \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx + o(1) \tag{3.5}$$

by Lemma 7. Taking $v = u_j$ in (3.3), we get

$$\|u_j\|^p = \lambda |u|^p_p + \int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha}dx + o(1), \tag{3.6}$$

since (u_j) is bounded in $X_p^s(\Omega)$ and converges to u in $L^p(\Omega)$. Testing (1.4) with $v=u$, we have

$$\|u\|^p = \lambda |u|_p^p + \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha} dx. \tag{3.7}$$

It follows from (3.4)-(3.7) and (1.5) that

$$\|\widetilde{u}_j\|^p = \int_\Omega \frac{|\widetilde{u}_j|^{p^*_{s,\alpha}}}{|x|^\alpha} dx + o(1) \leqslant \frac{\|\widetilde{u}_j\|^{p^*_{s,\alpha}}}{\mu_\alpha^{p^*_{s,\alpha}/p}} + o(1),$$

so

$$\|\widetilde{u}_j\|^p (\mu_\alpha^{p^*_{s,\alpha}/p} - \|\widetilde{u}_j\|^{p^*_{s,\alpha}-p}) \leqslant o(1). \tag{3.8}$$

On the other hand,

$$\begin{aligned}
c &= \frac{1}{p}\|u_j\|^p - \frac{\lambda}{p}|u|_p^p - \frac{1}{p^*_{s,\alpha}} \int_\Omega \frac{|u_j|^{p^*_{s,\alpha}}}{|x|^\alpha} dx + o(1) && \text{by (3.2)} \\
&= \frac{ps-\alpha}{(N-\alpha)p}(\|u_j\|^p - \lambda |u|_p^p) + o(1) && \text{by (3.6)} \\
&= \frac{ps-\alpha}{(N-\alpha)p}(\|\widetilde{u}_j\|^p + \|u\|^p - \lambda |u|_p^p) + o(1) && \text{by (3.4)} \\
&= \frac{ps-\alpha}{(N-\alpha)p}(\|\widetilde{u}_j\|^p + \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha} dx) + o(1) && \text{by (3.7)} \\
&\geqslant \frac{ps-\alpha}{(N-\alpha)p}\|\widetilde{u}_j\|^p + o(1),
\end{aligned}$$

so

$$\limsup_{j\to\infty} \|\widetilde{u}_j\|^p \leqslant \frac{(N-\alpha)p}{(ps-\alpha)} c < \mu_\alpha^{(N-\alpha)/(ps-\alpha)}. \tag{3.9}$$

It follows from (3.8) and (3.9) that $\|\widetilde{u}_j\| \to 0$. $\square$

3.1. Case 1: $N \geqslant sp^2$ and $0 < \lambda < \lambda_1$

By the definition of λ_1, we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}|u|_p^p - \frac{1}{p^*_{s,\alpha}} \int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha} dx \\
&\geqslant \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - \frac{1}{p^*_{s,\alpha}\mu_\alpha^{p^*_{s,\alpha}/p}}\|u\|^{p^*_{s,\alpha}},
\end{aligned}$$

so the origin is a strict local minimizer of I_λ. Fix $\delta > 0$ so small that $B_{\theta\delta}(0) \subset\subset \Omega$, so that $\operatorname{supp} u_{\varepsilon,\delta} \subset \Omega$ by (2.7). Noting that

$$I(Ru_{\varepsilon,\delta}) = \frac{R^p}{p}(\|u_{\varepsilon,\delta}\|^p - \lambda|u_{\varepsilon,\delta}|_p^p) - \frac{R^{p^*_{s,\alpha}}}{p^*_{s,\alpha}}\int_\Omega \frac{|u_{\varepsilon,\delta}|^{p^*_{s,\alpha}}}{|x|^\alpha}dx \to -\infty \qquad \text{as } R \to +\infty,$$

fix $R_0 > 0$ so large that $I(R_0 u_{\varepsilon,\delta}) < 0$. Then let

$$\Gamma = \{\gamma \in C([0,1], X_p^s(\Omega)) : \gamma(0) = 0, \gamma(1) = R_0 u_{\varepsilon,\delta}\}$$

and set

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0.$$

Since $t \mapsto tR_0 u_{\varepsilon,\delta}$ is a path in Γ,

$$\begin{aligned} c &\leqslant \max_{t\in[0,1]} I(tR_0 u_{\varepsilon,\delta}) \\ &= \frac{sp-\alpha}{(N-\alpha)p}\left[\frac{\|u_{\varepsilon,\delta}\|^p - \lambda|u_{\varepsilon,\delta}|_p^p}{\left(\int_\Omega \frac{|u_{\varepsilon,\delta}|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}}\right]^{\frac{N-\alpha}{ps-\alpha}} \\ &= \frac{sp-\alpha}{(N-\alpha)p}\mu_{\varepsilon,\delta}(\lambda)^{(N-\alpha)/(ps-\alpha)}. \end{aligned} \tag{3.10}$$

By (2.11), we have

$$\mu_{\varepsilon,\delta}(\lambda) \leqslant \begin{cases} \mu_\alpha + \left(C - \frac{\lambda}{C}|\log\varepsilon|\right)\varepsilon^{sp}, & \text{if } N = sp^2, \\ \mu_\alpha - \left(\frac{\lambda}{C} - C\varepsilon^{(N-sp^2)/(p-1)}\right)\varepsilon^{sp}, & \text{if } N > sp^2, \end{cases}$$

so $\mu_{\varepsilon,\delta}(\lambda) < \mu_\alpha$ if $\varepsilon > 0$ is sufficiently small. So

$$c < \frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)}$$

by (3.10), hence I satisfies the $(PS)_c$ condition by Proposition 5. Then c is a critical level of I by the mountain pass theorem.

3.2. Case 2: $N > sp^2$ and $\lambda > \lambda_1$ is not one of the eigenvalues λ_k

We have $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$, and then $i(\Psi^{\lambda_k}) = i(\mathscr{M}\backslash\Psi_{\lambda_{k+1}}) = k$ by (1.7). In what follows,

$$\pi(u) = \frac{u}{\|u\|}, \quad \pi_p(u) = \frac{u}{|u|_p}, \quad u \in X_p^s(\Omega)\backslash\{0\}$$

are the radial projections onto

$$\mathscr{M} = \{u \in X_p^s(\Omega) : \|u\| = 1\}, \ \mathscr{M}_p = \{u \in X_p^s(\Omega) : |u|_p = 1\},$$

respectively.

PROPOSITION 8. ([17]) *If $\lambda_k < \lambda_{k+1}$, then Ψ^{λ_k} has a compact symmetric subset E with $i(E) = k$ such that*

$$\left|(-\Delta)_p^s v\right|_\infty \leqslant C, \qquad \forall v \in E,$$

where $C = C(N,\Omega,p,s,k) > 0$. In particular,

$$|v|_\infty \leqslant C, \qquad \forall v \in E.$$

For $v \in E$ and $\delta > 0$, let $v_\delta = v\eta_\delta$, where η_δ is the cut-off function in Lemma 3 and let

$$E_\delta = \{\pi(v_\delta) : v \in E\}.$$

PROPOSITION 9. ([17]) *There exists a constant $C = C(N,\Omega,p,s,k) > 0$ such that for all sufficiently small $\delta > 0$,*

$$\frac{1}{C} \leqslant |w|_q \leqslant C, \qquad \forall w \in E_\delta, \qquad 1 \leqslant q \leqslant \infty, \tag{3.11}$$

$$\sup_{w \in E_\delta} \Psi(w) \leqslant \lambda_k + C\delta^{N-sp}, \tag{3.12}$$

$E_\delta \cap \Psi_{\lambda_{k+1}} = \emptyset$, $i(E_\delta) = k$, and supp $w \subset B_{2\theta\delta}(0)^c$ for all $w \in E_\delta$. In particular, the supports of w and $\pi(u_{\varepsilon,\delta})$ are disjoint and hence $\pi(u_{\varepsilon,\delta}) \notin E_\delta$.

Since

$$\begin{aligned}\int_\Omega |v_\delta|^p dx &\geqslant \int_{\Omega\setminus B_{3\theta\delta}(0)} |v|^p dx \\ &= \int_\Omega |v|^p dx - \int_{B_{3\theta\delta}(0)} |v|^p dx \\ &\geqslant \frac{1}{\lambda_k} - C\delta^N,\end{aligned}$$

where $C = C(N,p,s,\alpha,\Omega,k) > 0$ is a constant, combining with (1.8), it implies that

$$\int_\Omega \frac{|v_\delta|^{p_{s,\alpha}^*}}{|x|^\alpha} dx \geqslant \frac{1}{C}$$

if $\delta > 0$ is sufficiently small. By Lemma 3 and Proposition 8, we have

$$\|v_\delta\|^p \leqslant 1 + C\delta^{N-ps}.$$

So

$$\int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha} dx = \frac{\displaystyle\int_\Omega \frac{|v_\delta|^{p^*_{s,\alpha}}}{|x|^\alpha} dx}{\|v_\delta\|^{p^*_{s,\alpha}}} \geqslant \frac{1}{C}, \qquad \forall w \in E_\delta. \tag{3.13}$$

We are now ready to apply Theorem 2.1 to obtain a nontrivial critical point of I in the case where $\lambda > \lambda_1$ is not one of the eigenvalues λ_k. Fix λ' such that $\lambda_k < \lambda' < \lambda < \lambda_{k+1}$, and let $\delta > 0$ be so small that the conclusions of Proposition 9 hold with $\lambda_k + C\delta^{N-sp} < \lambda'$, in particular,

$$\Psi(w) < \lambda', \qquad \forall w \in E_\delta. \tag{3.14}$$

Then take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}}$, and note that A_0 and B_0 are disjoint nonempty closed symmetric subsets of $\mathscr{M}$ such that

$$i(A_0) = i(\mathscr{M} \backslash B_0) = k$$

by Proposition 9 and (1.7). Now let $0 < \varepsilon \leqslant \delta/2$, let $R > r > 0$, let $v_0 = \pi(u_{\varepsilon,\delta}) \in \mathscr{M} \backslash E_\delta$ and let A, B and X be as in Theorem 2.1.

For $u \in \Psi_{\lambda_{k+1}}$,

$$I(ru) \geqslant \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^p - \frac{r^{p^*_{s,\alpha}}}{p^*_{s,\alpha}\mu_\alpha^{p^*_{s,\alpha}/p}}.$$

Since $\lambda < \lambda_{k+1}$, it follows that $\inf I(B) > 0$ if r is sufficiently small.

Next we show that $I \leqslant 0$ on A if R is sufficiently large. For $w \in E_\delta$ and $t \geqslant 0$,

$$I(tw) \leqslant \frac{t^p}{p}\left(1 - \frac{\lambda}{\Psi(w)}\right) \leqslant 0$$

by (3.14). Now let $w \in E_\delta$ and $0 \leqslant t \leqslant 1$, and set $u = \pi((1-t)w + tv_0)$. Clearly, $\|(1-t)w + tv_0\| \leqslant 1$, and since the supports of w and v_0 are disjoint by Proposition 9,

$$\int_\Omega \frac{|(1-t)w + tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx = (1-t)^{p^*_{s,\alpha}} \int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha} dx + t^{p^*_{s,\alpha}} \int_\Omega \frac{v_0^{p^*_{s,\alpha}}}{|x|^\alpha} dx,$$

$$\begin{aligned} |u|_p^p &= \frac{|(1-t)w + tv_0|_p^p}{\|(1-t)w + tv_0\|^p} \\ &\geqslant \frac{(1-t)^p}{\Psi(w)} \\ &\geqslant \frac{(1-t)^p}{\lambda'}. \end{aligned}$$

In view of (3.13) and since

$$\int_\Omega \frac{v_0^{p^*_{s,\alpha}}}{|x|^\alpha}dx = \frac{\displaystyle\int_\Omega \frac{u_{\varepsilon,\delta}^{p^*_{s,\alpha}}}{|x|^\alpha}dx}{\|u_{\varepsilon,\delta}\|^{p^*_{s,\alpha}}} \geqslant \frac{1}{C} \tag{3.15}$$

by Lemma 4, it follows that

$$\begin{aligned}\int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx &= \frac{\displaystyle\int_\Omega \frac{|(1-t)w+tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx}{\|(1-t)w+tv_0\|^{p^*_{s,\alpha}}} \\ &\geqslant \int_\Omega \frac{|(1-t)w+tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx \\ &\geqslant \frac{t^{p^*_{s,\alpha}}}{C}\end{aligned}$$

if ε is sufficiently small, where $C = C(N,\Omega,p,s,\alpha,k) > 0$. Then we have

$$\begin{aligned}I(Ru) &= \frac{R^p}{p}\|u\|^p - \frac{\lambda R^p}{p}|u|_p^p - \frac{R^{p^*_{s,\alpha}}}{p^*_{s,\alpha}}\int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx \\ &\leqslant -\frac{R^p}{p}\left[\frac{\lambda}{\lambda'}(1-t)^p - 1\right] - \frac{t^{p^*_{s,\alpha}}}{p^*_{s,\alpha}C}R^{p^*_{s,\alpha}}.\end{aligned}$$

The above expression is clearly non-positive if $t \leqslant 1-(\lambda'/\lambda)^{1/p} := t_0$. For $t > t_0$, it is non-positive if R is sufficiently large.

In view of Theorem 2.1 and Proposition 5, it only remains to show that

$$\sup I(X) < \frac{ps-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(ps-\alpha)},$$

if ε is sufficiently small. Noting that

$$X = \{\rho\pi((1-t)w+tv_0) : w \in E_\delta, 0 \leqslant t \leqslant 1, 0 \leqslant \rho \leqslant R\}.$$

Set again $u = \pi((1-t)w+tv_0)$, it is obviously that $I(\rho u) \leqslant 0$, for all $0 \leqslant \rho \leqslant R$, if $0 \leqslant t \leqslant t_0$. So we only need to consider the case $1 \geqslant t > t_0$. Then

$$\begin{aligned}\sup_{0\leqslant\rho\leqslant R} I(\rho u) &\leqslant \sup_{\rho\geqslant 0}\left[\frac{\rho^p}{p}(1-\lambda|u|_p^p) - \frac{\rho^{p^*_{s,\alpha}}}{p^*_{s,\alpha}}\int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right] \\ &= \frac{ps-\alpha}{(N-\alpha)p}\left[\frac{(1-\lambda|u|_p^p)^+}{\left(\displaystyle\int_\Omega \frac{|u|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}}\right]^{(N-\alpha)/(ps-\alpha)}\end{aligned}$$

$$= \frac{ps-\alpha}{(N-\alpha)p}\left[\frac{(\|(1-t)w+tv_0\|^p-\lambda|(1-t)w+tv_0|_p^p)^+}{\left(\int_\Omega \frac{|(1-t)w+tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}}\right]^{(N-\alpha)/(ps-\alpha)} \tag{3.16}$$

Since $w=0$ in $B_{2\theta\delta}(0)$ by Proposition 9 and $v_0=0$ in $B_{\theta\delta}(0)^c$ by (2.7), it follows that

$$\begin{aligned}
&\|(1-t)w+tv_0\|^p \\
&\quad\leqslant (1-t)^p\int_{A_1}\frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}dxdy+t^p\int_{A_2}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\qquad +2\int_{A_3}\frac{|(1-t)w(x)-tv_0(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad =: (1-t)^pI_1+t^pI_2+2I_3,
\end{aligned} \tag{3.17}$$

where

$$A_1=B_{\theta\delta}(0)^c\times B_{\theta\delta}(0)^c,\qquad A_2=B_{2\theta\delta}(0)\times B_{2\theta\delta}(0),\qquad A_3=B_{2\theta\delta}(0)^c\times B_{\theta\delta}(0).$$

We estimate I_3 using the following elementary inequality: given $\kappa>1$ and $p-1<q<p$, there exists a constant $C=C(\kappa,q)>0$ such that

$$|a+b|^p\leqslant \kappa|a|^p+|b|^p+C|a|^{p-q}|b|^q,\qquad \forall a,b\in\mathbb{R}.$$

Taking $\kappa=\lambda/\lambda'$ and, thanks to $N>sp^2$, choosing $q\in(N(p-1)/(N-ps),p)$, we get

$$\begin{aligned}
I_3 &\leqslant \tfrac{\lambda}{\lambda'}(1-t)^p\textstyle\int_{A_3}\frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}dxdy+t^p\int_{A_3}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad +C\textstyle\int_{A_3}\frac{|w(x)|^{p-q}v_0(y)^q}{|x-y|^{N+ps}}dxdy=:\frac{\lambda}{\lambda'}(1-t)^pI_4+t^pI_5+CJ_q.
\end{aligned} \tag{3.18}$$

Clearly, $I_1+2I_4\leqslant\|w\|^p=1$ and $I_2+2I_5\leqslant\|v_0\|^p=1$. By (3.11) and

$$\begin{aligned}
|x-y| &\geqslant |x|-\theta\delta \\
&\geqslant |x|/2 \qquad \text{on } A_3,
\end{aligned}$$

we have

$$\begin{aligned}
J_q &\leqslant \frac{C}{\|u_{\varepsilon,\delta}\|^q}\int_{A_3}\frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+ps}}dxdy \\
&\leqslant \frac{C}{\delta^{ps}}\int_{\mathbb{R}^N}u_{\varepsilon,\delta}(y)^q dy,
\end{aligned}$$

since (2.9) and (1.8) imply that $\displaystyle\int_\Omega\frac{u_{\varepsilon,\delta}^{p^*_{s,\alpha}}}{|x|^\alpha}dx$, and hence also $\|u_{\varepsilon,\delta}\|$, is bounded away from zero if ε is sufficiently small. Recalling (2.6), it holds $G_{\varepsilon,\delta}(t)\leqslant t$ for all $t\geqslant 0$,

and thus

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}(y)^q dy \leqslant \int_{\mathbb{R}^N} U_\varepsilon(y)^q dy$$
$$= \varepsilon^{N-(N-ps)q/p} \int_{\mathbb{R}^N} U(y)^q dy,$$

and the last integral is finite by (2.4) since $q > N(p-1)/(N-sp)$. So combining (3.17) and (3.18), we have

$$\|(1-t)w + tv_0\|^p \leqslant \frac{\lambda}{\lambda'}(1-t)^p + t^p + C\varepsilon^{N-(N-ps)q/p}. \tag{3.19}$$

On the other hand, since the supports of w and v_0 are disjoint,

$$|(1-t)w + tv_0|_p^p = (1-t)^p |w|_p^p + t^p |v_0|_p^p,$$
$$\int_\Omega \frac{|(1-t)w + tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx = (1-t)^{p^*_{s,\alpha}} \int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha} dx + t^{p^*_{s,\alpha}} \int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx. \tag{3.20}$$

By (3.13), $\int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha} dx$ is bounded away from zero, and (3.15) implies that so is $\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx$ if ε is sufficiently small, so the last expression in (3.20) is bounded away from zero for $1 \geqslant t > t_0$. It follows from (3.19), (3.20) and $|w|_p^p = 1/\Psi(w) > 1/\lambda'$ by (3.14), that

$$\frac{\|(1-t)w + tv_0\|^p - \lambda |(1-t)w + tv_0|_p^p}{\left(\int_\Omega \frac{|(1-t)w + tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx\right)^{p/p^*_{s,\alpha}}} \leqslant \frac{1 - \lambda |v_0|_p^p}{\left(\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha} dx\right)^{p/p^*_{s,\alpha}}} + C\varepsilon^{N-(N-ps)q/p}.$$

Since $v_0 = u_{\varepsilon,\delta}/\|u_{\varepsilon,\delta}\|$, the right-hand side is less than or equal to

$$\mu_{\varepsilon,\delta}(\lambda) + C\varepsilon^{N-(N-ps)q/p} \leqslant \mu_\alpha - \left(\frac{\lambda}{C} - C\varepsilon^{(N-sp^2)/(ps-1)} - C\varepsilon^{(N-ps)/(1-q/p)}\right)\varepsilon^{sp}$$

by (2.11). Since $N > sp^2$ and $q < p$, it follows from this that the last expression in (3.16) is strictly less than $\frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)}$ if ε is sufficiently small.

3.3. Case 3: $(N-sp^2)(N-\alpha) > sp(sp-\alpha)$ and $\lambda = \lambda_k$

We note that $(N-sp^2)(N-\alpha) > sp(sp-\alpha)$ implies that $N > sp^2$, and the case where $\lambda > \lambda_1$ is not from the sequence (λ_k), was covered in the proof of Case 2, so we assume that $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$. Take $\delta > 0$ so small that the conclusions of Proposition 9 hold with $\lambda_k + C\delta^{N-sp} < \lambda_{k+1}$, in particular, $\Psi(w) < \lambda_{k+1}$ for all $w \in E_\delta$, and take $A_0 = E_\delta$, $B_0 = \Psi_{\lambda_{k+1}}$ and $v_0 = \pi(u_{\varepsilon,\delta}) \in \mathscr{M} \backslash E_\delta$ as in the last

subsection. Then let $0<\varepsilon\leqslant\delta/2$, $R>r>0$, and let A, B and X be as in Theorem 2.1. As before, $\inf I(B)>0$ if r is sufficiently small and

$$I(R\pi((1-t)w+tv_0))\leqslant 0,\qquad \forall w\in E_\delta,\qquad 0\leqslant t\leqslant 1$$

if R is sufficiently large. On the other hand,

$$\begin{aligned} I(tw) &\leqslant \frac{t^p}{p}\left(1-\frac{\lambda}{\Psi(w)}\right)\\ &\leqslant CR^p\delta^{N-sp},\qquad \forall w\in E_\delta,\qquad 0\leqslant t\leqslant R \end{aligned}$$

by (3.12), where C denotes a generic positive constant independent of ε and δ. It follows that

$$\begin{aligned} \sup I(A) &\leqslant CR^p\delta^{N-sp}\\ &< \inf I(B) \end{aligned}$$

if δ is sufficiently small. As in the last proof, it only remains to verify that (see (3.16))

$$\sup_{(w,t)\in E_\delta\times[0,1]}\frac{\|(1-t)w+tv_0\|^p-\lambda_k|(1-t)w+tv_0|_p^p}{\left(\int_\Omega\frac{|(1-t)w+tv_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}}<\mu_\alpha \tag{3.21}$$

if ε and δ are suitably small. We estimate the integral I_3 in (3.17) using the elementary inequality

$$|a+b|^p\leqslant|a|^p+|b|^p+C(|a|^{p-1}|b|+|a||b|^{p-1}),\qquad \forall a,b\in\mathbb{R} \tag{3.22}$$

to get

$$\begin{aligned} I_3 &\leqslant (1-t)^p\int_{A_3}\frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}dxdy+t^p\int_{A_3}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\quad+C(1-t)^{p-1}\int_{A_3}\frac{|w(x)|^{p-1}v_0(y)}{|x-y|^{N+ps}}dxdy+C(1-t)\int_{A_3}\frac{|w(x)|v_0(y)^{p-1}}{|x-y|^{N+ps}}dxdy\\ &=:(1-t)^pI_4+t^pI_5+C(1-t)^{p-1}J_1+C(1-t)J_{p-1}. \end{aligned} \tag{3.23}$$

As before, $I_1+2I_4\leqslant 1$, $I_2+2I_5\leqslant 1$ and for $q=1,p-1$,

$$\begin{aligned} J_q &:=\int_{A_3}\frac{|w(x)|^{p-q}v_0(y)^q}{|x-y|^{N+ps}}dxdy\\ &\leqslant C\int_{A_3}\frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+ps}}dxdy\\ &\leqslant\frac{C}{\delta^{sp}}\int_{B_{\theta\delta}(0)}U_\varepsilon(y)^q dy \end{aligned}$$

$$\leqslant \frac{C\varepsilon^{N-(N-sp)q/p}}{\delta^{sp}} \int_{B_{\theta\delta/\varepsilon}(0)} U(y)^q dy.$$

We take $\delta = \varepsilon^\beta$ with $\beta \in (0,1)$ and use (2.4) to estimate the last integral to get

$$J_q \leqslant C\varepsilon^{(N-sp)[p(p-q-1)\beta+q]/p(p-1)}.$$

So combining (3.17) and (3.23) gives

$$\|(1-t)w+tv_0\|^p \leqslant (1-t)^p + t^p + \widetilde{J}_1 + \widetilde{J}_{p-1}, \tag{3.24}$$

where

$$\widetilde{J}_q := C(1-t)^{p-q} J_q \leqslant C(1-t)^{p-q}\varepsilon^{(N-sp)[p(p-q-1)\beta+q]/p(p-1)}.$$

Young inequality then gives

$$\widetilde{J}_q \leqslant \frac{\kappa}{3}(1-t)^{p^*_{s,\alpha}} + C\varepsilon^{sp+\Gamma_q(\beta)}\kappa^{-\gamma_q} \tag{3.25}$$

for any $\kappa > 0$, where

$$\Gamma_q(\beta) = \frac{[(N-sp^2)(N-\alpha)-sp(sp-\alpha)](p-q)(p-1)-(N-\alpha)p(N-sp)(p-q-1)(\beta_0-\beta)}{[(sp-\alpha)p+(N-sp)q](p-1)},$$

and

$$\beta_0 = \frac{N-sp^2}{N-sp}, \qquad \gamma_q = \frac{(N-sp)(p-q)}{(N-\alpha)p-(N-sp)(p-q)}.$$

Then we have

$$\begin{aligned}\|(1-t)w+tv_0\|^p \leqslant (1-t)^p + t^p + \frac{2\kappa}{3}(1-t)^{p^*_{s,\alpha}} \\ + C\varepsilon^{sp}\left(\frac{\varepsilon^{\Gamma_1(\beta)}}{\kappa^{\gamma_1}} + \frac{\varepsilon^{\Gamma_{p-1}(\beta)}}{\kappa^{\gamma_{p-1}}}\right)\end{aligned} \tag{3.26}$$

by (3.24) and (3.25). Using $(N-sp^2)(N-\alpha) > sp(sp-\alpha)$, we fix $\beta < \beta_0$ so close to β_0 that $\Gamma_q(\beta) > 0$ for $q = 0, 1, p-1, p$. By (3.12) and Young inequality,

$$\begin{aligned}\lambda_k(1-t)^p|w|_p^p &\geqslant (1-t)^p\left(1 - C\varepsilon^{(N-sp)\beta}\right) \\ &\geqslant (1-t)^p - \frac{\kappa}{3}(1-t)^{p^*_{s,\alpha}} - C\varepsilon^{sp+\Gamma_0(\beta)}\kappa^{-\gamma_0}.\end{aligned} \tag{3.27}$$

By (3.26), (3.20) and (3.27), the quotient $Q(w,t)$ in (3.21) satisfies

$$Q(w,t) \leqslant \frac{(1-\lambda_k|v_0|_p^p)t^p + \kappa(1-t)^{p^*_{s,\alpha}} + C\varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma}}{\left[(1-t)^{p^*_{s,\alpha}}\int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha}dx + t^{p^*_{s,\alpha}}\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right]^{p/p^*_{s,\alpha}}}, \tag{3.28}$$

where

$$\Gamma(\beta) = \min\{\Gamma_0(\beta), \Gamma_1(\beta), \Gamma_{p-1}(\beta)\} > 0,$$
$$\gamma = \max\{\gamma_0, \gamma_1, \gamma_{p-1}\} = \frac{N-sp}{sp-\alpha}.$$

As before, the denominator is bounded away from zero if ε is sufficiently small, so it follows that

$$\begin{aligned}\sup_{(w,t)\in E_\delta\times[0,1]} Q(w,t) &\leqslant C(t_0^p + \kappa + \varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma}) \\ &< \mu_\alpha\end{aligned}$$

for some $t_0 > 0$ if κ and ε are sufficiently small. For $t \geqslant t_0$, rewriting the right-hand side of (3.28) as

$$\frac{\dfrac{1-\lambda_k|v_0|_p^p}{\left(\displaystyle\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}} + \dfrac{\kappa(1-t)^{p^*_{s,\alpha}} + C\varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma}}{t^p\left(\displaystyle\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx\right)^{p/p^*_{s,\alpha}}}}{\left[(1-t)^{p^*_{s,\alpha}}\dfrac{\displaystyle\int_\Omega \frac{|w|^{p^*_{s,\alpha}}}{|x|^\alpha}dx}{t^{p^*_{s,\alpha}}\displaystyle\int_\Omega \frac{|v_0|^{p^*_{s,\alpha}}}{|x|^\alpha}dx} + 1\right]^{p/p^*_{s,\alpha}}}$$

gives $Q(w,t) \leqslant g((1-t)^{p^*_{s,\alpha}})$, where

$$g(\tau) = \frac{\mu_{\varepsilon,\varepsilon^\beta}(\lambda_k) + C(\kappa\tau + \varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma})}{(1+C^{-1}\tau)^{p/p^*_{s,\alpha}}}, \qquad C = C(N,p,s,t_0,\alpha).$$

Since $0 \leqslant (1-t)^{p^*_{s,\alpha}} < 1$, then

$$Q(w,t) \leqslant \mu_{\varepsilon,\varepsilon^\beta}(\lambda_k) + C(\kappa + \varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma}).$$

If $\mu_{\varepsilon,\varepsilon^\beta}(\lambda_k) < \mu_s/2$ for some sequence $\varepsilon_j \to 0$, then the right-hand side is less than μ_s for sufficiently small κ and $\varepsilon = \varepsilon_j$ with sufficiently large j, so we may assume that $\mu_{\varepsilon,\varepsilon^\beta}(\lambda_k) \geqslant \mu_s/2$ for all sufficiently small ε. Then it is easily seen that if

$$\kappa < \frac{p\mu_s}{2p^*_{s,\alpha}(C+1)},$$

then $g'(\tau) \leqslant 0$ for all $\tau \in [0,1]$ and hence the maximum of $g((1-t)^{p^*_{s,\alpha}})$ on $[t_0,1]$ occurs at $t=1$. So, we reach

$$\begin{aligned}Q(w,t) &\leqslant \mu_{\varepsilon,\varepsilon^\beta}(\lambda_k) + C\varepsilon^{sp+\Gamma(\beta)}\kappa^{-\gamma} \\ &\leqslant \mu_s - \left(\frac{\lambda_k}{C} - C\varepsilon^{\Gamma_p(\beta)} - C\varepsilon^{\Gamma(\beta)}\kappa^{-\gamma}\right)\varepsilon^{sp}\end{aligned}$$

by (2.11), then the desired conclusion follows for sufficiently small κ and ε.

3.4. Case 4: $N(N-sp^2)(N-\alpha) > (N-sp)sp(sp-\alpha)$, $\partial\Omega \in C^{1,1}$, and $\lambda = \lambda_k$

By Iannizzotto, Mosconi and Squassina [14, Theorem 4.4], there exists a constant $C = C(N,\Omega,p,s) > 0$ such that for any $v \in X_p^s(\Omega)$ with $(-\Delta)_p^s v \in L^\infty(\Omega)$,

$$|v(x)| \leqslant C\left|(-\Delta)_p^s v\right|_\infty^{1/(p-1)} d^s(x), \qquad \forall x \in \mathbb{R}^N, \tag{3.29}$$

where $d(x) = dist(x, \mathbb{R}^N \backslash \Omega)$.

LEMMA 10. [17] *Assume that $\partial\Omega \in C^{1,1}$. Given $\alpha,\beta > 1$, there exists a constant $C = C(N,\Omega,p,s,\alpha,\beta) > 0$ such that if $B_{\beta r}(0) \subset \{x \in \Omega : d(x) < \alpha r\}$, then for any $v \in X_p^s(\Omega)$ with $(-\Delta)_p^s v \in L^\infty(\Omega)$,*

$$|v(x) - v(y)| \leqslant C\left|(-\Delta)_p^s v\right|_\infty^{1/(p-1)} |x-y|^s, \qquad \forall x \in B_r(0), \qquad \forall y \in \Omega \backslash B_{\beta r}(0).$$

Let η_δ be as the cut-off function in Lemma 3.

LEMMA 11. [17] *Assume that $\partial\Omega \in C^{1,1}$. Then there exists a constant $C = C(N,\Omega,p,s) > 0$ such that for any $v \in X_p^s(\Omega)$ such that $(-\Delta)_p^s v \in L^\infty(\Omega)$ and $\delta > 0$ such that $B_{6\theta\delta}(0) \subset \{x \in \Omega : d(x) < 12\theta\delta\}$,*

$$\|v\eta_\delta\|^p \leqslant \|v\|^p + C\left|(-\Delta)_p^s v\right|_\infty^{p/(p-1)} \delta^N. \tag{3.30}$$

Since $\partial\Omega \in C^{1,1}$, for all sufficiently small $\delta > 0$, the ball $B_{6\theta\delta}(0)$ is contained in $\{x \in \Omega : d(x) < 12\theta\delta\}$ after a translation. Then by Lemma 11 and Proposition 8,

$$\|v_\delta\|^p \leqslant 1 + C\delta^N, \qquad v \in E,$$

and using this inequality in the proof of Proposition 9 (see [17, Proposition 3.2]) shows that (3.12) can now be strengthened to

$$\sup_{w \in E_\delta} \Psi(w) \leqslant \lambda_k + C\delta^N. \tag{3.31}$$

Proceeding as in the last subsection, we have to verify (3.21) for suitably small ε and δ. Since the argument is similar, we only point out where it differs. Let $v \in E$ and let

$$w = \pi(v_\delta) = v_\delta / \|v_\delta\|.$$

As noted in the proof of Proposition 9, $\|v_\delta\|$ is bounded away from zero, so

$$J_q \leqslant C \int_{A_3} \frac{|v_\delta(x)|^{p-q} u_{\varepsilon,\delta}(y)^q}{|x-y|^{N+ps}} dxdy,$$

where $A_3 = B_{2\theta\delta}(0)^c \times B_{\theta\delta}(0)$. By Lemma 10, (3.29) and Proposition 9, moreover, since

$$|x-y| \geqslant |x|/2$$

$$\geqslant \theta\delta \qquad \text{on } A_3,$$

we get

$$\begin{aligned}|v_\delta(x)|^{p-q} &\leqslant |v(x)|^{p-q}\\ &\leqslant C(|v(x)-v(y)|^{p-q}+|v(y)|^{p-q})\\ &\leqslant C(|x-y|^{s(p-q)}+\delta^{s(p-q)})\\ &\leqslant C|x-y|^{s(p-q)},\end{aligned}$$

so

$$\begin{aligned}J_q &\leqslant C\int_{A_3}\frac{u_{\varepsilon,\delta}(y)^q}{|x|^{N+sq}}dxdy\\ &\leqslant \frac{C}{\delta^{sq}}\int_{B_{\theta\delta}(0)}U_\varepsilon(y)^q dy\\ &\leqslant C\varepsilon^{\{p[(p-q-1)N+sq]\beta+(N-sp)q\}/p(p-1)}.\end{aligned}$$

Then (3.25) holds with

$$\Gamma_q(\beta)=\frac{[N(N-sp^2)(N-\alpha)-(N-sp)sp(sp-\alpha)](p-q)(p-1)-(N-\alpha)p(N-sp)[(p-q-1)N+sq](\beta_0-\beta)}{[(sp-\alpha)p+(N-sp)q](p-1)(N-sp)},$$

so does (3.27) by (3.31). Using

$$N(N-sp^2)(N-\alpha)>(N-sp)sp(sp-\alpha),$$

we fix $\beta<\beta_0$ so close to β_0 that $\Gamma_q(\beta)>0$ for $q=0,1,p-1,p$ and proceed as before.

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Here we only give the proof of Theorem 1.2 (b), since the proof of (a) is similar and simpler. By Proposition 5, I_T satisfies the $(PS)_c$ condition for all

$$c<\frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)},$$

so we apply Theorem 2.2 with

$$b=\frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)}.$$

By Perera, Squassina and Yang [20, Proposition 3.5], the sublevel set $\Psi^{\lambda_{k+m}}$ has a compact symmetric subset A_0 with

$$i(A_0)=k+m.$$

We take $B_0 = \Psi_{\lambda_{k+1}}$, so that

$$i(S_1 \backslash B_0) = k$$

by (1.7). Set $0 < r < R$ and let A, B and X be as in Theorem 2.2. For $u \in B_0$,

$$I(ru) \geqslant \frac{r^p}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{r^{p^*_{s,\alpha}}}{p^*_{s,\alpha}\mu_\alpha^{p^*_{s,\alpha}/p}}$$

by (1.5). Since $\lambda < \lambda_{k+1}$ and $p^*_{s,\alpha} > p$, it follows that $\inf I_T(B) > 0$ if r is sufficiently small. For $u \in A_0 \subset \Psi^{\lambda_{k+m}}$,

$$I(Ru) \leqslant \frac{R^p}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{R^{p^*_{s,\alpha}}}{p^*_{s,\alpha}V_\alpha(\Omega)^{(sp-\alpha)/(N-sp)}\lambda_{k+1}^{p^*_{s,\alpha}/p}}$$

by (1.8), so there exists $R > r$ such that $I \leqslant 0$ on A. For $u \in X$,

$$\begin{aligned} I(u) &\leqslant \frac{\lambda_{k+1} - \lambda}{p}\int_\Omega |u|^p dx - \frac{1}{p^*_{s,\alpha}V_\alpha(\Omega)^{(sp-\alpha)/(N-sp)}}\left(\int_\Omega |u|^p dx\right)^{p^*_{s,\alpha}/p} \\ &\leqslant \sup_{\rho \geqslant 0}\left[\frac{\lambda_{k+1} - \lambda}{p}\rho - \frac{\rho^{p^*_{s,\alpha}/p}}{p^*_{s,\alpha}V_\alpha(\Omega)^{(sp-\alpha)/(N-sp)}}\right] \\ &= \frac{sp-\alpha}{(N-\alpha)p}V_\alpha(\Omega)(\lambda_{k+1} - \lambda)^{(N-\alpha)/(sp-\alpha)}. \end{aligned}$$

So

$$\begin{aligned} \sup I(X) &\leqslant \frac{sp-\alpha}{(N-\alpha)p}V_\alpha(\Omega)(\lambda_{k+1} - \lambda)^{(N-\alpha)/(sp-\alpha)} \\ &< \frac{sp-\alpha}{(N-\alpha)p}\mu_\alpha^{(N-\alpha)/(sp-\alpha)} \end{aligned}$$

by (1.9). Therefore, Theorem 2.2 gives now m distinct pairs of (nontrivial) critical points $\pm u_j^\lambda$, $j = 1, \cdots, m$ of I such that

$$0 < I(u_j^\lambda) \leqslant \frac{sp-\alpha}{(N-\alpha)p}V_\alpha(\Omega)(\lambda_{k+1} - \lambda)^{(N-\alpha)/(sp-\alpha)} \to 0 \qquad as\ \lambda \nearrow \lambda_{k+1}. \tag{4.1}$$

Then

$$\begin{aligned} \int_\Omega \frac{|u_j^\lambda|^{p^*_{s,\alpha}}}{|x|^\alpha}dx &= \frac{(N-\alpha)p}{sp-\alpha}\left[I(u_j^\lambda) - \frac{1}{p}I'(u_j^\lambda)u_j^\lambda\right] \\ &= \frac{(N-\alpha)p}{sp-\alpha}I(u_j^\lambda) \to 0 \end{aligned}$$

and hence $u_j^\lambda \to 0$ in $L^p(\Omega)$ also by (1.8), so

$$\|u_j^\lambda\|^p = pI(u_j^\lambda) + \lambda|u_j^\lambda|_p^p + \frac{p}{p^*_{s,\alpha}}\int_\Omega \frac{|u_j^\lambda|^{p^*_{s,\alpha}}}{|x|^\alpha}dx \to 0.$$

The proof of Theorem 1.2 is completed. □

REFERENCES

[1] A. AMBROSETTI, M. STRUWE, *A note on the problem* $-\Delta u = \lambda u + |u|^{2^*-2}u$, Manuscr. Math., **54**, 4 (1986), 373–379.

[2] G.M. BISCI, *Fractional equations with bounded primitive*, Appl. Math. Lett., **27**, (2014), 53–58.

[3] V. BENCI, *On critical point theory for indefinite functionals in the presence of symmetrics*, Trans. Amer. Math. Soc., **274**, 2 (1982), 533–572.

[4] H. BRÉZIS, L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36**, 4 (1983), 437–477.

[5] L. BRASCO, E. PARINI, *The second eigenvalue of the fractional* p*-Laplacian*, Adv. Calc. Var., **9**, 4 (2016), 323–355.

[6] W. CHEN, C. LI, B. OU, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., **59**, 3 (2006), 330–343.

[7] P. DRÁBEK, Y.X. HUANG, *Multiplicity of positive solutions for some quasilinear elliptic equation in* $\mathbb{R}^N$ *with critical Sobolev exponent*, J. Differential Equations, **140**, 1 (1997), 106–132.

[8] M. FERRARA, G.M. BISCI, B.L. ZHANG, *Existence of weak solutions for non-local fractional problems via Morse theory*, Discrete Contin. Dyn. Syst. Ser. B, **19**, 8 (2014), 2483–2499.

[9] G. FRANZINA, G. PALATUCCI, *Fractional* p*-eigenvalues*, Riv. Mat. Univ. Parma, **2**, 5 (2014), 315–328.

[10] E.R. FADELL, P.H. RABINOWITZ, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math., **45**, 2 (1978), 139–174.

[11] J.V. GONCALVES, C.O. ALVES, *Existence of positive solutions for* m*-Laplacian equations in* $\mathbb{R}^N$ *involving critical Sobolev exponents*, Nonlinear Anal., **32**, 1 (1998), 53–70.

[12] N. GHOUSSOUB, C. YUAN, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc., **352**, 12 (2000), 5703–5743.

[13] A. IANNIZZOTTO, S. LIU, K. PERERA, M. SQUSSSINA, *Existence results for fractional* p*-Laplacian problems via Morse theory*, Adv. Calc. Var., **9**, 2 (2016), 101–125.

[14] A. IANNIZZOTTO, S. MOSCONI, M. SQUASSINA, *Global Hölder regularity for the fractional* p*-Laplacian*, Rev. Mat. Iberoam., **32**, 4 (2016), 1353–1392.

[15] A. IANNIZZOTTO, M. SQUASSINA, *Weyl-type laws for fractional* p*-eigenvalue problems*, Asymptot. Anal., **88**, (2014), 233–245.

[16] S. MARANO, S. MOSCONI, *Asymptotic for optimizers of the fractional Hardy-Sobolev inequality*, arXiv preprint, (2016), arXiv:1609.01869.

[17] S. MOSCONI, K. PERERA, M. SQUASSINA AND Y. YANG, *The Brezis-Nirenberg problem for the fractional* p*-Laplacian*, Calc. Var. Partial Differential Equations, **55**, 4 (2016), 105.

[18] S.H. LIANG, J.H. ZHANG, *Multiplicity of solutions for the noncooperative Schrödinger-Kirchhoff system involving the fractional* p*-Laplacian in* $\mathbb{R}^N$, Z. Angew. Math. Phys., **68**, 3 (2017), 1–18.

[19] K. PERERA, *Nontrivial critical groups in* p*-Laplacian problems via the Yang index*, Topol. Methods Nonlinear Anal., **21**, 2 (2003), 301–309.

[20] K. PERERA, M. SQUASSINA, Y. YANG, *Bifurcation and multiplicity results for critical fractional* p*-Laplacian problems*, Math. Nachr., **289**, 2-3 (2016), 332–342.

[21] K. PERERA, W. ZOU, p*-Laplacian problems involving critical Hardy-Sobolev exponents*, arXiv preprint, (2016), arXiv: 1609.01804.

[22] X. ROS-OSTON, J. SERRA, *Nonexistence results for nonlocal equations with critical and supercritical nonlinearities*, Comm. Partial Differential Equations, **40**, 1 (2015), 115–133.

[23] X. ROS-OTON, J. SERRA, *Fractional Laplacian: Pohozaev identity and nonexistence results*, R. Math. Acad. Sci., **350**, 9-10 (2012), 505–508.

[24] R. SERVADEI, *The Yamabe equation in a non-local setting*, Adv. Nonlinear Anal., **2**, 3 (2013), 235–270.

[25] R. SERVADEI, *A critical fractional Laplace equation in the resonant case*, Topol. Methods Nonlinear Anal., **43**, 1 (2014), 251–267.

[26] E.A. SILVA, S.H. SOARES, *Quasilinear Dirichlet problems in* $\mathbb{R}^N$ *with critical growth*, Nonlinear Anal., **43**, 1 (2001), 1–20.

[27] R. SERVADEI, E. VALDINOCI, *The Brézis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc., **367**, 1 (2015), 67–102.

[28] R. SERVADEI, E. VALDINOCI, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal., **12**, 6 (2013), 2445–2464.
[29] R. SERVADEI, E. VALDINOCI, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc., **367**, 1 (2005), 67–102.
[30] E.A. SILVA, M.S. XAVIER, *Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **20**, 2 (2003), 341–358.
[31] Y. YANG, K. PERERA, *N-Laplacian problems with critical Trudinger-Moser nonlinearities*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **XVI**, 5 (2016), 1123–1138.

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