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BOX–COUNTING DIMENSION OF OSCILLATORY SOLUTIONS TO THE EMDEN–FOWLER EQUATION

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Abstract. The box-counting dimension of graphs of oscillatory solutions to the Emden-Fowler equation is studied. The half-linear equation is also considered.

1. Introduction

In this paper, we study the box-counting dimension of graphs of the oscillatory solutions to the Emden-Fowler equation

$$y'' + f(x)|y|^{\gamma-1}y = 0, \quad x \in (0,x_0], \tag{1.1}$$

where $\gamma > 0$, $\gamma \neq 1$ and

$$f \in C^2(0,x_0],\ f(x) > 0,\ f'(x) < 0,\ x \in (0,x_0],\ \lim_{x\to+0} f(x) = \infty. \tag{1.2}$$

The main result of the paper is stated in Theorem 1.1 below. A function $\lambda x^{-\sigma}$, $\lambda > 0$, $\sigma > 0$ is a typical example of $f(x)$ satisfying (1.2).

A solution y of (1.1) is said to be *oscillatory near* $x = 0$ if there exists $\{z_n\}_{n=1}^{\infty}$ such that $y(z_n) = 0$ for $n \in \mathbf{N}$ and $z_n \to 0$ as $n \to \infty$. Otherwise, it is said to be *nonoscillatory near* $x = 0$.

A study of the oscillatory solutions to the Emden-Fowler equation (1.1) has a long history. See, for example, [1, 2, 6, 7, 8, 10, 17]. For instance, the following result is well-known. (Consider the transformation $Y(t) = ty(t^{-1})$ on (1.1) and apply the celebrated results in [1] and [2].)

THEOREM A. *Let $\gamma > 0$, $\gamma \neq 1$, $f \in C(0,x_0]$ and $f(x) > 0$ for $x \in (0,x_0]$. Then every solution $y \in C^2(0,x_0]$ of (1.1) is oscillatory near $x = 0$ if and only if $x^{\gamma^*} f(x) \notin L^1(0,x_0]$, where $\gamma^* = \max\{\gamma, 1\}$.*

J. S. W. Wong [18] studied the rectifiability of oscillatory solutions to (1.1). A solution y of (1.1) is said to be *rectifiable* (resp. *nonrectifiable*) *oscillatory near* $x = 0$ if y is oscillatory near $x = 0$ and the length of y is finite (resp. infinite), that is,

$$\int_0^{x_0} \sqrt{1 + |y'(x)|^2}\,dx < \infty \quad (\text{resp. } = \infty).$$

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THEOREM B ([18, Theorem 1]). *Let $\gamma > 0$ and $\gamma \neq 1$. Assume that $f \in C^2(0,x_0]$, $f(x) > 0$ for $x \in (0,x_0]$, $\lim_{x\to+0} f(x) = \infty$, $f^{-1}f'_+ \in L^1(0,x_0]$ and $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')' \in L^1(0,x_0]$, where $f'_+(x) = \max\{f'(x),0\}$ and $\gamma^* = \max\{\gamma,1\}$. Then every nontrivial solution y of (1.1) is oscillatory near $x=0$ and the following properties hold*:

(i) *y is rectifiable if $f^{1/(\gamma+3)} \in L^1(0,x_0]$;*

(ii) *y is nonrectifiable if $f^{1/(\gamma+3)} \notin L^1(0,x_0]$.*

Proposition 1.1 below shows that the box-counting dimension of rectifiable curves is 1. In this paper, we will obtain the box-counting dimension of nonrectifiable oscillatory solutions to (1.1). For linear differential equations, it has been studied by Pašić [11], Kwong, Pašić and Wong [9], and Pašić and Tanaka [12]. However, to the authors' knowledge, there is no study about it for the Emden-Fowler equations.

Let $\Gamma \subset \mathbf{R}^2$ be a bounded set. We define the *box-counting dimension* (*Minkowski-Bouligand dimension*) of Γ by

$$\dim_{\mathrm{B}} \Gamma = 2 - \lim_{\varepsilon\to+0} \frac{\log|\Gamma_\varepsilon|}{\log\varepsilon},$$

provided the limit exists, where Γ_ε denotes the ε-neighborhood of Γ defined by

$$\Gamma_\varepsilon = \{(x,y) \in \mathbf{R}^2 : d((x,y),\Gamma) \leqslant \varepsilon\},$$

$d((x,y),\Gamma)$ denotes the Euclidean distance from (x,y) to Γ, and $|\Gamma_\varepsilon|$ denotes the two-dimensional Lebesgue measure of Γ_ε. For any $d \geqslant 0$, the *lower d-dimensional Minkowski content* of Γ and the *upper d-dimensional Minkowski content* of Γ are defined by

$$\mathscr{M}_*^d(\Gamma) := \liminf_{\varepsilon\to+0} \varepsilon^{-(2-d)}|\Gamma_\varepsilon| \quad \text{and} \quad \mathscr{M}^{*d}(\Gamma) := \limsup_{\varepsilon\to+0} \varepsilon^{-(2-d)}|\Gamma_\varepsilon|,$$

respectively. If $\mathscr{M}_*^d(\Gamma) = \mathscr{M}^{*d}(\Gamma)$, then it is said to be the *d-dimensional Minkowski content* of Γ and denoted by $\mathscr{M}^d(\Gamma)$. More details on the above definitions can be found in Falconer [5] and Tricot [16]. It is easy to see that if $0 < \mathscr{M}_*^d(\Gamma) \leqslant \mathscr{M}^{*d}(\Gamma) < \infty$, then $\dim_{\mathrm{B}} \Gamma = d$. We have the following well-known result, see e.g. Tricot [16, §9.1, Theorem].

PROPOSITION 1.1. *Let $\Gamma \subset \mathbf{R}^2$ be a simple curve of finite length. Then*

$$\lim_{\varepsilon\to+0} \frac{|\Gamma_\varepsilon|}{2\varepsilon} = \mathrm{length}(\Gamma),$$

where $\mathrm{length}(\Gamma)$ *denotes the length of* Γ.

Therefore, the box-counting dimension of rectifiable curves is 1.

For each $y \in C(0,x_0]$, we define the graph $\Gamma(y)$ of y by

$$\Gamma(y) = \{(x,y(x)) : 0 < x \leqslant x_0\}.$$

The main result of this paper is as follows.

THEOREM 1.1. *Let $\gamma > 0$ and $\gamma \neq 1$. Assume that (1.2) holds, $(f^{-(\gamma^*+2)/(\gamma^*+1)} f')' \in L^1(0,x_0]$ and $f^{1/(\gamma+3)} \notin L^1(0,x_0]$, where $\gamma^* = \max\{\gamma, 1\}$. Assume, moreover, that there exists $d \in (1,2)$ such that*

$$\liminf_{x\to+0} [f(x)]^{\frac{2(2-d)}{\gamma+3}} \int_0^x [f(\xi)]^{-\frac{1}{\gamma+3}} d\xi > 0,$$

$$\limsup_{x\to+0} \left[x[f(x)]^{-\frac{1}{\gamma+3}}\right]^{\frac{d-1}{2-d}} \int_x^{x_0} [f(\xi)]^{\frac{1}{\gamma+3}} d\xi < \infty.$$

Then every nontrivial solution y of (1.1) is oscillatory near $x=0$ and $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and hence $\dim_{\mathrm{B}} \Gamma(y) = d$.

We will prove Theorem 1.1 in Section 3.

Applying Theorem 1.1 to

$$y'' + \lambda x^{-\sigma} |y|^{\gamma-1} y = 0, \quad x \in (0,x_0], \tag{1.3}$$

we have the following corollary.

COROLLARY 1.1. *Let $\lambda > 0$, $\gamma > 0$, $\gamma \neq 1$ and $\sigma > \gamma+3$. Then every nontrivial solution y of (1.3) is oscillatory near $x=0$, $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and $\dim_{\mathrm{B}} \Gamma(y) = d$, where $d = \frac{3}{2} - \frac{\gamma+3}{2\sigma}$.*

REMARK 1.1. Corollary 1.1 answers the open problem raised by J. S. W. Wong [18]. Corollary 1.1 with $\gamma = 1$ was obtained by Pašić [11, Theorem 1.5]. From Theorem A, it follows that if $\sigma \leqslant \gamma^* + 1$, then (1.3) has a nonoscillatory solution near $x = 0$. Theorem B implies that if $\sigma > \gamma^* + 1$, then every nontrivial solution y of (1.3) is oscillatory near $x = 0$ and we have: y is rectifiable when $\gamma^* + 1 < \sigma < \gamma + 3$; y is nonrectifiable when $\sigma \geqslant \gamma + 3$. (See also [18, Corollary].) Proposition 1.1 shows that $\dim_{\mathrm{B}} \Gamma(y) = 1$ when $\gamma^* + 1 < \sigma < \gamma + 3$. Consequently, we do not know the box-counting dimension of oscillatory solutions for the case where $\sigma = \gamma + 3$. We formulate the conjecture as follows. We note that Pašić [11, Theorem 1.5] proved that $\dim_{\mathrm{B}} \Gamma(y) = 1$ for every nontrivial solution y of (1.3) with $\gamma = 1$ and $\sigma = \gamma + 3 = 4$.

CONJECTURE 1.1. *If $\lambda > 0$, $\gamma > 0$, $\gamma \neq 1$ and $\sigma = \gamma + 3$, then $\dim_{\mathrm{B}} \Gamma(y) = 1$ for every nontrivial solution y of (1.3).*

J. S. W. Wong [18] gave the asymptotic behavior of oscillatory solutions to (1.1) as the following form

$$y(x) = [f(x)]^{-\frac{1}{\gamma+3}} [V(x)]^{\frac{1}{\gamma+1}} S(\varphi(x)),$$

$$y'(x) = -[f(x)]^{\frac{1}{\gamma+3}} [V(x)]^{\frac{1}{2}} S(\varphi(x)),$$

where $V \in C^1(0,x_0]$ satisfies $\lim_{x\to+0} V(x) = V_0$ for some $V_0 > 0$ and $S \in C(\mathbf{R})$ is a periodic function having zeros. See Proposition 3.1 below. In this paper, we give a sufficient condition such that $\mathscr{M}_*^d(\Gamma(y)) > 0$ for each chirp-like function

$$y(x) = a(x)v(x)S(\varphi(x)), \quad x \in (0,x_0], \tag{1.4}$$

where $a \in C^1(0,x_0]$, $v \in C(0,x_0]$, $\varphi \in C^2(0,x_0]$ and $S \in C^1(\mathbf{R})$ satisfy

$$a(x) > 0,\ a'(x) \geqslant 0,\ x \in (0,x_0], \tag{1.5}$$

$$\lim_{x\to+0} v(x) = v_0 \quad \text{for some } v_0 > 0, \quad v(x) > 0,\ x \in (0,x_0], \tag{1.6}$$

$$\lim_{x\to+0} \varphi(x) = \infty,\ \varphi(x) > 0 \ \text{ and } \ \varphi'(x) < 0,\ x \in (0,x_0], \tag{1.7}$$

$$|S(t+\tau)| = |S(t)|,\ t \in \mathbf{R} \quad \text{for some } \tau > 0, \tag{1.8}$$

$$S(\tau_0) = 0,\ S(t) \neq 0,\ t \in (\tau_0, \tau_0+\tau) \quad \text{for some } \tau_0 \in \mathbf{R}. \tag{1.9}$$

Namely, we will prove the following result.

THEOREM 1.2. *Assume that* $a \in C^1(0,x_0]$, $v \in C(0,x_0]$ $\varphi \in C^1(0,x_0]$ *and* $S \in C^1(\mathbf{R})$ *satisfy* (1.5)–(1.9), *and that there exists* $\Phi \in C^1(0,x_0]$ *such that*

$$0 < \liminf_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} \leqslant \limsup_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} < \infty, \tag{1.10}$$

$$\Phi(x) > 0,\ \Phi'(x) < 0,\ x \in (0,x_0], \quad \limsup_{x\to+0} \left([\Phi(x)]^{-1}\right)' < \infty. \tag{1.11}$$

Assume, moreover, that there exists $d \in (1,2)$ *such that*

$$\liminf_{x\to+0} [\Phi(x)]^{2-d} \int_0^x a(\xi)d\xi > 0. \tag{1.12}$$

Then $\mathscr{M}_*^d(\Gamma(y)) > 0$ *for each chirp-like function* (1.4).

Theorem 1.2 with $v(x) \equiv 1$ and $\Phi(x) \equiv -\varphi'(x)$ has been obtained in [13]. The proof of Theorem 1.2 will be given in Section 2.

Using the result in [13, Lemma 28], we can obtain the following consequence, which will be shown in Section 2.

PROPOSITION 1.2. *Let* $y \in C^1(0,x_0]$ *be bounded on* $(0,x_0]$. *Assume that*

$$\limsup_{x\to+0} \left[x \sup_{\xi\in(0,x]} |y(\xi)|\right]^{\frac{d-1}{2-d}} \int_x^{x_0} |y'(\xi)|d\xi < \infty \tag{1.13}$$

for some $d \in (1,2)$. *Then* $\mathscr{M}^{*d}(\Gamma(y)) < \infty$.

Combining these results, we can obtain Theorem 1.1.

Next we consider the half-linear equation

$$(|y'|^{p-2}y')' + f(x)|y|^{p-2}y = 0, \quad x \in (0,x_0], \tag{1.14}$$

where $p > 1$ and f satisfies (1.2). There are a lot of oscillatory results for (1.14). See, for example, Došlý and P. Řehák [3]. Pašić and J. S. W. Wong [14] established the following result.

THEOREM C ([14, Theorem 6]). *Let $p > 1$. Assume that (1.2) holds and*

$$f^{-\theta}(f^{\theta-(1/p)})'' \in L^1(0,x_0] \quad \textit{for some } \theta \in (0,1/p), \tag{1.15}$$

$$0 < \liminf_{x\to+0} x^\sigma f(x) \leqslant \limsup_{x\to+0} x^\sigma f(x) < \infty \quad \textit{for some } \sigma > p.$$

Then every nontrivial solution y of (1.14) is oscillatory near $x = 0$ and the following properties hold:

(i) *if $p < \sigma < p^2$, then $0 < \mathscr{M}_*^1(\Gamma(y)) \leqslant \mathscr{M}^{*1}(\Gamma(y)) < \infty$ and $\dim_{\mathrm{B}} \Gamma(y) = 1$;*

(ii) *if $\sigma > p^2$, then $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and $\dim_{\mathrm{B}} \Gamma(y) = d$, where $d = 1 + \frac{1}{p} - \frac{p}{\sigma}$.*

Pašić and J. S. W. Wong [14] gave the asymptotic behavior of oscillatory solutions of (1.14). Therefore, in the same way as (1.1), we can obtain the box-counting dimension of graphs of oscillatory solutions as follows.

THEOREM 1.3. *Let $p > 1$. Assume that (1.2) and (1.15) hold. Assume, moreover, that there exists $d \in (1,2)$ such that*

$$\liminf_{x\to+0} [f(x)]^{\frac{2-d}{p}} \int_0^x [f(\xi)]^{-\frac{1}{pp'}} d\xi > 0,$$

$$\limsup_{x\to+0} \left[x[f(x)]^{-\frac{1}{pp'}}\right]^{\frac{d-1}{2-d}} \int_x^{x_0} [f(\xi)]^{\frac{1}{p^2}} d\xi < \infty,$$

where p' is a positive number with $\frac{1}{p} + \frac{1}{p'} = 1$, that is, $p' = p/(p-1)$. Then every nontrivial solution y of (1.14) is oscillatory near $x = 0$ and $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and hence $\dim_{\mathrm{B}} \Gamma(y) = d$.

The proof of Theorem 1.3 will be given in Section 3.

EXAMPLE 1.1. We consider the equation

$$(|y'|^{p-2}y')' + \lambda x^{-\sigma}|y|^{p-2}y = 0, \quad x \in (0,x_0], \tag{1.16}$$

where $p > 1$, $\lambda > 0$ and $\sigma \in \mathbf{R}$. Theorem 1.3 implies that if $\sigma > p^2$, then every nontrivial solution y of (1.16) is oscillatory near $x = 0$ and $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and $\dim_{\mathrm{B}} \Gamma(y) = d$, where $d = 1 + \frac{1}{p} - \frac{p}{\sigma}$.

2. Lower and upper Minkowski contents

In this section we prove Theorem 1.2. To this end, we need the following result obtained in [9, pp. 2350].

LEMMA 2.1. *Let $y \in C(0,x_0]$ be a bounded function on $(0,x_0]$ and let $a_n \in (0,x_0]$ be a decreasing sequence of consecutive zeros of $y(x)$ such that $a_n \to 0$. Assume that there exists $\varepsilon_0 > 0$ and $k : (0,\varepsilon_0) \to \mathbf{N}$ such that*

$$a_n - a_{n+1} \leqslant \varepsilon \quad \text{for all } n \geqslant k(\varepsilon) \text{ and } \varepsilon \in (0,\varepsilon_0). \tag{2.1}$$

Then

$$|\Gamma_\varepsilon(y)| \geqslant \sum_{n \geqslant k(\varepsilon)} \max_{x \in [a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}), \quad \varepsilon \in (0,\varepsilon_0). \tag{2.2}$$

Proof of Theorem 1.2. By (1.10) and (1.11), there exist $x_1 \in (0,x_0]$, $C_1 > 0$ and $C_2 > 0$ such that

$$0 < C_1\Phi(x) \leqslant -\varphi'(x) \leqslant C_2\Phi(x), \quad x \in (0,x_1]. \tag{2.3}$$

Let $y(x)$ be a chirp function given by (1.4). Set $a_n = \varphi^{-1}(\tau_0 + n\tau)$ for all sufficiently large $n \in \mathbf{N}$, where φ^{-1} is the inverse function of φ. From (1.7), it follows that a_n is strictly decreasing and $a_n \to 0$ as $n \to \infty$. Then $y(a_n) = 0$ and $y(x) \neq 0$ on (a_{n+1}, a_n) for all sufficiently large $n \in \mathbf{N}$. We take $N \in \mathbf{N}$ such that $a_N \leqslant x_1$. By the mean value theorem, for each $n \geqslant N$, there exists $c_n \in (\tau_0 + n\tau, \tau_0 + (n+1)\tau)$ such that

$$a_n - a_{n+1} = \varphi^{-1}(\tau_0 + n\tau) - \varphi^{-1}(\tau_0 + (n+1)\tau) = \frac{-\tau}{\varphi'(\varphi^{-1}(c_n))}.$$

Since $\Phi(\varphi^{-1}(t))$ is increasing, by (2.3), we observe that

$$\frac{-\tau}{\varphi'(\varphi^{-1}(c_n))} \leqslant \frac{\tau}{C_1\Phi(\varphi^{-1}(c_n))} \leqslant \frac{\tau}{C_1\Phi(\varphi^{-1}(\tau_0 + n\tau))} = \frac{\tau}{C_1\Phi(a_n)}$$

for $n \geqslant N$. In the same way, we have

$$\frac{-\tau}{\varphi'(\varphi^{-1}(c_n))} \geqslant \frac{\tau}{C_2\Phi(a_{n+1})}, \quad n \geqslant N.$$

Consequently, we obtain

$$\frac{\tau}{C_2\Phi(a_{n+1})} \leqslant a_n - a_{n+1} \leqslant \frac{\tau}{C_1\Phi(a_n)}, \quad n \geqslant N. \tag{2.4}$$

Note that $1/\Phi(a_n)$ is decreasing. Moreover, $1/\Phi(a_n) \to 0$ as $n \to \infty$. Indeed, by (2.4), we have

$$0 < \frac{1}{\Phi(a_{n+1})} \leqslant \frac{C_2}{\tau}(a_n - a_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

Set

$$\varepsilon_0 = \frac{\tau}{C_1\Phi(a_{N+1})}.$$

For each $\varepsilon \in (0,\varepsilon_0)$, let $k(\varepsilon)$ be the smallest positive integer satisfying

$$\frac{\tau}{C_1\Phi(a_{k(\varepsilon)})} \leqslant \varepsilon. \tag{2.5}$$

From (2.4), it follows that $k(\varepsilon)$ satisfies (2.1). Therefore, Lemma 2.1 implies that (2.2) holds.

By (1.11), there exists $L > 0$ such that

$$\frac{-\Phi'(x)}{[\Phi(x)]^2} = \left([\Phi(x)]^{-1}\right)' \leqslant L, \quad x \in (0,x_0]. \tag{2.6}$$

Set $\psi(t) = \log\Phi(\varphi^{-1}(t))$ for $t \geqslant \varphi(x_0)$. By the mean value theorem, for each $t \geqslant \varphi(x_1)$, there exists $c \in (t,t+2\tau)$ such that

$$\psi(t+2\tau) - \psi(t) = 2\tau\psi'(c),$$

that is,

$$\log\Phi(\varphi^{-1}(t+2\tau)) - \log\Phi(\varphi^{-1}(t)) = \frac{2\tau[-\Phi'(\varphi^{-1}(c))]}{\Phi(\varphi^{-1}(c))[-\varphi'(\varphi^{-1}(c))]}.$$

Therefore, by (2.3) and (2.6), we have

$$\begin{aligned}\log\frac{\Phi(\varphi^{-1}(t+2\tau))}{\Phi(\varphi^{-1}(t))} \leqslant \frac{2\tau[-\Phi'(\varphi^{-1}(c))]}{\Phi(\varphi^{-1}(c))[C_1\Phi(\varphi^{-1}(c))]} &= 2\tau C_1^{-1}\frac{-\Phi'(\varphi^{-1}(c))}{[\Phi(\varphi^{-1}(c))]^2}\\ &\leqslant 2\tau C_1^{-1}L,\end{aligned}$$

which implies that

$$\Phi(\varphi^{-1}(t)) \geqslant C_3\Phi(\varphi^{-1}(t+2\tau)), \quad t \geqslant \varphi(x_1), \tag{2.7}$$

where $C_3 = e^{-2\tau C_1^{-1}L}$.

By the definition of $k(\varepsilon)$ and (2.5), we find that

$$\frac{\tau}{C_1\Phi(a_{k(\varepsilon)-1})} > \varepsilon.$$

Hence, from (2.7), it follows that

$$\begin{aligned}\tau\varepsilon^{-1} > C_1\Phi(a_{k(\varepsilon)-1}) &= C_1\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)-1)\tau))\\ &\geqslant C_1C_3\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)-1)\tau + 2\tau))\\ &= C_1C_3\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)+1)\tau))\\ &= C_1C_3\Phi(a_{k(\varepsilon)+1}),\end{aligned}$$

which means that

$$\Phi(a_{k(\varepsilon)+1}) < C_4\varepsilon^{-1}, \quad \varepsilon \in (0,\varepsilon_0), \tag{2.8}$$

where $C_4 = \tau/(C_1C_3)$.

Set

$$M = \max_{t\in\mathbf{R}} |S(t)| = \max_{t\in[0,\tau]} |S(t)|.$$

We can take $v_1 > 0$ such that $v(x) \geqslant v_1$ for $x \in (0,x_0]$ in view of (1.6). Since $a(x)$ is nondecreasing on $(0,x_0]$ by (1.5), we find that

$$\begin{aligned}\max_{x\in[a_{n+1},a_n]} |y(x)| &= \max_{x\in[a_{n+1},a_n]} a(x)v(x)|S(\varphi(x))| \\ &\geqslant v_1 a(a_{n+1}) \max_{x\in[a_{n+1},a_n]} |S(\varphi(x))| = v_1 M a(a_{n+1}).\end{aligned} \tag{2.9}$$

From (2.4) and (2.9), it follows that

$$\max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) \geqslant C_5 \frac{a(a_{n+1})}{\Phi(a_{n+1})} = C_5 \frac{a\big(\varphi^{-1}(\tau_0+(n+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0+(n+1)\tau)\big)}$$

for $n \geqslant N$, where $C_5 = \tau v_1 M C_2^{-1}$. Since $a(\varphi^{-1}(t))/\Phi(\varphi^{-1}(t))$ is nonincreasing, using (2.3), we observe that

$$\begin{aligned}\sum_{n\geqslant k(\varepsilon)} \max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) &\geqslant C_5 \sum_{n\geqslant k(\varepsilon)} \frac{a\big(\varphi^{-1}(\tau_0+(n+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0+(n+1)\tau)\big)} \\ &\geqslant C_5 \int_{k(\varepsilon)}^{\infty} \frac{a\big(\varphi^{-1}(\tau_0+(t+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0+(t+1)\tau)\big)}\,dt \\ &= \tau^{-1}C_5 \int_0^{\varphi^{-1}(\tau_0+(k(\varepsilon)+1)\tau)} \frac{a(\xi)}{\Phi(\xi)}[-\varphi'(\xi)]d\xi \\ &= C_6 \int_0^{a_{k(\varepsilon)+1}} a(\xi)\,d\xi,\end{aligned} \tag{2.10}$$

where $C_6 = \tau^{-1}C_1C_5$. By (1.12), there exists $C_7 > 0$ such that

$$\int_0^x a(\xi)\,d\xi \geqslant C_7[\Phi(x)]^{-(2-d)}, \quad x \in (0,x_0]. \tag{2.11}$$

Combining (2.2), (2.10), (2.11) and (2.8), we obtain

$$\begin{aligned}|\Gamma_\varepsilon(y)| \geqslant \sum_{n\geqslant k(\varepsilon)} \max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) &\geqslant C_6C_7\big[\Phi(a_{k(\varepsilon)+1})\big]^{-(2-d)} \\ &\geqslant C_6C_7\big[C_4\varepsilon^{-1}\big]^{-(2-d)} \\ &= c_1\varepsilon^{2-d}, \quad \varepsilon \in (0,\varepsilon_0),\end{aligned}$$

where $c_1 = C_6C_7C_4^{-(2-d)}$. Consequently, $\mathscr{M}_*^d(\Gamma(y)) \geqslant c_1 > 0$. $\square$

Now we give a proof of Proposition 1.2. That is a corollary of the following lemma obtained in [13, Lemma 28].

LEMMA 2.2. *Let $y \in C^1(0,x_0]$ be bounded on $(0,x_0]$. Assume that $y' \notin L^1(0,x_0]$ and (1.13) holds for some $d \in (1,2)$. Then there exists $c_2 > 0$ such that $|\Gamma_\varepsilon(y)| \leqslant c_2\varepsilon^{2-d}$ for $\varepsilon \in (0,1)$.*

Proof of Proposition 1.2. If $y' \notin L^1(0,x_0]$, then Lemma 2.2 implies that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$. Now we assume that $y' \in L^1(0,x_0]$. Then y is rectifiable. Indeed, since $\sqrt{1+x^2} \leqslant 1+|x|$ for $x \in \mathbf{R}$, we have

$$\int_0^{x_0} \sqrt{1+|y'(x)|^2}dx \leqslant \int_0^{x_0} (1+|y'(x)|)dx < \infty.$$

Hence, by Proposition 1.1, we conclude that $\mathscr{M}^{*d}(\Gamma(y)) = 0 < \infty$. $\square$

3. Box-counting dimension of oscillatory solutions

In this section, we give proofs of Theorems 1.1 and 1.3.

According to Drábek and Manásevich [4] and Takeuchi [15], for each $p,\ q \in (1,\infty)$, we define the generalized sine function $\sin_{p,q}$ and the corresponding generalized π by the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x (1-s^q)^{-1/p} ds, \quad 0 \leqslant x \leqslant 1$$

and

$$\pi_{p,q} := 2\sin_{p,q}^{-1} 1 = 2\int_0^1 (1-s^q)^{-1/p} ds,$$

respectively. We note here that $\sin_{p,q}(\pi_{p,q}/2) = 1$. The function $\sin_{p,q} x$ is increasing in $[0,\pi_{p,q}/2]$ onto $[0,1]$. We extend it to $(\pi_{p,q}/2,\pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q}-x)$ and to the whole real line $\mathbf{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Then $\sin_{p,q} x$ is a solution of

$$(\phi_p(S'))' + \frac{q}{p'}\phi_q(S) = 0, \quad S(0) = 0,\ S'(0) = 1.$$

We define the generalized cosine function $\cos_{p,q}$ by $\cos_{p,q} x := (\sin_{p,q} x)'$ for $x \in \mathbf{R}$. Then we have

$$|\sin_{p,q} x|^p + |\cos_{p,q} x|^q = 1, \quad x \in \mathbf{R}.$$

For Emden-Fowler equations, we have the following asymptotic behavior result obtained by J. S. W. Wong [18].

PROPOSITION 3.1. *Let $\gamma > 0$ and $\gamma \neq 1$. Assume that $f \in C^2(0,x_0]$, $f(x) > 0$ for $x \in (0,x_0]$, $\lim_{x\to+0} f(x) = \infty$, $f^{-1}f'_+ \in L^1(0,x_0]$ and $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')' \in L^1(0,x_0]$, where $f'_+(x) = \max\{f'(x),0\}$ and $\gamma^* = \max\{\gamma,1\}$. Then, for each nontrivial solution y of (1.1), there exist functions V, $\varphi \in C^1(0,x_0]$ such that*

$$y(x) = [f(x)]^{-\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{\gamma+1}} \sin_{2,\gamma+1}(\varphi(x)), \tag{3.1}$$

$$y'(x) = -[f(x)]^{\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{2}} \cos_{2,\gamma+1}(\varphi(x)), \tag{3.2}$$

$$\lim_{x\to+0} V(x) = V_0 \quad \textit{for some constant } V_0 > 0, \tag{3.3}$$

$$\lim_{x\to+0} \varphi(x) = \infty, \tag{3.4}$$

$$\varphi'(x) = -[f(x)]^{\frac{2}{\gamma+3}}[V(x)]^{\frac{\gamma-1}{2(\gamma+1)}} + \frac{1}{\gamma+1}\frac{f'(x)}{f(x)}\sin_{2,\gamma+1}(\varphi(x))\cos_{2,\gamma+1}(\varphi(x)) \tag{3.5}$$

for $x \in (0, x_0]$.

The following lemma was also obtained by J. S. W. Wong [18, Lemma 2]

LEMMA 3.1. *Assume that* $f \in C^2(0, x_0]$, $f(x) > 0$ *for* $x \in (0, x_0]$, $\lim_{x\to+0} f(x) = \infty$, *and* $(f^{-(\gamma^*+2)/(\gamma^*+1)} f')' \in L^1(0, x_0]$, *where* $\gamma^* = \max\{\gamma, 1\}$. *Then*

$$\lim_{x\to+0} [f(x)]^{-\frac{\gamma+5}{\gamma+3}} f'(x) = 0.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let y be a nontrivial solution of (1.1). Proposition 3.1 implies that there exist V, $\varphi \in C^1(0, x_0]$ such that (3.1)–(3.5) hold. Therefore, y is oscillatory near $x = 0$ and bounded on $(0, x_0]$. Set

$$a(x) = [f(x)]^{-\frac{1}{\gamma+3}}, \quad v(x) = [V(x)]^{\frac{1}{\gamma+1}}, \quad \Phi(x) = [f(x)]^{\frac{2}{\gamma+3}}.$$

From Lemma 3.1, (3.3) and (3.5), it follows that

$$\lim_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} = V_0^{\frac{\gamma-1}{2(\gamma+1)}} \tag{3.6}$$

and

$$\lim_{x\to+0} ([\Phi(x)]^{-1})' = -\frac{2}{\gamma+3} \lim_{x\to+0} [f(x)]^{-\frac{\gamma+5}{\gamma+3}} f'(x) = 0.$$

By (3.3), (3.4) and (3.6), there exists $x_1 \in (0, x_0]$ such that $v(x) > 0$, $\varphi(x) > 0$ and $\varphi'(x) < 0$ for $x \in (0, x_1]$. Applying Theorem 1.2, we have $\mathscr{M}_*^d(\Gamma(y)) > 0$.

Next we will show that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$, by Proposition 1.2. From (3.1) and (3.3), it follows that

$$|y(x)| \leqslant C_1 [f(x)]^{-\frac{1}{\gamma+3}}, \quad x \in (0, x_0]$$

for some $C_1 > 0$. Since $f'(x) < 0$ for $x \in (0, x_0]$, we have

$$\sup_{\xi\in(0,x]} |y(\xi)| \leqslant C_1 [f(x)]^{-\frac{1}{\gamma+3}}, \quad x \in (0, x_0].$$

By (3.2), we obtain

$$|y'(x)| \leqslant C_2 [f(x)]^{\frac{1}{\gamma+3}}, \quad x \in (0, x_0]$$

for some $C_2 > 0$. Hence, Proposition 1.2 shows that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$. $\square$

Finally, we give a proof of Theorem 1.3. To this end, we need the following Proposition 3.2 and Lemma 3.2, which were established by Pašić and Wong [14].

PROPOSITION 3.2. *Let $p>1$. Assume that (1.2) and (1.15) hold. Then, for each nontrivial solution y of (1.14), there exist functions V, $\varphi\in C^1(0,x_0]$ satisfying (3.3), (3.4) and*

$$y(x)=(p-1)^{\frac{1}{pp'}}[f(x)]^{-\frac{1}{pp'}}[V(x)]^{\frac{1}{p}}\sin_{p,p}(\varphi(x)), \tag{3.7}$$

$$y'(x)=-(p-1)^{\frac{1}{p^2}}[f(x)]^{\frac{1}{p^2}}[V(x)]^{\frac{1}{p}}\cos_{p,p}(\varphi(x)), \tag{3.8}$$

$$\varphi'(x)=-(p-1)^{-\frac{1}{p}}[f(x)]^{\frac{1}{p}}+\frac{1}{p}\frac{f'(x)}{f(x)}\sin_{p,p}(\varphi(x))|\cos_{p,p}(\varphi(x))|^{p-2}\cos_{p,p}(\varphi(x)) \tag{3.9}$$

for $x\in(0,x_0]$.

LEMMA 3.2. *Let $p>1$. Assume that (1.2) and (1.15) hold. Then*

$$\lim_{x\to+0}[f(x)]^{-\frac{1}{p}-1}f'(x)=0.$$

Proof of Theorem 1.3. Let y be a nontrivial solution of (1.14). By Proposition 3.2, there exist V, $\varphi\in C^1(0,x_0]$ such that (3.3), (3.4), (3.7), (3.8), and (3.9) hold. Then y is oscillatory near $x=0$ and bounded on $(0,x_0]$. Set

$$a(x)=[f(x)]^{-\frac{1}{pp'}},\quad v(x)=(p-1)^{\frac{1}{pp'}}[V(x)]^{\frac{1}{p}},\quad \Phi(x)=[f(x)]^{\frac{1}{p}}.$$

By Lemma 3.2 and (3.9), we find that

$$\lim_{x\to+0}\frac{-\varphi'(x)}{\Phi(x)}=(p-1)^{-\frac{1}{p}} \tag{3.10}$$

and

$$\lim_{x\to+0}([\Phi(x)]^{-1})'=-\frac{1}{p}\lim_{x\to+0}[f(x)]^{-\frac{1}{p}-1}f'(x)=0.$$

From (3.3), (3.4) and (3.10), it follows that $v(x)>0$, $\varphi(x)>0$ and $\varphi'(x)<0$ on $(0,x_1]$ for some $x_1\in(0,x_0]$. Therefore, Theorem 1.2 implies that $\mathscr{M}_*^d(\Gamma(y))>0$.

By (3.7), there exists a constant $C_1>0$ such that

$$|y(x)|\leqslant C_1[f(x)]^{-\frac{1}{pp'}},\quad x\in(0,x_0].$$

Since $f'(x)<0$ for $x\in(0,x_0]$, we have

$$\sup_{\xi\in(0,x]}|y(\xi)|\leqslant C_1[f(x)]^{-\frac{1}{pp'}},\quad x\in(0,x_0].$$

From (3.8), it follows that

$$|y'(x)|\leqslant C_2[f(x)]^{\frac{1}{p^2}},\quad x\in(0,x_0]$$

for some $C_2>0$. Hence, Proposition 1.2 implies that $\mathscr{M}^{*d}(\Gamma(y))<\infty$. $\square$

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