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ON THE SOLVABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS SUBJECT TO GENERALIZED BOUNDARY CONDITIONS

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Abstract. In this paper, we analyze nonlinear differential equations subject to generalized boundary conditions. More specifically, we provide a framework from which we can provide conditions, which are straightforward to check, for the solvability of a large number of nonlinear scalar boundary value problems. We begin by giving our general strategy which involves the reformulation of our boundary value problem as an operator equation. We then proceed to establish our results and compare them to closely related previous work.

1. Introduction

This paper is devoted to the study of nonlinear differential equations subject to generalized nonlinear boundary conditions or constraints. The class of problems we consider include, as a special case, differential equations subject to multi-point boundary conditions. A framework is provided which enables us to establish easily verifiable conditions which guarantee the existence of solutions to a significant class of problems. Two relevant papers devoted to the study of nonlinear differential equations subject to constraints are [12] and [13]. The work that we will now present allows us to establish the existence of solutions for problems that do lie within the scope of the results in either one of these two papers.

The literature concerning the study of boundary value problems is extensive. In [9], [14] and [15] the authors analyze boundary value problems subject to linear constraints. In [2], the reader will find results pertaining to three-point boundary value problems. The use of projection schemes appears in [7], [8], [9], [14] and [15].

In [3], [4], [11], and [12] the authors obtain existence results based on a global inverse function theorem. Readers interested in fractional differential equations may consult [1] and those who would like to see results involving discrete-time systems are referred to [6], [10], [11] and [13].

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2. Generalities

In this paper, we study nonlinear scalar boundary value problems which we approach by reformulating as an operator equation of the form

$$\mathscr{L}x = H(x) \tag{1}$$

where $\mathscr{L}$ is a linear operator, H is a nonlinear operator and both are defined on a Banach space. Suppose that $\mathscr{L}$ has an inverse, and $H = \Psi + G$. The strategy we will employ is to first give conditions under which $\mathscr{L} - \Psi$ is guaranteed an inverse. That is, we give conditions under which we can uniquely solve the equation

$$\mathscr{L}x - \Psi(x) = y \tag{2}$$

for any point y in the space that $\mathscr{L}$ and Ψ map into. Given a result of this type, we then study conditions under which (1) has a (possibly non-unique) solution by studying the operator $(\mathscr{L} - \Psi)^{-1}G$ and determining conditions under which it has a fixed point. This will rely on a Schauder's fixed point theorem argument.

3. Differential equations

We consider nonlinear differential equations on the interval $[0,1]$ of the form

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = G(x)(t) \tag{3}$$

subject to the boundary conditions

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t) + \eta_i(x) = \phi_i(x) \tag{4}$$

for $1 \leqslant i \leqslant n$.

Here $\psi : \mathbb{R} \to \mathbb{R}$, the maps η_i and ϕ_i for $1 \leqslant i \leqslant n$ are nonlinear real-valued maps from $\mathscr{C} = (C[0,1], \mathbb{R}, \|\cdot\|_\infty)$ into $\mathbb{R}$ where $\|\cdot\|_\infty$ denotes the supremum norm. Further $G : \mathscr{C} \to \mathscr{C}$ is a continuous map, $a_0, a_1, \ldots, a_n \in \mathscr{C}$ and $a_n(t) \neq 0$ for all $t \in [0,1]$. We use $\mathscr{C}^n$ to denote the subspace of $\mathscr{C}$ consisting of all n-times continuously differentiable functions on $[0,1]$. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $\omega_{ij} : [0,1] \to \mathbb{R}$ is a real-valued function of bounded variation. We will determine conditions under which we can guarantee at least one solution in $\mathscr{C}^n$ to (3)–(4).

To do so, we first consider a closely related problem. That is, we seek conditions under which we can uniquely solve

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t) + \eta_i(x) = v_i$$

for $1 \leqslant i \leqslant n$ and for any $h \in \mathscr{C}$ and $v \in \mathbb{R}^n$.

Define $L : \mathscr{C}^n \to \mathscr{C}$ by

$$[Lx](t) = a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t)$$

and $B : \mathscr{C}^n \to \mathbb{R}^n$ by

$$B(x) = \begin{bmatrix} \sum_{j=1}^n \int_0^1 x^{(j-1)}(t)d\omega_{1j}(t) \\ \sum_{j=1}^n \int_0^1 x^{(j-1)}(t)d\omega_{2j}(t) \\ \cdots \\ \sum_{j=1}^n \int_0^1 x^{(j-1)}(t)d\omega_{nj}(t) \end{bmatrix}.$$

The map $\mathscr{L} : \mathscr{C}^n \to \mathscr{C} \times \mathbb{R}^n$ will be defined as

$$\mathscr{L} = \begin{pmatrix} L \\ B \end{pmatrix}.$$

Before proceeding, we state the following remark which illustrates an important special case of (3)–(4) that can be dealt with using the framework of this section.

REMARK 1. Let $t_0 \in [0,1]$, $\beta \in \mathbb{R}$ and let the function $\omega : [0,1] \to \mathbb{R}$ be the step function

$$\omega(x) = \begin{cases} 0, & t < t_0, \\ \beta, & t \geqslant t_0, \end{cases}$$

So for any $x \in \mathscr{C}$, the Riemann-Stieltjes of x with respect to ω is given by

$$\int_0^1 x(t)d\omega(t) = \beta x(t_0).$$

Therefore, the boundary value problem (3)–(4) includes problems of the form

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = G(x)(t)$$

subject to the multipoint boundary conditions

$$\sum_{i=1}^q B_i \bar{x}(t_i) + \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \\ \cdots \\ \eta_n(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \cdots \\ \phi_n(x) \end{bmatrix}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ \cdots \\ x^{(n-1)}(t) \end{bmatrix}$$

$t_i \in [0,1]$, and B_i is a real-valued $n \times n$ matrix for all $1 \leqslant i \leqslant q$.

It is well known from the theory of linear differential equations that $\ker(L)$ is n-dimensional. Without loss of generality, choose a basis $\{u_1, u_2, \ldots, u_n\}$ for the kernel of L such that $\|u_1\| + \|u_2\| + \ldots + \|u_n\| \leqslant 1$ and let

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$$

Suppose that the functions of bounded variation $\omega_{ij} : [0,1] \to \mathbb{R}$ for $i, j \in \{1, 2, \ldots, n\}$ appearing in the boundary conditions are such that the $n \times n$ real-valued matrix $\mathscr{B} = [Bu_1|Bu_2|\ldots|Bu_n]$ is invertible.

Defining the constant B_0 as:

$$B_0 = \|\mathscr{B}^{-1}\|$$

then we have that for any $v \in \mathbb{R}^n$

$$\|u^T \mathscr{B}^{-1} v\| \leqslant B_0 |v|.$$

As a matter of notation, define $\eta : \mathscr{C} \to \mathbb{R}^n$ by $\eta(x) = \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \\ \cdots \\ \eta_n(x) \end{bmatrix}$.

THEOREM 1. *Suppose the map $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant K_1 and $\eta : \mathscr{C} \to \mathbb{R}^n$ is Lipschitz with constant K_2. If*

$$A_0 K_1 + B_0 K_2 < 1$$

then for each pair $h \in \mathscr{C}$, $v \in \mathbb{R}^n$, the boundary value problem

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\, d\omega_{ij}(t) + \eta_i(x) = v_i, \qquad 1 \leqslant i \leqslant n$$

has a unique solution.

Proof. Suppose $\mathscr{L}(x_0) = 0$ for some $0 \neq x_0 \in \mathscr{C}^n$ and $\{u_1, \ldots, u_n\}$ be the basis we chose for $\ker(L)$ above. Since $x_0 \in \ker(L)$ there exists a unique set of constants $c_1, \ldots, c_n \in \mathbb{R}$ with $c_i \neq 0$ for some $1 \leqslant i \leqslant n$ such that $x_0 = \sum_{i=1}^n c_i u_i$. Since $x_0 \in \ker(B)$ we have that

$$\begin{aligned} 0 &= Bx_0 \\ &= B\left(\sum_{i=1}^{n} c_i u_i\right) \\ &= \sum_{i=1}^{n} c_i B u_i \end{aligned}$$

contradicting the fact that $\{Bu_1,\dots,Bu_n\}$ is a linearly independent set in $\mathbb{R}^n$. Therefore, $x_0=0$ and we conclude that $\mathscr{L}:\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ is one-to-one.

Let $h\in\mathscr{C}$ and $v\in\mathbb{R}^n$. By the general theory of linear scalar ODEs it is well known that the solution $Lx=h$ has at least one solution in $\mathscr{C}^n$. Let $x_p\in\mathscr{C}^n$ be the particular solution to this equation given by variation of parameters. That is,

$$x_p(t)=\sum_{k=1}^{n}u_k(t)\int_0^t\frac{h(s)W_k[u_1,\dots,u_n](s)}{a_n(s)W[u_1,\dots,u_n](s)}ds$$

where W_k denotes the determinant of the matrix obtained by replacing the kth column of the matrix whose determinant is W with e_n (the standard basis vector with a 1 in the nth slot and $0s$ everywhere else).

By our assumption that $\{Bu_1,\dots,Bu_n\}$ is a basis for $\mathbb{R}^n$, there exists a unique set of constants $d_1,\dots,d_n$ such that

$$\sum_{i=1}^{n}d_iBu_i=v-B(x_p).$$

Then we have that

$$L\left(x_p+\sum_{i=1}^{n}d_iu_i\right)=h+0=h$$

and further that

$$\begin{aligned}B\left(x_P+\sum_{i=1}^{n}d_iu_i\right)&=B(x_p)+B\left(\sum_{i=1}^{n}d_iu_i\right)\\&=B(x_p)+\sum_{i=1}^{n}d_iBu_i\\&=B(x_p)+(v-B(x_p)).\end{aligned}$$

Therefore, $\mathscr{L}:\mathscr{C}^n\to\mathscr{C}\times\mathbb{R}^n$ is a bijection with

$$\mathscr{L}^{-1}(h,v)^T=\left(L|_{\ker(B)}\right)^{-1}h+u^T\mathscr{B}^{-1}v$$

where $\left(L|_{\ker(B)}\right)$ is the map L restricted to the kernel of B. Consequently, we have that $\|\mathscr{L}^{-1}\|\leqslant\max\{A_0,B_0\}$ where A_0 is an upper bound on the norm of the continuous linear map $\left(L|_{\ker(B)}\right)^{-1}$.

We now define $\Psi:\mathscr{C}\to\mathscr{C}\times\mathbb{R}^n$ by $\Psi(x)=\begin{bmatrix}-\psi(x)\\-\eta(x)\end{bmatrix}$. Note that since the map from $\mathbb{R}$ to $\mathbb{R}$ given by $t\mapsto\psi(t)$ is Lipschitz with constant K_1, then the map from $\mathscr{C}$ to $\mathscr{C}$ defined by $x\mapsto\psi\circ x$ is Lipschitz with constant K_1. For each pair $(h,v)\in\mathscr{C}\times\mathbb{R}^n$, we define the map $H_{(h,v)}:\mathscr{C}\to\mathscr{C}$ by

$$H_{(h,v)}(x)=\mathscr{L}^{-1}\Psi(x)+\mathscr{L}^{-1}[(h,v)^T].$$

Let $x_1, x_2 \in \mathscr{C}$. Then we have that

$$\begin{aligned}\|H_{(h,v)}(x_1) - H_{(h,v)}(x_2)\| &= \|\mathscr{L}^{-1}\Psi(x_1) - \mathscr{L}^{-1}\Psi(x_2)\| \\ &\leqslant A_0\|\psi(x_1) - \psi(x_2)\| + B_0\|\eta(x_1) - \eta(x_2)\| \\ &\leqslant (A_0K_1 + B_0K_2)\|x_1 - x_2\|.\end{aligned}$$

Then $H_{(h,v)}$ is a contraction on $\mathscr{C}$ and so by Banach's fixed point theorem it has a unique fixed point $x_0 \in \mathscr{C}$. Since $\mathscr{L}^{-1}$ maps into $\mathscr{C}^n$, we have that $x_0 \in \mathscr{C}^n$. Therefore, there exists a unique $x_0 \in \mathscr{C}^n$ satisfying $\mathscr{L}(x) - \Psi(x) = (h,v)^T$. Since $h \in \mathscr{C}$ and $v \in \mathbb{R}^n$ were arbitrary, we conclude that the operator $(\mathscr{L} - \Psi) : \mathscr{C}^n \to \mathscr{C} \times \mathbb{R}^n$ is invertible. From this it follows that for each pair $(h,v) \in \mathscr{C} \times \mathbb{R}^n$, the boundary value problem

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = h(t)$$

subject to

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\,d\omega_{ij}(t) + \eta_i(x) = v_i, \qquad 1 \leqslant i \leqslant n$$

has exactly one solution. □

The following lemma establishes an important result regarding the map $\mathscr{L}^{-1} : \mathscr{C} \times \mathbb{R}^n \to \mathscr{C}$. The importance of this lemma will become apparent when we provide our conditions for the solvability of (3)–(4).

LEMMA 1. *The operator $\mathscr{L}^{-1} : \mathscr{C} \times \mathbb{R}^n \to \mathscr{C}$ is compact.*

Proof. Let $M > 0$ and define $S = \{(h,v) \in \mathscr{C} \times \mathbb{R}^n : \|h\| + |v| \leqslant M\}$. Let $(h,v) \in S$. Then

$$\begin{aligned}\|\mathscr{L}^{-1}(h,v)\| &\leqslant \left\|\mathscr{L}^{-1}\right\|(\|h\| + |v|) \\ &\leqslant \max\{A_0, B_0\}(\|h\| + |v|) \\ &\leqslant \max\{A_0, B_0\}M.\end{aligned}$$

Therefore the set $\{\mathscr{L}^{-1}(S)\}$ is bounded. We now wish to show this set forms an equicontinuous set of functions.

Let $\varepsilon > 0$, and let $\delta = \frac{\varepsilon}{\max\{A_0,B_0\}M}$. Then for any $(h,v) \in \mathscr{C} \times \mathbb{R}^n$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$,

$$\begin{aligned}&\left|\left[\mathscr{L}^{-1}(h,v)\right](t_1) - \left[\mathscr{L}^{-1}(h,v)\right](t_2)\right| \\ &= \left|\left[\left(L|_{\ker(B)}\right)^{-1}h\right](t_1) - \left[\left(L|_{\ker(B)}\right)^{-1}h\right](t_2) + u^T(t_1)\mathscr{B}^{-1}v(t_1) - u^T(t_2)\mathscr{B}^{-1}v(t_2)\right| \\ &\leqslant \left|\left[\left(L|_{\ker(B)}\right)^{-1}h\right](t_1) - \left[\left(L|_{\ker(B)}\right)^{-1}h\right](t_2)\right| + \left|u^T(t_1)\mathscr{B}^{-1}v(t_1) - u^T(t_2)\mathscr{B}^{-1}v(t_2)\right| \\ &\leqslant \max\{A_0, B_0\}M|t_1 - t_2| \\ &< \varepsilon.\end{aligned}$$

Therefore the set of functions $\{\mathscr{L}^{-1}(S)\}$ is equicontinous and we conclude that $\mathscr{L}^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}$ is compact by the Arzelá–Ascoli theorem. $\square$

Recall that in the proof of theorem 1, we established that the operator $\mathscr{L}-\Psi$ is a bijection from $\mathscr{C}^n$ onto $\mathscr{C}\times\mathbb{R}^n$. We now state important properties of $(\mathscr{L}-\Psi)^{-1}$ under the conditions of theorem 1. The proof of the first corollary follows directly from corollary 2.3.2 in [5].

COROLLARY 1. *Suppose the conditions of theorem 1 hold. Then the map* $(\mathscr{L}-\Psi)^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}^n$ *is Lipschitz continuous with constant*

$$K\equiv\frac{\max\{A_0,B_0\}}{1-(A_0K_1+B_0K_2)}.$$

COROLLARY 2. *Under the conditions of theorem 1, the map* $(\mathscr{L}-\Psi)^{-1}:\mathscr{C}\times\mathbb{R}^n\to\mathscr{C}^n$ *is compact. This follows from the fact that we can write*

$$(\mathscr{L}-\Psi)^{-1}=\mathscr{L}^{-1}\Big(\Psi\circ(\mathscr{L}-\Psi)^{-1}+I\Big).$$

Therefore it is clear from this representation that $(\mathscr{L}-\Psi)^{-1}$ *is compact as the composition of a compact operator with a continuous one.*

We now proceed to establish conditions for the solvability of the boundary value problem

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\ldots+a_0(t)x(t)+\psi(x(t))=G(x)(t)$$

subject to the boundary conditions

$$\sum_{j=1}^{n}\int_0^1 x^{(j-1)}(t)d\omega_{ij}(t)+\eta_i(x)=\phi_i(x)$$

for $1\leqslant i\leqslant n$.

We define $\phi:\mathscr{C}\to\mathbb{R}^n$ by $\phi=\begin{bmatrix}\phi_1\\ \phi_2\\ \cdots\\ \phi_n\end{bmatrix}$ and $\mathscr{G}:\mathscr{C}\to\mathscr{C}\times\mathbb{R}^n$ by $\mathscr{G}=\begin{bmatrix}G\\ \phi\end{bmatrix}$.

In doing so, we are now ready to state sufficient conditions under which we can guarantee the existence of at least one solution to the nonlinear boundary value problem $(3)-(4)$.

Before stating theorem 2, define M_0 as the norm of the unique solution to the boundary value problem

$$a_n(t)x^{(n)}(t)+a_{n-1}(t)x^{(n-1)}(t)+\ldots+a_0(t)x(t)+\psi(x(t))=0$$

subject to the boundary conditions

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\, d\omega_{ij}(t) + \eta_i(x) = 0$$

for $1 \leqslant i \leqslant n$.

THEOREM 2. *Suppose the map* $\psi : \mathbb{R} \to \mathbb{R}$ *is Lipschitz with constant* K_1 *and* $\eta : \mathscr{C} \to \mathbb{R}^n$ *is Lipschitz with constant* K_2. *If*

$$A_0 K_1 + B_0 K_2 < 1$$

and there exists a constant M *such that for* $\|x\| \leqslant M$, $\|\mathscr{G}(x)\| \leqslant K^{-1}(M - M_0)$. *Then there exists a solution to the boundary value problem*

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \ldots + a_0(t)x(t) + \psi(x(t)) = G(x)(t)$$

subject to the boundary conditions

$$\sum_{j=1}^{n} \int_0^1 x^{(j-1)}(t)\, d\omega_{ij}(t) + \eta_i(x) = \phi_i(x)$$

for $1 \leqslant i \leqslant n$.

Proof. Note that the map $(\mathscr{L} - \Psi)^{-1}\mathscr{G} : \mathscr{C} \to \mathscr{C}$ is compact as the composition of a compact operator with a continuous one. Define $B = \{z \in \mathscr{C} : \|z\| \leqslant M\}$. Let $x \in B$. Then

$$\begin{aligned}
\|(\mathscr{L} - \Psi)^{-1}\mathscr{G}(x)\| &\leqslant \|(\mathscr{L} - \Psi)^{-1}\mathscr{G}(x) - M_0\| + M_0 \\
&\leqslant K\|\mathscr{G}(x)\| + M_0 \\
&\leqslant K(K^{-1}(M - M_0)) + M_0 \\
&= M.
\end{aligned}$$

Since $(\mathscr{L} - \Psi)^{-1}\mathscr{G}(B) \subseteq B$ and B is clearly closed, bounded, and convex we have that $(\mathscr{L} - \Psi)^{-1}\mathscr{G}$ has at least one fixed point in $\mathscr{C}$ by Schauder's fixed point theorem. That is, there exists at least one $x_0 \in \mathscr{C}$ such that $(\mathscr{L} - \Psi)^{-1}\mathscr{G}(x_0) = x_0$. Since $(\mathscr{L} - \Psi)^{-1}$ maps into $\mathscr{C}^n$, we have that x_0 must be an element of $\mathscr{C}^n$. This is equivalent to there existing at least one $x_0 \in \mathscr{C}^n$ such that $\mathscr{L}(x_0) - \Psi(x_0) = \mathscr{G}(x_0)$. $\square$

The following corollary is immediate:

COROLLARY 3. *Suppose the map* $\psi : \mathbb{R} \to \mathbb{R}$ *is Lipschitz with constant* K_1 *and* $\eta : \mathscr{C} \to \mathbb{R}^n$ *is Lipschitz with constant* K_2. *If*

$$A_0 K_1 + B_0 K_2 < 1$$

and

$$\lim_{\|x\|\to\infty} \frac{\|\mathscr{G}(x)\|}{\|x\|} = 0$$

then the boundary value problem (3)–(4) *has a solution.*

In the next section, we consider advantages this framework provides us in cases where we attempt to analyze problems that are seemingly well-suited to using the framework outlined in [12] and [13].

4. Comparison to previous results

To view advantages of using this framework as opposed to previous results, consider another set of special cases of the general boundary value problem (3)–(4). That is, problems already in self-adjoint form. This is a necessity if we are to attempt to use the analysis of [12] and [13].

REMARK 2. Consider differential equations on $[0,1]$ of the form:

$$(p(t)x'(t))' + q(t)x(t) + \psi(x(t)) = G(x(t)) \tag{5}$$

subject to the boundary conditions

$$\begin{aligned} \alpha x(0) + \beta x'(0) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\,d\omega_{1j}(t) + \eta_1(x) = 0 \\ \gamma x(1) + \delta x'(1) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\,d\omega_{2j}(t) + \eta_2(x) = 0 \end{aligned} \tag{6}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$, η_1 and η_2 are nonlinear Lipschitz functions from $\mathscr{C}^2$ into $\mathbb{R}$. The function $\omega_{ij} : [0,1] \to \mathbb{R}$ is a function of bounded variation for $i = 1,2$ and $j = 1,2$. We assume that p, p', and q are continuous, $p(t) > 0$ for all $t \in [0,1]$. We assume the map G is a continuous function from $\mathscr{C}$ into $\mathscr{C}$ satisfying

$$\lim_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|} = 0.$$

Using results from [12] and [13], we would have to treat the linear integral boundary conditions appearing in (6) as part of the nonlinear component of the problem. Let $\hat{K}_2$ be the Lipschitz constant of the map defined by

$$x \mapsto \begin{bmatrix} \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\,d\omega_{1j}(t) + \eta_1(x) \\ \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t)\,d\omega_{2j}(t) + \eta_2(x) \end{bmatrix}$$

with respect to the norms used in those previous papers. Suppose $\{u_1, u_2\}$ is a basis for the solution space of

$$(p(t)x'(t))' + q(t)x(t) = 0$$

and without loss of generality suppose that

$$\left(\int_0^1 |u_1(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^1 |u_2(t)|^2 dt\right)^{\frac{1}{2}} \leqslant 1.$$

Further, suppose the 2×2 matrix $\hat{\mathscr{B}} = \begin{bmatrix} \alpha u_1(0) + \beta u_1'(0), & \alpha u_2(0) + \beta u_2'(0) \\ \gamma u_1(1) + \delta u_1'(1), & \gamma u_2(1) + \delta u_2'(1) \end{bmatrix}$ is invertible. Then if

$$\left(\sup_{v \in \mathbb{R}^2} |\hat{\mathscr{B}}^{-1} v|\right) \hat{K}_2 \geqslant 1$$

it would be impossible to establish the existence of solutions to (5)–(6) using any of the results appearing in [12] or [13].

When we formulate (5)–(6) within the framework presented in section 3, it is clear that if the map

$$x \mapsto \begin{bmatrix} (p(t)x'(t))' + q(t)x(t) \\ \alpha x(0) + \beta x'(0) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t) d\omega_{1j}(t) \\ \gamma x(1) + \delta x'(1) + \sum_{j=1}^{2} \int_0^1 x^{(j-1)}(t) d\omega_{2j}(t) \end{bmatrix}$$

is a bijection from its domain onto $\mathscr{C} \times \mathbb{R}^2$ then the boundary value problem would have a solution provided the Lipschitz constants for η_1, η_2, and ψ are sufficiently small. The magnitude of the linear integral boundary conditions is completely irrelevant.

We would now like to point out that the map G that we have just described can be generated in a variety of ways. For example G could be of the form

$$G(x)(t) = g(x(t))$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous or of the type

$$G(x)(t) = \int_0^1 k(t, x(s)) ds$$

where $k : \mathbb{R}^2 \to \mathbb{R}$.

In addition to the advantages that we have just discussed, we would like to point out that if the nonlinearities η_i appearing above are of the form

$$\eta_i = \sum_{j=1}^{N} f_{i,j}(x(t_j))$$

then in order to use results appearing in [12] or [13] it must be assumed that the operator G is compact. This restriction is no longer present when using the results that we have just presented.

REFERENCES

[1] B. AHMAD, J. J. NIETO, *Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations*, Abstract and Applied Analysis **2009**, (2009).

[2] A. BOUCHERIF, *Nonlinear three-point boundary value problems*, J. Math Anal. Appl. **77**, 2 (1980), 577–600.

[3] K. BROWN, *Nonlinear boundary value problems and a global inverse function theorem*, Annali di Matematica Pura ed Applicata **106**, 1 (1975), 205–217.

[4] K. BROWN, S. LIN, *Periodically perturbed conservative systems and a global inverse function theorem*, Nonlinear Anal.: Theory, Methods & Applications **4**, 1 (1980), 193–201.

[5] P. DRÁBEK, J. MILOTA, *Methods of Nonlinear Analysis: Applications to Differential Equations*, Birkhäuser Verlag AG, Basel, 2007.

[6] R. MA, *Nonlinear discrete Sturm–Liouville problems at resonance*, Nonlinear Anal.: Theory, Methods & Applications **67**, 11 (2007), 3050–3057.

[7] D. MARONCELLI, *Scalar multi-point boundary value problems at resonance*, Differential Equations and Applications **7**, 4 (2015), 449–468.

[8] J. RODRÍGUEZ, *Galerkin's method for ordinary differential equations subject to generalized nonlinear boundary conditions*, J. Differential Equations **97**, 1 (1992), 112–126.

[9] J. RODRÍGUEZ, *Nonlinear differential equations under Stieltjes boundary conditions*, Nonlinear Anal.: Theory, Methods & Applications **7**, 1 (1983), 107–116.

[10] J. RODRÍGUEZ, *Nonlinear discrete Sturm–Liouville problems*, J. Math Anal. Appl. **308**, 1 (2005), 380–391.

[11] J. RODRÍGUEZ, Z. ABERNATHY, *Nonlinear discrete Sturm-Liouville problems with global boundary conditions*, Journal of Difference Equations and Applications **18**, 3 (2012), 431–445.

[12] J. RODRÍGUEZ, Z, ABERNATHY, *On the solvability of nonlinear Sturm-Liouville problems*, J. Math Anal. Appl. **387**, 1 (2012), 310–319.

[13] J. RODRÍGUEZ, A. J. SUAREZ, *On nonlinear perturbations of Sturm-Liouville problems in discrete and continuous settings*, Differential Equations and Applications **8**, 3 (2016), 319–334.

[14] J. RODRÍGUEZ, P. TAYLOR, *Multipoint boundary value problems for nonlinear ordinary differential equations*, Nonlinear Analysis: Theory, Methods & Applications **68**, 11 (2008), 3465–3474.

[15] M. URABE, *Galerkin's prodcedure for nonlinear periodic systems*, Archive for Rational Mechanics and Analysis **20**, (1965), 120–152.

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