Differential
Equations
& Applications
Volume 11, Number 1 (2019), 87–106 doi:10.7153/dea-2019-11-02

A GRONWALL INEQUALITY AND THE CAUCHY–TYPE PROBLEM BY MEANS OF ψ–HILFER OPERATOR

JOSÉ VANTERLER DA COSTA SOUSA AND EDMUNDO CAPELAS DE OLIVEIRA

(*Communicated by JinRong Wang*)

Abstract. In this paper, we propose a generalized Gronwall inequality through the fractional integral with respect to another function. The Cauchy-type problem for a nonlinear differential equation involving the ψ-Hilfer fractional derivative and the existence and uniqueness of solutions are discussed. Finally, through generalized Gronwall inequality, we prove the continuous dependence of data on the Cauchy-type problem.

1. Introduction

Over the decades, the fractional calculus has been building a great history and consolidating itself in several scientific areas such as: mathematics, physics and engineering, among others. The emergence of new fractional integrals and derivatives, makes the wide number of definitions becomes increasingly larger and clears its numerous applications [10, 12, 19]. Recently, Sousa and Oliveira [20] introduced the so-called ψ-Hilfer fractional derivative with respect to another function, in order to unify the wide number of fractional derivatives in a single fractional operator and consequently, open a window for new applications.

Problems of initial values for the so-called fractional order differential equations that emerge and describe linear and nonlinear phenomena have obtained much attention in the scientific community and specially in engineering. Recently, there are several researchers [5, 7, 9, 11, 13, 14, 16, 29] that have used fractional differential equations to model natural phenomena. It is remarkable the growth and the importance of fractional differential equations both in the theoretical sense and in the practical sense [7, 9, 12, 19]. In this sense, due to this exponential growth of the fractional calculus added to differential equations, the number of researchers involved in the investigation of existence, uniqueness, Ulam-Hyers stabilities of solutions of fractional differential equations of the type: impulsive, functional and evolution with non-instantaneous and instantaneous impulses, has been the target of research and has gained increasing prominence in the scientific community [18, 21, 22, 23]. Recently, Sousa and Oliveira [24, 25] investigated the Ulam-Hyers-Rassias, Ulam-Hyers and δ-Ulam-Hyers-Rassias

Mathematics subject classification (2010): 26A33, 36A08, 34A12, 34A40.
Keywords and phrases: ψ-Hilfer fractional derivative, Cauchy-type problem, existence and uniqueness, continuous dependence, generalized Gronwall inequality.

© ELEMENT, Zagreb
Paper DEA-11-02

stabilities of solutions of impulsive fractional differential equations and integrodifferential equations in the sense of ψ-Hilfer fractional derivative, using the fixed-point method. We can also highlight the essential and important paper that Sousa and Oliveira [26] carried out by means of the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer fractional derivative through the Gronwall inequality presented here. On the other hand, an extension of Gronwall inequality was shown to be important in the existence and δ-Ulam-Hyers-Rassias stability of mild solution of fractional differential equations with noninstantaneous impulses in a $\mathrm{P}\delta$-normed Banach space [27]. Gronwall inequality is one of the tools used to study the existence, uniqueness and data continuous dependence of the Cauchy's problem solutions and, in addition, other important applications might be found [6].

Consider the following Cauchy-type problem for the fractional differential equation [5]

$$D_{a+}^{\alpha,\beta} y(x) = f(x, y(x)), \, 0 < \alpha < 1, \, 0 \leqslant \beta \leqslant 1 \tag{1}$$

$$I_{a+}^{1-\gamma} y(a) = y_a, \qquad \gamma = \alpha + \beta(1-\alpha), \tag{2}$$

where $I_{a+}^{1-\gamma}(\cdot)$ is the fractional integral in the Riemann-Liouville sense and $D_{a+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative.

In this paper, we propose a generalized Gronwall inequality using the fractional integral of a function f with respect to another function $\psi(\cdot)$, and from it to study the data continuous dependence of the Cauchy-type problem Eq. (1)–Eq. (2) as well as proposing and discussing the existence and uniqueness by means of ψ-Hilfer fractional derivative [20].

The paper is organized as follows:. in section 2, we begin with the definition of some spaces of functions, the weighted spaces among them. In this section, we present the definition of fractional integral of a function f with respect to another function ψ, in a addition to the definitions of ψ-Riemann-Liouville fractional derivative, ψ-Caputo fractional derivative and ψ-Hilfer fractional derivative. From these definitions, some important results are introduced for the development of the paper. Finally, the definition of the Lipschitz condition, is presented. In section 3, we discuss the generalized Gronwall inequality by means of fractional integral with respect to another ψ function. Also, two consonant results of the generalized Gronwall inequality theorem are demonstrated. In section 4, we present the version of the Cauchy-type problem for the ψ-Hilfer fractional derivative, showing: this problem is equivalent to solve a Volterra integral equation. From this, we study the existence and uniqueness of Cauchy-type problem. In section 5, using the generalized Gronwall inequality, we address the data continuous dependence of the Cauchy-type problem. Concluding remarks close the paper.

2. Preliminaries

First, we present the weighted spaces and some definitions and important results for the development of the paper.

Let $[a,b]$ $(0<a<b<\infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and $C[a,b]$, $AC^n[a,b]$, $C^n[a,b]$ be the spaces of continuous functions, n-times absolutely continuous and n-times continuously differentiable functions on $[a,b]$, respectively.

The space of continuous function f on $[a,b]$ with the norm is defined by [12]

$$\|f\|_{C[a,b]}=\max_{t\in[a,b]}|f(t)|.$$

On the other hand, we have n-times absolutely continuous functions given by

$$AC^n[a,b]=\left\{f:[a,b]\to\mathbb{R};\ f^{(n-1)}\in AC([a,b])\right\}.$$

The weighted spaces $C_{\gamma,\psi}[a,b]$ and $C_{1-\gamma,\psi}[a,b]$ of functions f on $[a,b]$ are defined by

$$C_{\gamma;\psi}[a,b]=\left\{f:(a,b]\to\mathbb{R};\ (\psi(t)-\psi(a))^{\gamma}f(t)\in C[a,b]\right\},\ 0\leqslant\gamma<1,$$

$$C_{1-\gamma;\psi}[a,b]=\left\{f:(a,b]\to\mathbb{R};\ (\psi(t)-\psi(a))^{1-\gamma}f(t)\in C[a,b]\right\},\ 0\leqslant\gamma<1,$$

respectively, with the norm

$$\|f\|_{C_{\gamma;\psi}[a,b]}=\left\|(\psi(t)-\psi(a))^{\gamma}f(t)\right\|_{C[a,b]}=\max_{t\in[a,b]}\left|(\psi(t)-\psi(a))^{\gamma}f(t)\right| \tag{3}$$

and

$$\|f\|_{C_{1-\gamma;\psi}[a,b]}=\left\|(\psi(t)-\psi(a))^{1-\gamma}f(t)\right\|_{C[a,b]}=\max_{t\in[a,b]}\left|(\psi(t)-\psi(a))^{1-\gamma}f(t)\right|. \tag{4}$$

The weighted space $C^n_{\gamma;\psi}[a,b]$ of function f on $[a,b]$ is defined by

$$C^n_{\gamma;\psi}[a,b]=\left\{f:(a,b]\to\mathbb{R};\ f(t)\in C^{n-1}[a,b];\ f^{(n)}(t)\in C_{\gamma;\psi}[a,b]\right\},\ 0\leqslant\gamma<1$$

with the norm

$$\|f\|_{C^n_{\gamma;\psi}[a,b]}=\sum_{k=0}^{n-1}\left\|f^{(k)}\right\|_{C[a,b]}+\left\|f^{(n)}\right\|_{C_{\gamma;\psi}[a,b]}.$$

For $n=0$, we have, $C^0_{\gamma}[a,b]=C_{\gamma}[a,b]$.

As one of the objectives of this paper is to propose a generalized Gronwall inequality, we present the Gronwall inequality recently introduced [2]. Let u, v be two integrable functions and g a continuous function, with domain $[a,b]$. Assume that: u and v are nonnegative; g is nonnegative and nondecreasing. If

$$u(t)\leqslant v(t)+g(t)\rho^{1-\alpha}\int_a^t\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1}u(\tau)\,d\tau,$$

then

$$u(t)\leqslant v(t)+\int_a^t\sum_{k=1}^{\infty}\frac{\rho^{1-k\alpha}[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{k\alpha-1}v(\tau)\,d\tau,$$

$\forall t \in [a,b]$. We suggest [3, 8, 31], where other formulations of the Gronwall inequality can be found via fractional integrals.

Let $\alpha > 0$, $n \in \mathbb{N}$, with $I = [a,b]$ $(-\infty \leqslant a < x < b \leqslant \infty)$ a finite or infinite interval, f an integrable function defined on I and $\psi \in C^1([a,b],\mathbb{R})$ an increasing function such that $\psi'(x) \neq 0$, $\forall x \in I$. The fractional integrals of a function f with respect to another function are given by [12, 19]

$$I_{a+}^{\alpha;\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{\alpha-1} f(t)\,dt \tag{5}$$

and

$$I_{b-}^{\alpha;\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)\,(\psi(t) - \psi(x))^{\alpha-1} f(t)\,dt. \tag{6}$$

In this sense, from Eq. (5) and Eq. (6), the Riemann-Liouville fractional derivatives of a function f of order α with respect to another function ψ, are defined by [12, 19]

$$\begin{aligned} \mathscr{D}_{a+}^{\alpha;\psi} f(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha;\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)\,(\psi(x) - \psi(t))^{n-\alpha-1} f(t)\,dt \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathscr{D}_{b-}^{\alpha;\psi} f(x) &= \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha;\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b \psi'(t)\,(\psi(t) - \psi(x))^{n-\alpha-1} f(t)\,dt, \end{aligned} \tag{8}$$

respectively and $n = [\alpha] + 1$.

REMARK 1. From Eq. (5) and Eq. (6), we also consider a fractional derivative version, the so-called ψ-Caputo fractional derivative with respect to another function, recently introduced by Almeida [1].

DEFINITION 1. [20] Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $I = [a,b]$ an interval such that $-\infty \leqslant a < b \leqslant \infty$ and $f, \psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivatives (left-sided and right-sided) ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ and ${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi}(\cdot)$ of a function of order α and type $0 \leqslant \beta \leqslant 1$ respectively, are defined by

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \tag{9}$$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x). \tag{10} $$

The ψ-Hilfer fractional derivatives as above defined, can be written in the following forms

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\gamma-\alpha;\psi} \mathscr{D}_{a+}^{\gamma;\psi} f(x) \tag{11} $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\gamma-\alpha;\psi} (-1)^{n} \mathscr{D}_{b-}^{\gamma;\psi} f(x), \tag{12} $$

respectively with $\gamma = \alpha + \beta(n-\alpha)$ and $I_{a+}^{\gamma-\alpha;\psi}(\cdot)$, $D_{a+}^{\gamma;\psi}(\cdot)$, $I_{b-}^{\gamma-\alpha;\psi}(\cdot)$, $D_{b-}^{\gamma;\psi}(\cdot)$ as defined in Eq. (5), Eq. (7), Eq. (6) and Eq. (8).

DEFINITION 2. [5] Assume that $f(x,y(x))$ is defined on set $(a,b] \times H$, $H \subset \mathbb{R}$. A function $f(x,y(x))$ satisfies Lipschitz condition with respect to y, if all $x \in (a,b]$ and for $y_1, y_2 \in H$,

$$ |f(x,y_1) - f(x,y_2)| \leqslant A |y_1 - y_2|, \tag{13} $$

where $A > 0$ is Lipschitz constant.

DEFINITION 3. Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, the weighted space $C_{1-\gamma;\psi}^{\alpha,\beta}[a,b]$ is defined by

$$ C_{1-\gamma;\psi}^{\alpha,\beta}[a,b] = \left\{ f \in C_{1-\gamma;\psi}[a,b] : {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \in C_{1-\gamma;\psi}[a,b] \right\}, $$

with $\gamma = \alpha + \beta(1-\alpha)$.

LEMMA 1. *If $\alpha > 0$ and $0 \leqslant \mu < 1$, then $I_{a+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{\mu;\psi}[a,b]$ to $C_{\mu;\psi}[a,b]$. In addition, if $\mu \leqslant \alpha$, then $I_{a+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{\mu;\psi}[a,b]$ to $C[a,b]$.*

LEMMA 2. *Let $\alpha > 0$ and $\delta > 0$.*

1. *If $f(x) = (\psi(x) - \psi(a))^{\delta-1}$, then*

$$ I_{a+}^{\alpha;\psi} f(x) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(x) - \psi(a))^{\alpha+\delta-1} $$

2. *If $g(x) = (\psi(b) - \psi(x))^{\delta-1}$, then*

$$ I_{b-}^{\alpha;\psi} g(x) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(b) - \psi(x))^{\alpha+\delta-1} $$

Proof. See [12]. □

THEOREM 1. *If $f \in C^n[a,b]$, $n-1<\alpha<n$ and $0 \leqslant \beta \leqslant 1$, then*

$$I_{a+}^{\alpha;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = f(x) - \sum_{k=1}^{n} \frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a)$$

and

$$I_{b-}^{\alpha;\psi}\,{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = f(x) - \sum_{k=1}^{n} \frac{(-1)^k(\psi(b)-\psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{b-}^{(1-\beta)(n-\alpha);\psi} f(b).$$

Proof. See [20]. □

THEOREM 2. *Let $f \in C^1[a,b]$, $\alpha>0$ and $0 \leqslant \beta \leqslant 1$, we have*

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f(x) = f(x) \text{ and } {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} I_{b-}^{\alpha;\psi} f(x) = f(x).$$

Proof. See [20]. □

3. The Gronwall inequality

The Gronwall inequality plays an important role in the study of the qualitative theory of integral and differential equations [2, 5, 17, 28], as well as solving Cauchy-type problems of non-linear differential equations. In order to work with continuous dependence of differential equations via ψ-Hilfer fractional derivative, in this section, we present the first main result, the generalized Gronwall inequality by means of the fractional integral with respect to another function ψ and other important results.

THEOREM 3. *Let u, v be two integrable functions and g continuous, with domain $[a,b]$. Let $\psi \in C^1[a,b]$ an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a,b]$. Assume that*

1. *u and v are nonnegative;*
2. *g in nonnegative and nondecreasing.*

If

$$u(t) \leqslant v(t) + g(t)\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} u(\tau)\,d\tau,$$

then

$$u(t) \leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha k-1} v(\tau)\,d\tau, \tag{14}$$

$\forall t \in [a,b]$.

Proof. Let

$$A\phi(t) = g(t)\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\phi(\tau)\,d\tau, \tag{15}$$

$\forall t \in [a,b]$, for locally integral functions ϕ. Then,

$$u(t) \leqslant v(t) + Au(t).$$

Iterating, for $n \in \mathbb{N}$, we can write

$$u(t) \leqslant \sum_{k=0}^{n-1} A^k v(t) + A^n u(t).$$

The next step, is usually mathematical induction, that if ϕ is a nonnegative function, then

$$A^k u(t) \leqslant \int_a^t \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha k-1}u(\tau)\,d\tau. \tag{16}$$

We know that relation Eq. (16) is true for $n=1$. Suppose that the formula is true for some $k=n\in\mathbb{N}$, then the induction hypothesis implies

$$\begin{aligned} A^{k+1}u(t) = {} & A\left(A^k u(t)\right) \leqslant A\left(\int_a^t \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha k-1}u(\tau)\,d\tau\right) \\ = {} & g(t)\int_a^t \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1} \\ & \times\left(\int_a^\tau \frac{[g(\tau)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\psi'(s)[\psi(\tau)-\psi(s)]^{\alpha k-1}u(s)\,ds\right)d\tau. \end{aligned} \tag{17}$$

By hypothesis, g is a nondecreasing function, that is $g(\tau)\leqslant g(t)$, for all $\tau\leqslant t$, then from Eq. (17), we get

$$\begin{aligned} & A^{k+1}u(t) \\ \leqslant {} & \frac{[\Gamma(\alpha)]^k}{\Gamma(\alpha k)}[g(t)]^{k+1}\int_a^t\int_a^\tau \psi'(\tau)[\psi(t)-\psi(\tau)]^{\alpha-1}\psi'(s)[\psi(\tau)-\psi(s)]^{\alpha k-1}u(s)\,ds\,d\tau. \end{aligned} \tag{18}$$

By Dirichlet's formula, the Eq. (18) can be rewritten as

$$\begin{aligned} & A^{k+1}u(t) \\ \leqslant {} & \frac{[\Gamma(\alpha)]^k}{\Gamma(\alpha k)}[g(t)]^{k+1}\int_a^t \psi'(\tau)u(\tau)\int_\tau^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}[\psi(s)-\psi(\tau)]^{\alpha k-1}\,ds\,d\tau. \end{aligned} \tag{19}$$

Note that,

$$\begin{aligned} & \int_\tau^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}[\psi(s)-\psi(\tau)]^{\alpha k-1}\,ds \\ = {} & \int_\tau^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}\left[1-\frac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}\right]^{\alpha-1}[\psi(s)-\psi(\tau)]^{\alpha k-1}\,ds. \end{aligned}$$

Introducing a change of variables $u=\dfrac{\psi(s)-\psi(\tau)}{\psi(t)-\psi(\tau)}$ and using the definition of beta function and the relation with gamma function $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we have

$$\begin{aligned}&\int_\tau^t \psi'(s)\left[\psi(t)-\psi(s)\right]^{\alpha-1}\left[\psi(s)-\psi(\tau)\right]^{\alpha k-1}ds\\&=\left[\psi(t)-\psi(\tau)\right]^{k\alpha+\alpha-1}\int_0^1\left[1-u\right]^{\alpha-1}u^{k\alpha-1}du\\&=\left[\psi(t)-\psi(\tau)\right]^{k\alpha+\alpha-1}\frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma(\alpha+k\alpha)}.\end{aligned}\tag{20}$$

Replacing Eq. (19) in Eq. (20), we get

$$A^{k+1}u(t)\leqslant\int_a^t\frac{\left[g(t)\Gamma(\alpha)\right]^{k+1}}{\Gamma(\alpha(k+1))}\psi'(\tau)u(\tau)\left[\psi(t)-\psi(\tau)\right]^{\alpha(k+1)-1}d\tau.$$

Let us now prove that $A^n u(t)\to 0$ as $n\to\infty$. As g is a continuous function on $[a,b]$, then by Weierstrass theorem [4, 15], there exist a constant $M>0$ such that $g(t)\leqslant M$ for all $t\in[a,b]$. Then, we obtain

$$A^n u(t)\leqslant\int_a^t\frac{\left[M\Gamma(\alpha)\right]^n}{\Gamma(\alpha n)}\psi'(\tau)u(\tau)\left[\psi(t)-\psi(\tau)\right]^{\alpha n-1}d\tau.$$

Consider the series

$$\sum_{n=1}^{\infty}\frac{\left[M\Gamma(\alpha)\right]^n}{\Gamma(\alpha n)},$$

satisfying the relation

$$\lim_{n\to\infty}\frac{\Gamma(\alpha n)(\alpha n)^\alpha}{\Gamma(\alpha n+\alpha)}=1.\tag{21}$$

Using the ratio test to the series and the asymptotic approximation [30], so we obtain

$$\lim_{n\to\infty}\frac{\Gamma(\alpha n)}{\Gamma(\alpha n+\alpha)}=0.$$

Therefore, the series converges and we conclude that

$$u(t)\leqslant\sum_{k=0}^{\infty}A^k v(t)\leqslant v(t)+\int_a^t\sum_{k=1}^{\infty}\frac{\left[g(t)\Gamma(\alpha)\right]^k}{\Gamma(\alpha k)}\psi'(\tau)\left[\psi(t)-\psi(\tau)\right]^{\alpha k-1}v(\tau)d\tau.\quad\square$$

COROLLARY 1. *Let $\alpha>0$, $I=[a,b]$ and $f,\psi\in C^1([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t)\neq 0$ for all $t\in I$. Suppose $b\geqslant 0$ and v is a nonnegative function locally integrable on $[a,b]$ and suppose also that u is nonnegative and locally integrable on $[a,b]$ with*

$$u(t)\leqslant v(t)+b\int_a^t\psi'(\tau)\left[\psi(t)-\psi(\tau)\right]^{\alpha-1}u(\tau)d\tau,\ \forall t\in[a,b].$$

Then, we can write

$$u(t) \leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[b\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau) [\psi(t) - \psi(\tau)]^{\alpha k - 1} v(\tau) d\tau, \ \forall t \in [a,b].$$

COROLLARY 2. *Under the hypothesis of* Theorem 3*, let* v *be a nondecreasing function on* $[a,b]$*. Then, we have*

$$u(t) \leqslant v(t) \mathbb{E}_\alpha \left(g(t) \Gamma(\alpha) [\psi(t) - \psi(a)]^\alpha \right), \ \forall t \in [a,b],$$

where $\mathbb{E}_\alpha(\cdot)$ *is the Mittag-Leffler function defined by* $\mathbb{E}_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$ *with* $Re(\alpha) > 0$.

Proof. In fact, as v is nondecreasing, so, for all $\tau \in [a,t]$, we have $v(\tau) \leqslant v(t)$ and we can write

$$\begin{aligned} u(t) &\leqslant v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau) [\psi(t) - \psi(\tau)]^{\alpha k - 1} v(\tau) d\tau \\ &= v(t) \left[1 + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau) [\psi(t) - \psi(\tau)]^{\alpha k - 1} v(\tau) d\tau \right] \\ &= v(t) \left[1 + \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \frac{[\psi(t) - \psi(a)]^{\alpha k}}{k\alpha} \right] \\ &= v(t) \mathbb{E}_\alpha \left(g(t) \Gamma(\alpha) [\psi(t) - \psi(a)]^\alpha \right). \quad \square \end{aligned}$$

REMARK 2.

- Note that, choosing $g(t) = Lsgn(t)$, $\psi(t) = |t|$ and $a = -t$ with $-1 \leqslant t \leqslant 1$ and substituting in Theorem 3, we have as particular case Theorem 2.1 [3]. On the other hand, making the same procedure in the Corollary 2, we have as particular case the Corollary 2.4 [3];

- Choosing $\psi(t) = t$ and $a = 0$ and substituting in Theorem 3, we have as particular case Theorem 1 in the Riemann-Liouville fractional integral sense. In an analogous way, it is valid for Corollary 2, that by replacing we get the Corollary 2 [31];

- Choosing $\psi(t) = \ln t$, $a = 1$ and substituting in Theorem 3, we have as particular case Theorem 13. 4 Hadamard fractional integral sense. In an analogous way, it is valid for Corollary 2, that by replacing we get the Corollary 13.2 [8];

- In the previous items, we present some particular cases from the generalized Gronwall inequality. However, recently, Sousa and Oliveira [20], introduced ψ-Hilfer fractional derivative, and presented a class of fractional integrals and

fractional derivatives, which can be obtained from such definitions, with the particular choice of the function ψ. In this sense, the Gronwall inequality given above can be considered as a wide class of inequality of Gronwall, that is, in other words, it's a generalization of the possible inequalities of Gronwall introduced via fractional integral.

4. Existence and uniqueness

In this section, we prove the second ma

in result of the paper, the existence and uniqueness of solutions of the Cauchy-type problem Eq. (22)–Eq. (23), using the fact that Volterra integral equation is equivalent to the Cauchy-type problem in the $C_{1-\gamma;\psi}[a,b]$ weighted space, by means of ψ-Hilfer fractional derivative.

To this end, we consider the following initial value problem

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} y(x) = f(x,y(x)), \quad 0<\alpha<1, \quad 0\leqslant\beta\leqslant 1 \tag{22} $$

$$ I_{a+}^{1-\gamma;\psi} y(a) = y_a, \qquad \gamma=\alpha+\beta(1-\alpha), \tag{23} $$

where ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} y(x)$ is the $\psi-$Hilfer fractional derivative, $f:[a,b)\times\mathbb{R}\to\mathbb{R}$ and y_a is a constant.

Applying the fractional integral operator $I_{a+}^{\alpha;\psi}(\cdot)$ on both sides of the fractional equation Eq. (22) and using Theorem 1, we get

$$ y(x) = \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a+}^{(1-\beta)(1-\alpha);\psi} f(a) + I_{a+}^{\alpha;\psi} f(x,y(x)). \tag{24} $$

On the other hand, if y satisfies Eq. (24), then y satisfies Eq. (22)–Eq. (23). However, applying the fractional derivative operator ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ on both sides of Eq. (24), we have

$$ \begin{aligned} {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} y(x) = {}& {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}\left(\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a+}^{(1-\beta)(1-\alpha);\psi} f(a)\right) \\ & + {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f(x,y(x)). \end{aligned} \tag{25} $$

Using the Theorem 2 and the formula [7, 20]

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\psi(x)-\psi(a))^{\gamma-1} = 0, \quad 0<\gamma<1, $$

we obtain

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} y(x) = f(x,y(x)). $$

Then, we conclude that, $y(x)$ satisfies initial value problem Eq. (22)–Eq. (23) if and only if $y(x)$ satisfies the second kind Volterra integral equation

$$ y(x) = y_a \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t,y(t))\,dt. \tag{26} $$

LEMMA 3. *Let $\psi \in C^1([a,b],\mathbb{R})$ be a function such that ψ is increasing and $\psi \neq 0$, $\forall x \in [a,b]$. If $\gamma = \alpha + \beta(1-\alpha)$ where $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$, then ψ-Riemann fractional integral operator $I_{a+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{1-\gamma;\psi}[a,b]$ to $C_{1-\gamma;\psi}[a,b]$:*

$$\left\|I_{a+}^{\alpha;\psi} f\right\|_{C_{1-\gamma;\psi}[a,b]} \leqslant M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(x)-\psi(a))^{\alpha}, \tag{27}$$

where, M is the bound of a bounded function f.

Proof. From Lemma 1, the result follows. Now we prove the estimative Eq. (27). By the weighted space given in Eq. (3), we have

$$\begin{aligned}
\left\|I_{a+}^{\alpha;\psi} f\right\|_{C_{1-\gamma;\psi}[a,b]} &= \left\|(\psi(x)-\psi(a))^{1-\gamma} I_{a+}^{\alpha;\psi} f\right\|_{C[a,b]} \\
&\leqslant \left\|(\psi(x)-\psi(a))^{1-\gamma} f\right\|_{C[a,b]} \\
&\quad \times \left\|(\psi(x)-\psi(a))^{1-\gamma} I_{a+}^{\alpha;\psi} (\psi(x)-\psi(a))^{\gamma-1}\right\|_{C[a,b]} \\
&= \|f\|_{C_{1-\gamma;\psi}[a,b]} \left\|I_{a+}^{\alpha;\psi} (\psi(x)-\psi(a))^{\gamma-1}\right\|_{C_{1-\gamma;\psi}[a,b]} \\
&\leqslant M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \frac{(\psi(x)-\psi(a))^{\gamma}}{\psi(x)-\psi(a)} (\psi(x)-\psi(a))^{\alpha}.
\end{aligned}$$

As $0 < \gamma < 1$, then $\dfrac{(\psi(x)-\psi(a))^{\gamma}}{\psi(x)-\psi(a)} < 1$. So, we conclude that

$$\left\|I_{a+}^{\alpha;\psi} f\right\|_{C_{1-\gamma;\psi}[a,b]} \leqslant M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(x)-\psi(a))^{\alpha}. \quad \square$$

THEOREM 4. *Let $\gamma = \alpha + \beta(1-\alpha)$ where $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(x,y) \in C_{1-\gamma;\psi}[a,b]$ for any $y \in C_{1-\gamma;\psi}[a,b]$ and satisfies Lipschitz condition* Eq. (13) *with respect to y. Then there exists a unique solution $y(x)$ for the Cauchy-type problem* Eq. (22)–Eq. (23) $C_{1-\gamma;\psi}^{\alpha,\beta}[a,b]$.

Proof. The Volterra integral equation makes sense in any interval $[a,x_1] \subset [a,b]$. So, we choose x_1 such that

$$A \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(x)-\psi(a))^{\alpha} < 1, \tag{28}$$

holds and first we prove the existence of unique solution $y \in C_{1-\gamma;\psi}[a,x_1]$. From the Picard's sequences, we consider

$$y_0(x) = y_a \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}, \qquad \gamma = \alpha + \beta(1-\alpha) \tag{29}$$

and

$$y_m(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t, y_{m-1}(t))\, dt, \quad m \in \mathbb{N}. \tag{30}$$

We now show that $y_m(x) \in C_{1-\gamma;\psi}[a,b]$. From Eq. (29), it follows that $y_0(x) \in C_{1-\gamma;\psi}[a,b]$. By Lemma 1 $I_{a+}^{\alpha;\psi} f$ is bounded from $C_{1-\gamma;\psi}[a,b]$ to $C_{1-\gamma;\psi}[a,b]$, which gives $y_m(x) \in C_{1-\gamma;\psi}[a,b]$, $m \in \mathbb{N}$ and $x \in [a,b]$.

Using Eq. (29) and Eq. (30), we have

$$\begin{aligned}
&\|y_1(x) - y_0(x)\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&= \left\| y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} f(t, y_0(t))\, dt - y_0(x) \right\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&= \left\| I_{a+}^{\alpha;\psi} f(x, y_0(x)) \right\|_{C_{1-\gamma;\psi}[a,x_1]}.
\end{aligned}$$

By means of Lemma 3, we have

$$\|y_1(x) - y_0(x)\|_{C_{1-\gamma;\psi}[a,x_1]} \leqslant M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(x)-\psi(a))^{\alpha}. \tag{31}$$

Further, we get

$$\begin{aligned}
\|y_2(x) - y_1(x)\|_{C_{1-\gamma;\psi}[a,x_1]} &= \left\| I_{a+}^{\alpha;\psi} [f(x, y(x)) - f(x, y_0(x))] \right\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&\leqslant \frac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} A \|y(x) - y_0(x)\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&\leqslant M \frac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} \left(A \frac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} \right).
\end{aligned}$$

Continuing this process m-times, we can write

$$\begin{aligned}
&\|y_m(x) - y_{m-1}(x)\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&\leqslant M \frac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} \left(A \frac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} \right)^{m-1}.
\end{aligned} \tag{32}$$

Taking the limit $m \to \infty$ on both sides of Eq. (32) and remember the following condition $A \dfrac{\Gamma(\gamma)(\psi(x_1)-\psi(a))^{\alpha}}{\Gamma(\gamma+\alpha)} < 1$, we get

$$\|y_m(x) - y(x)\|_{C_{1-\gamma;\psi}[a,x_1]} \to 0. \tag{33}$$

Again, by Lemma 3, it follows that

$$\begin{aligned}
&\left\| I_{a+}^{\alpha;\psi} f(x, y_m(x)) - I_{a+}^{\alpha;\psi} f(x, y(x)) \right\|_{C_{1-\gamma;\psi}[a,x_1]} \\
&\leqslant \left\| (\psi(x_1)-\psi(a))^{1-\gamma} (f(x, y_m(x)) - f(x, y(x))) \right\|_{C[a,x_1]} \\
&\quad \times \left\| (\psi(x_1)-\psi(a))^{1-\gamma} I_{a+}^{\alpha;\psi} (\psi(x_1)-\psi(a))^{\gamma-1} \right\|_{C[a,x_1]} \\
&\leqslant A \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(x_1)-\psi(a))^{\alpha} \|y_m(x) - y(x)\|_{C_{1-\gamma;\psi}[a,x_1]}
\end{aligned}$$

and hence by Eq. (33)

$$\left\| I_{a+}^{\alpha;\psi} f\left(x, y_m(x)\right) - I_{a+}^{\alpha;\psi} f\left(x, y(x)\right) \right\|_{C_{1-\gamma;\psi}[a,x_1]} \to 0 \tag{34}$$

as $m \to \infty$.

Using Eq. (33) and Eq. (34), we conclude that $y(x)$ is the solution of integral equation Eq. (26) in $C_{1-\gamma;\psi}[a,x_1]$.

On the other hand, considering two solutions $y(x)$ and $z(x)$ of the integral equation Eq. (26) on $[a,x_1]$, we prove that the solution $y(x)$ is unique. Substituting them into Eq. (34) and using Lemma 1 with Lipschitz condition Eq. (13), we get

$$\begin{aligned} \|y(x) - z(x)\|_{C_{1-\gamma;\psi}[a,x_1]} &\leqslant \left\| I_{a+}^{\alpha;\psi} \left(\psi(x) - \psi(a)\right)^{\gamma-1} \right\|_{C_{1-\gamma;\psi}[a,x_1]} \\ &\quad \times \|f(x, y(x)) - f(x, z(x))\|_{C_{1-\gamma;\psi}[a,x_1]} \\ &\leqslant A \frac{\Gamma(\gamma)\left(\psi(x) - \psi(a)\right)^{\alpha}}{\Gamma(\gamma+\alpha)} \|y(x) - z(x)\|_{C_{1-\gamma;\psi}[a,x_1]}. \end{aligned} \tag{35}$$

From this it follows that $A \dfrac{\Gamma(\gamma)\left(\psi(x) - \psi(a)\right)^{\alpha}}{\Gamma(\gamma+\alpha)} \geqslant 1$, contradicting the condition Eq. (28). Thus there exists $y(x) = y_1(x) \in C_{1-\gamma;\psi}[a,x_1]$ as a unique solution on $[a,x_1]$.

Now, consider the interval $[x,x_2]$, where $x_2 = x_1 + h_1$, $h_1 > 0$ such that $x_2 < b$. Now the integral equation Eq. (35) takes the form

$$\begin{aligned} y(x) = &\frac{y_a}{\Gamma(\gamma)} \left(\psi(x) - \psi(a)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t, y(t))\, dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_{a}^{x_1} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t, y(t))\, dt, \end{aligned} \tag{36}$$

with $x \in [x_1, x_2]$.

As seen previously, $y(x)$ is uniquely defined on $[a,x_1]$ and the integral equation Eq. (36) is known and can be written as follows

$$y(x) = y_0^*(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t, y(t))\, dt, \tag{37}$$

with $x \in [x_1, x_2]$, where

$$y_0^*(x) = \frac{y_a}{\Gamma(\gamma)} \left(\psi(x) - \psi(a)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{a}^{x_1} \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t, y(t))\, dt \tag{38}$$

is a known function. Using the same argument as above, we deduce that there exist a unique solution $y(x) = y_2(x) \in C_{1-\gamma;\psi}[x_1, x_2]$ on $[x_1, x_2]$. Taking interval $[x_2, x_3]$, where $x_3 = x_2 + h_2$, $h_2 > 0$ such that $x_3 < b$, and repeating the above steps, we obtain a unique solution $y(x) \in C_{1-\gamma;\psi}[a,b]$, of the integral equation Eq. (26) such that $y(x) = y_j(x) \in C_{1-\gamma;\psi}\left[x_{j-1}, x_j\right]$, for $j = 1, 2, \ldots, l$ and $a = x_0 < x_2 < \cdots < x_l = b$. Using

differential equation Eq. (22) and Lipschitz condition Eq. (13), we obtain

$$\begin{aligned}\left\| {}^{H}D_{a+}^{\alpha,\beta;\psi} y_m(x) - {}^{H}D_{a+}^{\alpha,\beta;\psi} y(x) \right\|_{C_{1-\gamma;\psi}[a,b]} &= \| f(x, y_m(x)) - f(x, y(x)) \|_{C_{1-\gamma;\psi}[a,b]} \\ &\leqslant A \| y_m(x) - y(x) \|_{C_{1-\gamma;\psi}[a,b]} . \end{aligned} \tag{39}$$

Therefore, by Eq. (33) and Eq. (39) implies that ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} y(x) \in C_{1-\gamma;\psi}[a,b]$ and thus we conclude the result. □

5. Continuous dependence

Working with initial value problem in which it can models a physical phenomenon, it is desirable that any little perturbation in the initial data does not influence the solution. In this section, we study the data continuous dependence of the Cauchy problem solution for a fractional differential equation using ψ-Hilfer derivative via the generalized Gronwall inequality, obtained in Eq. (14).

THEOREM 5. *Let $f, \psi \in C([a,b], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$ and f satisfying Lipschitz condition* Eq. (13) *in $\mathbb{R}$. Let $\alpha > 0$, $\delta > 0$ such that $0 < \alpha - \delta < \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. For $a \leqslant x \leqslant h < b$, assume that y is the solution of the initial value problem* (IVP) Eq. (22) *and $\overset{*}{y}$ is the solution of the following* IVP

$$ {}^{H}\mathbb{D}_{a+}^{\alpha-\delta,\beta;\psi} \overset{*}{y}(x) = f\left(x, \overset{*}{y}(x)\right), \quad 0 < \alpha < 1, \quad 0 \leqslant \beta \leqslant 1 \tag{40}$$

$$ I_{a+}^{1-\gamma-\delta(\beta-1);\psi} \overset{*}{y}(x)_{x=a} = \overset{*}{y}_a, \quad \gamma = \alpha + \beta(1-\alpha). \tag{41}$$

Then, for $a < x \leqslant h$,

$$\left| \overset{*}{y}(x) - y(x) \right| \leqslant B(x) + \int_a^x \left[\sum_{n=1}^{\infty} \left(\frac{A}{\Gamma(\alpha)} \Gamma(\alpha - \delta) \right)^n \frac{(\psi(x) - \psi(t))^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))} B(t) \right] dt$$

holds, where

$$\begin{aligned} B(x) = &\left| \frac{\overset{*}{y}_a (\psi(x) - \psi(a))^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma + \delta(\beta-1))} - \frac{y_a (\psi(x) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \right| \\ &+ \| f \| \left| \frac{(\psi(x) - \psi(a))^{\alpha-\delta}}{\Gamma(\alpha - \delta + 1)} - \frac{(\psi(x) - \psi(a))^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} \right| \\ &+ \| f \| \left| \frac{(\psi(x) - \psi(a))^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(\psi(x) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)} \right| \end{aligned} \tag{42}$$

and

$$\| f \| = \max_{x \in [a,b]} | f(x, y(x)) | .$$

Proof. IVPs Eq. (22)–Eq. (23) and Eq. (40)–Eq. (41), have similar integral solutions and are given by

$$y(x)=\frac{y_a(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}f(t,y(t))\,dt$$

and

$$\overset{*}{y}(x)=\frac{\overset{*}{y}_a(\psi(x)-\psi(a))^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma+\delta(\beta-1))}+\frac{1}{\Gamma(\alpha-\delta)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-\delta-1}f(t,\overset{*}{y}(t))dt$$

respectively. It follows that

$$\begin{aligned}
&\left|\overset{*}{y}(x)-y(x)\right|\\
\leqslant&\left|\frac{\overset{*}{y}_a(\psi(x)-\psi(a))^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma+\delta(\beta-1))}-\frac{y_a(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\right|\\
&+\left|\frac{1}{\Gamma(\alpha-\delta)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-\delta-1}f\left(t,\overset{*}{y}(t)\right)dt\right.\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}f(t,y(t))\,dt\right|\\
=&\left|\frac{\overset{*}{y}_a(\psi(x)-\psi(a))^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma+\delta(\beta-1))}-\frac{y_a(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\right|\\
&+\left|\int_a^x \psi'(t)\left[\frac{(\psi(x)-\psi(t))^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}-\frac{(\psi(x)-\psi(t))^{\alpha-\delta-1}}{\Gamma(\alpha)}\right]f\left(t,\overset{*}{y}(t)\right)dt\right.\\
&+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-\delta-1}\left[f\left(t,\overset{*}{y}(t)\right)-f(t,y(t))\right]dt\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\left[(\psi(x)-\psi(t))^{\alpha-\delta-1}-(\psi(x)-\psi(t))^{\alpha-1}\right]f(t,y(t))\,dt\right|\\
\leqslant&\left|\frac{\overset{*}{y}_a(\psi(x)-\psi(a))^{\gamma+\delta(\beta-1)-1}}{\Gamma(\gamma+\delta(\beta-1))}-\frac{y_a(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\right|\\
&+\|f\|\left|\int_a^x \psi'(t)\left[\frac{(\psi(x)-\psi(t))^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}-\frac{(\psi(x)-\psi(t))^{\alpha-\delta-1}}{\Gamma(\alpha)}\right]dt\right|\\
&+\frac{A}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-\delta-1}\left|\overset{*}{y}(t)-y(t)\right|dt\\
&+\|f\|\left|\int_a^x \psi'(t)\left[\frac{(\psi(x)-\psi(t))^{\alpha-\delta-1}}{\Gamma(\alpha)}-\frac{(\psi(x)-\psi(t))^{\alpha-1}}{\Gamma(\alpha)}\right]dt\right|\\
=&\,B(x)+\frac{A}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-\delta-1}\left|\overset{*}{y}(t)-y(t)\right|dt
\end{aligned}$$

where $B(x)$ is defined by Eq. (42). Applying the Gronwall inequality Eq. (14), we conclude that

$$\left|\overset{*}{y}(x)-y(x)\right| \leqslant B(x)+\int_a^x \left[\sum_{n=1}^{\infty}\left(\frac{A\Gamma(\alpha-\delta)}{\Gamma(\alpha)}\right)^n \frac{\psi'(t)(\psi(x)-\psi(t))^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))}B(t)\right]dt.$$

□

Next, we consider the fractional differential equation Eq. (22) with small change in the initial condition Eq. (23)

$$I_{a+}^{1-\gamma;\psi}y(x)_{x=a}=y_a+\varepsilon,\ \gamma=\alpha+\beta(1-\alpha),$$

where ε is an arbitrary positive constant. We state and prove the result as follows.

THEOREM 6. *Suppose that assumptions of* Theorem 3 *hold. Suppose* $y(x)$ *and* $\overset{*}{y}(x)$ *are solutions of* IVP Eq. (22)–Eq. (23) *and* Eq. (40)–Eq. (41) *respectively. Then*

$$\left|y(x)-\overset{*}{y}(x)\right| \leqslant |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}\mathbb{E}_{\alpha,\gamma}\left[A(\psi(x)-\psi(a))^{\alpha}\right],\ x(a,b]$$

holds, where $\mathbb{E}_{\alpha,\gamma}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\gamma)}$ *is the Mittag-Leffler function with* $Re(\alpha)>0$.

Proof. By Theorem 3, we have $y(x)=\lim_{m\to\infty}y_m(x)$ with $y_0(x)$ as defined in Eq. (40) and Eq. (5.2), respectively. Clearly, we can write $\overset{*}{y}(x)=\lim_{m\to\infty}\overset{*}{y}_m(x)$ and

$$\overset{*}{y}_0(x)=\frac{(y_a+\varepsilon)}{\Gamma(\gamma)}(\psi(x)-\psi(a))^{\gamma-1}, \tag{43}$$

and

$$\overset{*}{y}_m(x)=\overset{*}{y}_0(x)+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}f\left(t,\overset{*}{y}_{m-1}(t)\right)dt.$$

From Eq. (26) and Eq. (43), we have

$$\begin{aligned}\left|y_0(x)-\overset{*}{y}_0(x)\right| &= \left|\frac{y_a}{\Gamma(\gamma)}(\psi(x)-\psi(a))^{\gamma-1}-\frac{(y_a+\varepsilon)}{\Gamma(\gamma)}(\psi(x)-\psi(a))^{\gamma-1}\right| \\ &\leqslant |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\end{aligned} \tag{44}$$

Using relations Eq. (30) and Eq. (43), the Lipschitz condition Eq. (13) and the inequality Eq. (44), we get

$$
\begin{aligned}
&\left|y_1(x)-\overset{*}{y}_1(x)\right| \\
&= \left|\frac{\varepsilon(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}(f(t,y_0(t))-f(t,\overset{*}{y}_0(t)))dt\right| \\
&\leqslant |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{A}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|y_0(t)-\overset{*}{y}_0(t)\right|dt \\
&\leqslant |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{A|\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}dt \\
&= |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+A|\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma+\alpha-1}}{\Gamma(\gamma+\alpha)}.
\end{aligned}
$$

From this it follows that

$$
\left|y_1(x)-\overset{*}{y}_1(x)\right| \leqslant |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}\sum_{j=0}^{1}A^j\frac{(\psi(x)-\psi(a))^{\alpha j}}{\Gamma(\gamma+\alpha j)}.
$$

On the other hand, we have

$$
\begin{aligned}
&\left|y_2(x)-\overset{*}{y}_2(x)\right| \\
&= \left|\frac{\varepsilon(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}(f(t,y_1(t))-f(t,\overset{*}{y}_1(t)))dt\right| \\
&\leqslant |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{A}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|y_1(t)-\overset{*}{y}_1(t)\right|dt \\
&\leqslant |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{A|\varepsilon|}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}(\psi(t)-\psi(a))^{\gamma-1} \\
&\quad\times\sum_{j=0}^{1}A^j\frac{(\psi(t)-\psi(a))^{\alpha j}}{\Gamma(\gamma+\alpha j)}dt \\
&= |\varepsilon|\frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}+\frac{A|\varepsilon|}{\Gamma(\alpha)}\sum_{j=0}^{1}\frac{A^j}{\Gamma(\gamma+\alpha j)} \\
&\quad\times\left\{\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}(\psi(t)-\psi(a))^{\gamma+\alpha j-1}dt\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant |\varepsilon| \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} + \frac{A|\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\alpha)\Gamma(\gamma)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\,dt \\
&\quad + \frac{A^2|\varepsilon|(\psi(x)-\psi(a))^{\gamma+\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+\gamma)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\,dt \\
&\leqslant |\varepsilon| \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} + \frac{A|\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}(\psi(x)-\psi(a))^{\alpha}}{\Gamma(\alpha+\gamma)} \\
&\quad + \frac{A^2|\varepsilon|(\psi(x)-\psi(a))^{\gamma-1}(\psi(x)-\psi(a))^{2\alpha}}{\Gamma(2\alpha+\gamma)} \\
&= |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1} \sum_{j=0}^{2} A^j \frac{(\psi(x)-\psi(a))^{\alpha j}}{\Gamma(\alpha j+\gamma)}.
\end{aligned}
$$

Using induction on j, we get

$$
\left| y_m(x) - \overset{*}{y}_m(x) \right| \leqslant |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1} \sum_{j=0}^{m} A^j \frac{(\psi(x)-\psi(a))^{\alpha j}}{\Gamma(\alpha j+\gamma)}. \tag{45}
$$

Taking the limit $m \to \infty$ in Eq. (45), we conclude that

$$
\begin{aligned}
\left| y_m(x) - \overset{*}{y}_m(x) \right| &\leqslant |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1} \sum_{j=0}^{m} A^j \frac{(\psi(x)-\psi(a))^{\alpha j}}{\Gamma(\alpha j+\gamma)} \\
&= |\varepsilon|(\psi(x)-\psi(a))^{\gamma-1} \mathbb{E}_{\alpha,\gamma}\left(A(\psi(x)-\psi(a))^{\alpha}\right). \quad \square
\end{aligned}
$$

6. Concluding remarks

In this paper, combining the fractional integral with respect to another ψ function and the classical Gronwall inequality [6], we proposed a generalized Gronwall inequality. For this inequality, we discussed the existence and uniqueness of solutions of the Cauchy-type problem by means of ψ-Hilfer fractional derivative recently introduced [20]. On the other hand, as application, the data continuous dependence of the Cauchy-type problem was studied via generalized Gronwall inequality.

Acknowledgement. The authors are grateful Espaço da Escrita, Coordenadoria Geral da Universidade – Unicamp, for the language services provided and also by the suggestions of the anonymous reviewers who improved the manuscript.

REFERENCES

[1] R. ALMEIDA, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonl. Sci. Numer. Simulat. **44** (2017) 460–481.

[2] R. ALMEIDA, *A Gronwall inequality for a general Caputo fractional operator*, Math. Inequal. Appl. **20**(4) (2017) 1089–1105.

[3] M. ASHYRALIYEV, *On Gronwall's type integral inequalities with singular kernels*, Filomat **31** (4) (2017) 1041–1049.

[4] R. COURANT, *Differential and integral calculus*, Vol. 2, John Wiley & Sons, New York, 2011.

[5] D. DHAIGUDE, S. P. BHAIRAT, *Existence and uniqueness of solution of Cauchy-type problem for Hilfer fractional differential equations*, (2017) arXiv:1704.02174.

[6] S. S. DRAGOMIR, *Some Gronwall Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2003.

[7] K. M. FURATI, M. KASSIM, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl. **64** (6) (2012) 1616–1626.

[8] Z. GONG, D. QIAN, C. LI, P. GUO, *On the Hadamard type fractional differential system*, Frac. Dyn. Control, (2012) 159–171.

[9] D. GUYOMAR, B. DUCHARNE, G. SEBALD, D. AUDIGER, *Fractional derivative operators for modeling the dynamic polarization behavior as a function of frequency and electric field amplitude*, IEEE transactions on ultrasonics, ferro electrics, and frequency control, **56** (3) (2009) 437–443.

[10] R. HERRMANN, *Fractional Calculus: An Introduction for Physicists*, World Scientific Publishing Company, Singapore, 2011.

[11] R. HILFER, Y. LUCHKO, Z. TOMOVSKI, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **12** (3) (2009) 299–318.

[12] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[13] A. A. KILBAS, S. A. MARZAN, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Diff. Equ. **41** (1) (2005) 84–89.

[14] A. KILBAS, S. MARZAN, *Cauchy problem for differential equation with Caputo derivative*, Fract. Cal. Appl. Anal., **7** (3) (2004) 297p–321p.

[15] E. L. LIMA, *Real Analysis (in portuguese)*, Rio de Janeiro: Instituto de Matemática Pura e Aplicada, CNPq, 2004.

[16] J. NANWARE, D. DHAIGUDE, *Existence and uniqueness of solution of Riemann-Liouville fractional differential equations with integral boundary conditions*, Int. J. Nonlinear Sci., **14** (4) (2012) 410–415.

[17] D. S. OLIVEIRA, E. CAPELAS DE OLIVEIRA, *Hilfer-Katugampola fractional derivatives*, Comput. Appl. Math., **37** (2018) 3672–3690.

[18] E. CAPELAS DE OLIVEIRA, J. VANTERLER DA C. SOUSA, *Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations*, Results in Math., **73** (3) (2018) 111.

[19] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional integrals and derivatives*, Theory and Applications, Gordon and Breach, Yverdon, 1993.

[20] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018) 72–91.

[21] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *Fractional order pseudoparabolic partial differential equation: Ulam–Hyers stability*, Bull. Braz. Math. Soc., New Series, (2018), `https://doi.org/10.1007/s00574-018-0112-x`.

[22] J. VANTERLER DA C. SOUSA, D. SANTOS OLIVEIRA, E. CAPELAS DE OLIVEIRA, *On the existence and stability for non-instantaneous impulsive fractional integrodifferential equation*, Math. Meth. Appl. Sci., (Accepted for publication) (2018).

[23] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *Existence, uniqueness, estimation and continuous dependence of the solutions of a nonlinear integral and an integrodifferential equations of fractional order*, `arXiv:1806.01441`, (2018).

[24] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation*, Appl. Math. Lett. **81** (2018) 50–56.

[25] J. VANTERLER DA C. SOUSA, KISHOR D. KUCCHE, E. CAPELAS DE OLIVEIRA, *Stability of ψ-Hilfer impulsive fractional differential equations*, Appl. Math. Lett. **88** (2019) 73–80.

[26] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator*, J. Fixed Point Theory Appl. **20** (3) (2018) 96.

[27] J. VANTERLER DA C. SOUSA, D. SANTOS OLIVEIRA, E. CAPELAS DE OLIVEIRA, *A note on the mild solutions of Hilfer impulsive fractional differential equations*, `arXiv:1811.09256`, (2018).

[28] F.-F. WANG, D.-Y. CHEN, X.-G. ZHANG, Y. WU, *The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay*, Appl. Math. Lett. **53** (2016) 45–51.

[29] J. WANG, Y. ZHOU, *Existence of mild solutions for fractional delay evolution systems*, Appl. Math. Comput. **218** (2) (2011) 357–367.

[30] R. WONG, *Asymptotic Approximations of Integrals*, SIAM. Philadelphia, 2001.

[31] H. YE, J. GAO, Y. DING, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2) (2007) 1075–1081.

(Received September 12, 2017)

José Vanterler da Costa Sousa
Department of Applied Mathematics
Institute of Mathematics, Statistics and Scientific Computation
University of Campinas – UNICAMP
rua Sérgio Buarque de Holanda 651, 13083–859, Campinas SP, Brazil
e-mail: ra160908@ime.unicamp.br

Edmundo Capelas de Oliveira
Department of Applied Mathematics
Institute of Mathematics, Statistics and Scientific Computation
University of Campinas – UNICAMP
rua Sérgio Buarque de Holanda 651, 13083–859, Campinas SP, Brazil
e-mail: capelas@ime.unicamp.br