Differential
Equations
& Applications
Volume 12, Number 2 (2020), 143–163 doi:10.7153/dea-2020-12-10

ALMOST PERIODIC HOMOGENIZATION OF THE KLEIN–GORDON TYPE EQUATION

LAZARUS SIGNING

(*Communicated by D. Hilhorst*)

Abstract. In this paper, the homogenization problem for the Klein-Gordon type equation is studied in the almost periodic setting. The propagation speed and the potential are spatial and time dependent almost periodically varying functions. One convergence theorem is proved and we derive the macroscopic homogenized model verified by the mean wave function.

1. Introduction

We consider a non empty smooth bounded open subset Ω of $\mathbb{R}^N_x$ (the N-numerical space $\mathbb{R}^N$ of variables $x=(x_1,\dots,x_N)$, where N is a given positive integer), and the real numbers T and ε with $T>0$ and $0<\varepsilon<1$. Let $f\in H^1_0(\Omega)\otimes W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$ and $g\in L^\infty\left(\mathbb{R}^N_y;\mathbb{R}\right)$, $W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$ being the Sobolev space of functions in $L^\infty\left(\mathbb{R}^N_y;\mathbb{R}\right)$ with their derivatives of order 1 ($H^1_0(\Omega)\otimes W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$ is the space of functions Φ of $\Omega\times\mathbb{R}^N$ into $\mathbb{R}$ of the form $\Phi=\sum_{finite}\varphi_i\psi_i$ with $\varphi_i\in H^1_0(\Omega)$ and $\psi_i\in W^{1,\infty}\left(\mathbb{R}^N_y;\mathbb{R}\right)$). Let us put

$$f^\varepsilon(x)=f\left(x,\frac{x}{\varepsilon}\right) \text{ and } g^\varepsilon(x)=g\left(\frac{x}{\varepsilon}\right) \text{ for } x\in\Omega.$$

The functions f^ε and g^ε above-mentioned belong to $H^1_0(\Omega)$ and $L^\infty(\Omega)$ respectively. Next, we consider the Cauchy-Dirichlet boundary value problem

$$\frac{\partial^2 u_\varepsilon}{\partial t^2}-\mathbf{div}\left(\mathbf{c}^\varepsilon\nabla u_\varepsilon\right)+\mathbf{w}^\varepsilon u_\varepsilon=0 \text{ in } \Omega\times]0,T[, \tag{1.1}$$

$$u_\varepsilon=0 \text{ on } \partial\Omega\times]0,T[, \tag{1.2}$$

$$u_\varepsilon(0)=\varepsilon f^\varepsilon \text{ in } \Omega, \tag{1.3}$$

$$\frac{\partial u_\varepsilon}{\partial t}(0)=g^\varepsilon \text{ in } \Omega, \tag{1.4}$$

where the square of the propagation speed of the light in the vacuum $\mathbf{c}$ and the potential $\mathbf{w}$ verify

$$\mathbf{c} \text{ and } \mathbf{w}\in\mathscr{B}^1\left(\mathbb{R}_\tau;L^\infty\left(\mathbb{R}^N_y\right)\right) \tag{1.5}$$

Mathematics subject classification (2010): 35B27, 35B40, 81Q05.
Keywords and phrases: Homogenization, Klein-Gordon type equation, sigma-convergence.

© ELEMENT, Zagreb
Paper DEA-12-10

and,

$$\mathbf{c}(y,\tau) \geqslant \alpha \text{ for all } \tau \in \mathbb{R} \tag{1.6}$$

and for almost all $y \in \mathbb{R}^N$, and where $\mathbf{c}^\varepsilon(x,t) = \mathbf{c}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$, $\mathbf{w}^\varepsilon(x,t) = \mathbf{w}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ ($(x,t) \in \Omega \times]0,T[$), $\mathbb{R}_\tau$ being the numerical space $\mathbb{R}$ of variables τ ($\mathscr{B}^1\left(\mathbb{R}_\tau; L^\infty\left(\mathbb{R}^N_y\right)\right)$ is the space of continuously differentiable functions of $\mathbb{R}$ into $L^\infty\left(\mathbb{R}^N_y\right)$ which are bounded with their derivatives). Now, for $t \in [0,T]$ let $a_\varepsilon(t;.,.)$ be the sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$ defined by

$$a_\varepsilon(t;u,v) = \int_\Omega \mathbf{c}^\varepsilon \nabla u \cdot \overline{\nabla v} dx + \int_\Omega \mathbf{w}^\varepsilon u \overline{v} dx \qquad \left(u, v \in H^1(\Omega)\right).$$

In view of (1.5)-(1.6), we see that for all $0 < \varepsilon < 1$,

$$\mathbf{c}^\varepsilon, \mathbf{w}^\varepsilon \in \mathscr{C}^1([0,T]; L^\infty(\Omega))$$

and

$$a_\varepsilon(t;v,v) + \|\mathbf{w}\|_\infty \|v\|^2_{L^2(\Omega)} \geqslant \alpha \|v\|^2_{H^1_0(\Omega)} \qquad \left(v \in H^1_0(\Omega)\right),$$

α being the constant in (1.6). Thus, the hypotheses of [10, Theorem 1.1, p. 294] are fulfilled. Therefore the initial boundary value problem (1.1)-(1.4) admits a unique solution u_ε in $\mathscr{C}\left([0,T]; H^1_0(\Omega)\right) \cap \mathscr{C}^1\left([0,T], L^2(\Omega)\right)$. The aim here is to investigate the limiting behaviour of u_ε solution of (1.1)-(1.4) when ε goes to zero, under the hypothesis that the coefficient $\mathbf{c}$ and the potential $\mathbf{w}$ vary almost periodically in time and space.

The homogenization problem for the linear Klein-Gordon type equation has been discussed in the book of Bensoussan, Lions and Papanicolaou [1] for the periodic setting using the asymptotic expansions. Later in 1992, Brahim-Otsmane, Francfort and Murat in [5] investigated the non-periodic case via the Γ-convergence techniques. For further results on this topic, one can refer to the book of Cioranescu and Donato [6]. This paper deals with the homogenization of an evolution hyperbolic problem with time-dependent coefficients via the *sigma-convergence*.

The model (1.1)-(1.4) under investigation in this paper is connected with the relativistic version of the Schrödinger type equation describing the motion of spinless particles.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notations. If X and F denote a locally compact space and a Banach space respectively, then we write $\mathscr{C}(X;F)$ for continuous mappings of X into F, and $\mathscr{B}(X;F)$ for those mappings in $\mathscr{C}(X;F)$ that are bounded. We shall assume $\mathscr{B}(X;F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x \in X} \|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathscr{C}(X) = \mathscr{C}(X;\mathbb{C})$ and $\mathscr{B}(X) = \mathscr{B}(X;\mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X;F)$ and $L^p_{loc}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1 \ldots dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results on the sigma-convergence of almost periodic structures, whereas in Section 3 one convergence theorem is established for (1.1)-(1.4).

2. Preliminaries

2.1. Almost periodic functions

2.1.1. Bohr almost periodic functions

DEFINITION 2.1. Let $u \in \mathscr{B}(\mathbb{R}^m)$ (m being a positive integer) and let $\varepsilon > 0$ be a real number. The vector $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ is said to be an *ε-period of u* if

$$\sup_{x \in \mathbb{R}^m} |u(x+s) - u(x)| \leqslant \varepsilon.$$

We denote by $E(u, \varepsilon)$ the set of all ε-periods of u.

DEFINITION 2.2. A function $u \in \mathscr{B}(\mathbb{R}^m)$ is said to be *almost periodic in the sense of Bohr* if for all $\varepsilon > 0$, there exists some $l = l(\varepsilon) = (l_1, \ldots, l_m) \in \mathbb{R}^m$ with $l_j > 0$ $(1 \leqslant j \leqslant m)$ such that for every $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$

$$([a_1, a_1 + l_1[\times [a_2, a_2 + l_2[\times \cdots \times [a_m, a_m + l_m[) \cap E(u, \varepsilon) \neq \varnothing.$$

EXAMPLE 1. For any $k = (k_1, \ldots, k_m) \in \mathbb{R}^m$ we set

$$\gamma_k(y) = \mathbf{e}^{2i\pi k \cdot y} \qquad (y \in \mathbb{R}^m).$$

The function γ_k is almost periodic in the sense of Bohr. Indeed, for all $y = (y_1, \ldots, y_m)$ and $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ we have

$$|\gamma_k(y+s) - \gamma_k(y)| = |\gamma_k(s) - 1|.$$

Further, for $\varepsilon > 0$ we put $l_j = \frac{2}{|k_j|}$ if $k_j \neq 0$ and $l_j = \varepsilon$ if $k_j = 0$ $(1 \leqslant j \leqslant m)$. For any $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ there exists $(n_1, \ldots, n_m) \in \mathbb{Z}^m$ such that

$$|k_j| a_j \leqslant n_j < |k_j| a_j + 2 \qquad (1 \leqslant j \leqslant m).$$

On the other hand, let $s = (s_1, \ldots, s_m)$ with $s_j = \frac{n_j}{|k_j|}$ if $k_j \neq 0$ and $s_j = a_j$ if $k_j = 0$. We have

$$\sup_{y \in \mathbb{R}^m} |\gamma_k(y+s) - \gamma_k(y)| = |\gamma_k(s) - 1| = 0 < \varepsilon.$$

Thus, $s \in ([a_1, a_1 + l_1[\times [a_2, a_2 + l_2[\times \cdots \times [a_m, a_m + l_m[) \cap E(u, \varepsilon)$.

PROPOSITION 2.1. *A function $u \in \mathscr{B}(\mathbb{R}^m)$ is almost periodic in the sense of Bohr if and only if the translations $\tau_a u$ $(a \in \mathbb{R}^m)$ form a relative compact set in $\mathscr{B}(\mathbb{R}^m)$ ($\tau_a u(y) = u(y+a)$ for all a and $y \in \mathbb{R}^m$).*

The proof of the preceding proposition can be found in the book of Besicovitch [2] and Guichardet [8] for $m=1$. The general case $m \geqslant 1$ is just a simple adaptation of the particular one $m=1$.

We denote by $AP(\mathbb{R}^m)$ the espace of Bohr's almost periodic functions on $\mathbb{R}^m$ which is a C^*-algebra with identity, the involution being the complex conjugation.

As usual, we denote by $\widehat{\mathbb{R}}^m$ the dual group of the additif group $\mathbb{R}^m$, that is, the group of all continuous homomorphisms of $\mathbb{R}^m$ into the unit circle $U=\{\xi \in \mathbb{C} : |\xi|=1\}$. Endowed with the topology of compact convergence on $\mathbb{R}^m$, $\widehat{\mathbb{R}}^m$ is a locally compact abelian group. The elements of $\widehat{\mathbb{R}}^m$ are the so-called continuous characters of $\mathbb{R}^m$. Let us remark that

$$\widehat{\mathbb{R}}^m = \{\gamma_k : k \in \mathbb{R}^m\}$$

where the functions γ_k are defined in Example 1. Moreover, the γ_k belong to $AP(\mathbb{R}^m)$, thus $\widehat{\mathbb{R}}^m \subset AP(\mathbb{R}^m)$.

Let $T(\mathbb{R}^m)$ be the algebra of all trigonometric polynomial on $\mathbb{R}^m$. $T(\mathbb{R}^m)$ consists of all functions u of the form

$$u(y) = \sum_{k \in R} c_k \gamma_k(y) \qquad (y \in \mathbb{R}^m),$$

where R is a finite subset (depending on u) of $\mathbb{R}^m$ and $c_k \in \mathbb{C}$ for all $k \in R$. Moreover, in virtue of the Bohr approximation theorem $T(\mathbb{R}^m)$ is dense in $AP(\mathbb{R}^m)$.

Let us state now the notion of the mean value on $AP(\mathbb{R}^m)$.

PROPOSITION 2.2. *For any $u \in AP(\mathbb{R}^m)$, the closed convex hull of $\{\tau_a u : a \in \mathbb{R}^m\}$ in $\mathscr{B}(\mathbb{R}^m)$ contains one and only one constant function whose value we denote by $M(u)$. Further, the mapping $u \to M(u)$ of $AP(\mathbb{R}^m)$ into $\mathbb{C}$ verifies the following properties:*

i) M is a positive linear form;

ii) M is continuous;

iii) $M(1)=1$;

iv) M is translation invariant, i.e., $M(\tau_a u) = M(u)$ for all $u \in AP(\mathbb{R}^m)$ and all $a \in \mathbb{R}^m$.

The proof of this proposition can be found in [8, Proposition 5.5].

REMARK 2.1. Let $\omega = (0,\dots,0)$ (the neutral element of $\mathbb{R}^m$). We have

$$M(\gamma_k) = 0,$$

for any $k \in \mathbb{R}^m \setminus \{\omega\}$ (γ_k is defined in Example 1). Indeed, by Proposition 2.2, for any $\eta > 0$ there exists some $\alpha_j > 0$ $(1 \leqslant j \leqslant n)$ with $\sum_{j=1}^n \alpha_j = 1$, and some $a_j \in \mathbb{R}^m$ $(1 \leqslant j \leqslant n)$ such that

$$\left| M(\gamma_k) - \sum_{j=1}^n \alpha_j \gamma_k(y + a_j) \right| < \eta, \tag{2.1}$$

for all $y \in \mathbb{R}^m$. Since $k = (k_1, \ldots, k_m) \neq \omega$, there exists some $j_0 \in \{1, \ldots, m\}$ such that $k_{j_0} \neq 0$. Further, for any $t \in \mathbb{R}$ we choose in (2.1) a particular $y = (y_1, \ldots, y_m)$ with $y_{j_0} = t$ and $y_j = 0$ for $j \neq j_0$. This leads to

$$\left| M(\gamma_k) - \mathbf{e}^{2i\pi k_{j_0} t} \sum_{l=1}^{n} \alpha_l \gamma_k(a_l) \right| < \eta,$$

for all $t \in \mathbb{R}$. The preceding inequality implies that

$$\left| M(\gamma_k) - \frac{1}{r} \int_0^r \mathbf{e}^{2i\pi k_{j_0} t} dt \left(\sum_{l=1}^{n} \alpha_l \gamma_k(a_l) \right) \right| < \eta,$$

for all $r > 0$. Taking the limit as $r \to +\infty$ in the preceding inequality, we obtain

$$|M(\gamma_k)| \leqslant \eta,$$

for all $\eta > 0$. Thus $M(\gamma_k) = 0$.

Now, let $u \in AP(\mathbb{R}^m)$ and $\varepsilon > 0$. We define $u^\varepsilon \in \mathscr{B}(\mathbb{R}^m)$ by

$$u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \qquad (x \in \mathbb{R}^m),$$

and we have the following proposition:

PROPOSITION 2.3. *For any $u \in AP(\mathbb{R}^m)$, u^ε converges to $M(u)$ in $L^\infty(\mathbb{R}^m_x)$-weak $*$ as $\varepsilon \to 0$.*

Proof. Let $\varphi \in L^1(\mathbb{R}^m_x)$. We have to check that as $\varepsilon \to 0$

$$\int_{\mathbb{R}^m} u^\varepsilon \varphi dx \to M(u) \int_{\mathbb{R}^m} \varphi dx. \tag{2.2}$$

To this end, thanks to the density of $T(\mathbb{R}^m)$ in $AP(\mathbb{R}^m)$, it is enough to show (2.2) for $u = \gamma_k$ (Example 1), k being arbitrary in $\mathbb{R}^m$. But,

$$\int_{\mathbb{R}^m} \gamma_k^\varepsilon \varphi dx = \mathscr{F}\varphi\left(\frac{-k}{\varepsilon}\right),$$

where $\mathscr{F}$ denotes the Fourier transformation on $\mathbb{R}^m$. Hence, the result follows by Remark 2.1 and the Riemann-Lebesgue lemma.

2.1.2. Stepanoff almost periodic functions

Let $p \in \mathbb{R}$ with $p \geqslant 1$, and let $Y = \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ with $m \in \mathbb{N}^*$. We define $(L^p, l^\infty) = (L^p, l^\infty)(\mathbb{R}^m)$ to be the space of all $u \in L^p_{loc}(\mathbb{R}^m)$ such that

$$\|u\|_{p,\infty} = \sup_{k \in \mathbb{Z}^m} \left[\int_{k+Y} |u(y)|^p dy \right]^{\frac{1}{p}} < +\infty.$$

This is a vector subspace of $L^p_{loc}(\mathbb{R}^m)$, and $\|.\|_{p,\infty}$ is a norm on (L^p, l^∞). Further, (L^p, l^∞) equipped with this norm is a Banach space (see [7]). One can easily verify that a function $u \in L^p_{loc}(\mathbb{R}^m)$ lies in (L^p, l^∞) if and only if the translations $\tau_a u$ $(a \in \mathbb{R}^m)$ form a bounded set in $L^p_{loc}(\mathbb{R}^m)$. Further, if B_N denotes the open unit ball of $\mathbb{R}^m$, we put

$$N_{p,\infty}(u) = \sup_{a\in\mathbb{R}^m} \left[\int_{B_N} |u(y+a)|^p \, dy\right]^{\frac{1}{p}}, \qquad (u \in (L^p, l^\infty)).$$

This is a norm on (L^p, l^∞), and $N_{p,\infty}$ is equivalent to $\|\cdot\|_{p,\infty}$.

DEFINITION 2.3. A function $u \in L^p_{loc}(\mathbb{R}^m)$ is said to be *almost periodic in Stepanoff sense* if u belongs to (L^p, l^∞) and if the translations $\tau_a u$ $(a \in \mathbb{R}^m)$ form a relatively compact set in (L^p, l^∞).

We denote by $L^p_{AP} = L^p_{AP}(\mathbb{R}^m)$ the set of all almost periodic functions in the sense of Stepanoff which is a closed vector subspace of (L^p, l^∞). We assume L^p_{AP} to be equipped with the norm $\|.\|_{p,\infty}$ which makes it a Banach space. Further, L^p_{AP} is the closure in (L^p, l^∞) of the set of trigonometric polynomials $T(\mathbb{R}^m)$ (see [7]). Therefore, $AP(\mathbb{R}^m)$ is dense in L^p_{AP} and we have the following proposition.

PROPOSITION 2.4. *The mean value $M : AP(\mathbb{R}^m) \to \mathbb{C}$ is extended to a unique continuous linear mapping, still denoted by M, of L^p_{AP} into $\mathbb{C}$. Moreover, M is positive and translation invariant on L^p_{AP}.*

PROPOSITION 2.5. *Let Ω be a bounded open set in $\mathbb{R}^m$. Let $u \in L^p_{AP}$ $(1 \leqslant p < +\infty)$. We put $u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right)$ for all $x \in \Omega$ and we have $u^\varepsilon \to M(u)$ in $L^p(\Omega)$-weak as $\varepsilon \to 0$.*

Before stating the next proposition we require a notation.
For $u \in L^p(\Omega; AP(\mathbb{R}^m))$ with $1 \leqslant p < +\infty$, we put

$$\widetilde{u}(x) = M(u(x)) \qquad (x \in \Omega). \tag{2.3}$$

This defines a function $\widetilde{u} \in L^p(\Omega)$, and in particular if $u \in \mathscr{C}\left(\overline{\Omega}; AP(\mathbb{R}^m)\right)$ then $\widetilde{u} \in \mathscr{B}(\Omega)$. Moreover, the mapping $u \to \widetilde{u}$ sends continuously and linearly $L^p(\Omega; AP(\mathbb{R}^m))$ into $L^p(\Omega)$ and $\mathscr{C}\left(\overline{\Omega}; AP(\mathbb{R}^m)\right)$ into $\mathscr{B}(\Omega)$, since M is a continuous linear form on $AP(\mathbb{R}^m)$ (see Proposition 2.2).

PROPOSITION 2.6. *Let $u \in \mathscr{C}\left(\overline{\Omega}; AP(\mathbb{R}^m)\right)$. For $\varepsilon > 0$, we put $u^\varepsilon(x) = u\left(x, \frac{x}{\varepsilon}\right)$ for all $x \in \overline{\Omega}$. This defines a function $u^\varepsilon \in \mathscr{B}(\Omega)$, and as $\varepsilon \to 0$, we have $u^\varepsilon \to \widetilde{u}$ in $L^\infty(\Omega)$-weak $*$.*

PROPOSITION 2.7. *Let $1 \leqslant p < +\infty$ and $u \in L^p(\Omega; AP(\mathbb{R}^m))$. Then, as $\varepsilon \to 0$, we have $u^\varepsilon \to \widetilde{u}$ in $L^p(\Omega)$-weak.*

The preceding three propositions have their proofs in [14] (see also [19, Subsection 3.1]).

2.2. Almost periodic homogenization algebras

Let $u \in AP(\mathbb{R}^m)$. We define $Sp(u) = \{k \in \mathbb{R}^m : M(\overline{\gamma}_k u) \neq 0\}$, where γ_k is given by Example 1. $Sp(u)$ is a countable subset of $\mathbb{R}^m$ (see [8, p. 92]), $Sp(u) = \emptyset$ (empty set) if and only if u is the null function on $\mathbb{R}^m$. The set $Sp(u)$ will be called the spectrum of u.

Now, let $\mathscr{R}$ be a countable subgroup of $\mathbb{R}^m$. We set

$$AP_{\mathscr{R}}(\mathbb{R}^m) = \{u \in AP(\mathbb{R}^m) : Sp(u) \subset \mathscr{R}\},$$

and we verify easily that: $AP_{\mathscr{R}}(\mathbb{R}^m)$ is a closed subalgebra of $\mathscr{B}(\mathbb{R}^m)$ which is separable with the supremum norm and contains the constants. Further, if $u \in AP_{\mathscr{R}}(\mathbb{R}^m)$ then $\overline{u} \in AP_{\mathscr{R}}(\mathbb{R}^m)$. Thus, $AP_{\mathscr{R}}(\mathbb{R}^m)$ (with the supremum norm) is a commutative B*-algebra with identity (the constant function 1 on $\mathbb{R}^m$), the involution being the usual one of complex conjugation.

Throughout the rest of this study we shall always assume that $AP_{\mathscr{R}}(\mathbb{R}^m)$ is equipped with the supremum norm.

Now, let S be a subgroup of $\mathbb{R}^m$. We define

$$S^* = \{k \in \mathbb{R}^m : k \cdot y \in \mathbb{Z} \text{ for all } y \in S\},$$

where the dot denotes the Euclidian inner product in $\mathbb{R}^m$. The set S^* is a closed subgroup of $\mathbb{R}^m$. Further, if S is a réseau in $\mathbb{R}^m$ (i.e. a discrete subgroup of rank m), so also is S^* ([4, VII, p.7, prop.5]). Similarly, we may define $(S^*)^*$ and we have $(S^*)^* = \overline{S}$ (see [4, VII, p.7, prop.6]). Thus, if S is closed then $(S^*)^* = S$. Let us denote by $P_S(\mathbb{R}^m)$ the set of functions $u \in \mathscr{B}(\mathbb{R}^m)$ which are S-periodic, i.e., $u(y+k) = u(y)$ for all $y \in \mathbb{R}^m$ and all $k \in S$, where S is a réseau in $\mathbb{R}^m$. The space $P_S(\mathbb{R}^m)$ is a commutative B^*-algebra with identity and we have $P_S(\mathbb{R}^m) = AP_{S^*}(\mathbb{R}^m)$ (see, e.g., [14] and [19] for more details).

DEFINITION 2.4. An *almost periodic homogenization algebra* on $\mathbb{R}^m$ is an algebra $A = AP_{\mathscr{R}}(\mathbb{R}^m)$, where $\mathscr{R}$ is a countable nontrivial subgroup of $\mathbb{R}^m$.

Now, let A be an almost periodic homogenization algebra on $\mathbb{R}^m$. We denote by $\Delta(A)$ the spectrum of A, i.e., the set of all non-zero linear forms $s : A \to \mathbb{C}$ such that $s(uv) = s(u)s(v)$ for all $u, v \in A$. We shall always assume that $\Delta(A)$ is endowed with the Gelfand topology. We recall that the Gelfand transformation on A is the mapping $\mathscr{G} : A \to \mathscr{C}(\Delta(A))$ defined by $\mathscr{G}(u)(s) = s(u)$ for all $u \in A$ and all $s \in \Delta(A)$. For any $u \in A$, $\mathscr{G}(u)$ is called the Gelfand transformation of u and is denoted $\widehat{u}$. We recall also the commutative Gelfand-Naimark Theorem (see [9, p.277]) which states that the Gelfand transformation on A is an isometric $*$-isomorphism of A onto $\mathscr{C}(\Delta(A))$. On the other hand the Gelfand transformation on A satisfies the following basic properties (see, e.g., [14], [17] and [19] for details):

(i) If $u \in A$ is real valued then $\mathscr{G}(u)$ is real valued, and further if $u \geqslant 0$ then $\mathscr{G}(u) \geqslant 0$.

(ii) Let $p > 0$ and $u \in A$. Then $|u|^p \in A$ and $\mathscr{G}(|u|^p) = |\mathscr{G}(u)|^p$.

(iii) If $f \in \mathscr{C}(\Delta(A))$ is real valued then $\mathscr{G}^{-1}(f)$ is real valued, and further if $f \geqslant 0$ then $\mathscr{G}^{-1}(f) \geqslant 0$ ($\mathscr{G}^{-1} : \mathscr{C}(\Delta(A)) \to A$ is the inverse isomorphism of $\mathscr{G}$).

We have the following proposition.

PROPOSITION 2.8. *There exists a unique Radon measure β on $\Delta(A)$ such that*

$$M(u) = \int_{\Delta(A)} \mathscr{G}(u)(s)\, d\beta(s) \qquad (u \in A).$$

Moreover, β is positive and of total mass 1.

Proof. Let β be the mapping $f \to M\left(\mathscr{G}^{-1}(f)\right)$ of $\mathscr{C}(\Delta(A))$ into $\mathbb{C}$. By the preceding properties (i) and (iii), and Proposition 2.2, we see that β is a positive Radon measure on $\Delta(A)$ and $\|\beta\| = 1$ (since $\mathscr{G}(1) = 1$). Moreover, for any $u \in A$ we have

$$M(u) = \beta(\mathscr{G}(u)) = \langle \beta, \mathscr{G}(u) \rangle = \int_{\Delta(A)} \mathscr{G}(u)(s)\, d\beta(s).$$

It is clear that β is unique.

We recall that $\mathscr{B}\left(\mathbb{R}_y^N\right)$, $\mathscr{B}(\mathbb{R}_\tau)$ and $\mathscr{B}\left(\mathbb{R}_y^N \times \mathbb{R}_\tau\right)$ denote respectively the spaces of bounded continuous complex functions on $\mathbb{R}_y^N$, $\mathbb{R}_\tau$ and $\mathbb{R}_y^N \times \mathbb{R}_\tau$. It is well known that the above spaces with the supremum norm and the usual algebra operations are commutative $\mathscr{C}^*$-algebras with identity (the involution is here the usual complex conjugation).

Throughout the rest of paper, A_y and A_τ denote respectively the almost periodic homogenization algebras on $\mathbb{R}_y^N$ and $\mathbb{R}_\tau$. Therefore $A_y = AP_{\mathscr{R}_y}\left(\mathbb{R}_y^N\right)$ and $A_\tau = AP_{\mathscr{R}_\tau}(\mathbb{R}_\tau)$ where $\mathscr{R}_y$ and $\mathscr{R}_\tau$ are respectively nontrivial countable subgroups of $\mathbb{R}_y^N$ and $\mathbb{R}_\tau$. Let us put $\mathscr{R} = \mathscr{R}_y \times \mathscr{R}_\tau$, and $A = AP_{\mathscr{R}}\left(\mathbb{R}_y^N \times \mathbb{R}_\tau\right)$. The almost periodic homogenization algebra A on $\mathbb{R}_y^N \times \mathbb{R}_\tau$, coincides with the closure of $AP_{\mathscr{R}_y}\left(\mathbb{R}_y^N\right) \otimes AP_{\mathscr{R}_\tau}(\mathbb{R}_\tau)$ in $\mathscr{B}\left(\mathbb{R}_y^N \times \mathbb{R}_\tau\right)$ (see [17, Proposition 3.2]). Moreover, for all $u \in A_y$ and all $v \in A_\tau$, we have $u^\varepsilon \to M(u)$ in $L^\infty\left(\mathbb{R}_x^N\right)$-weak $*$ and $v^\varepsilon \to M(v)$ in $L^\infty(\mathbb{R}_t)$-weak $*$ as $\varepsilon \to 0$ $(\varepsilon > 0)$, where:

$$u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \quad \left(x \in \mathbb{R}^N\right),$$

$$v^\varepsilon(t) = v\left(\frac{t}{\varepsilon}\right) \quad (t \in \mathbb{R}),$$

the mapping $u \to M(u)$ of A_y (resp. A_τ) into $\mathbb{C}$ being given by Proposition 2.2. Thus, $M(u \otimes v) = M(u)M(v)$ for all $u \in A_y$ and all $v \in A_\tau$, since for any $w \in A$, we have $w^\varepsilon \to M(w)$ in $L^\infty\left(\mathbb{R}_{(x,t)}^{N+1}\right)$-weak $*$ as $\varepsilon \to 0$ $(\varepsilon > 0)$ where

$$w^\varepsilon(x,t) = w\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \quad \left((x,t) \in \mathbb{R}^N \times \mathbb{R}\right).$$

We denote by $\Delta(A_y)$, $\Delta(A_\tau)$ and $\Delta(A)$ the spectra of A_y, A_τ and A respectively, and by $\mathscr{G}$ the Gelfand transformation on A_y, A_τ and A.

The appropriate measures on $\Delta(A_y)$, $\Delta(A_\tau)$ and $\Delta(A)$ are the positive Radon measures β_y, β_τ and β (of total mass 1) on $\Delta(A_y)$, $\Delta(A_\tau)$ and $\Delta(A)$ respectively (given by Proposition 2.8), such that $M(u) = \int_{\Delta(A_y)} \mathscr{G}(u)\, d\beta_y$ for $u \in A_y$, $M(v) = \int_{\Delta(A_\tau)} \mathscr{G}(v)\, d\beta_\tau$ for $v \in A_\tau$ and $M(w) = \int_{\Delta(A)} \mathscr{G}(w)\, d\beta$ for $w \in A$. Points in $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) are denoted by s (resp. s_0). Furthermore, we have $\Delta(A) = \Delta(A_y) \times \Delta(A_\tau)$ (Cartesian product) and $\beta = \beta_y \otimes \beta_\tau$ (see [17, Theorem 3.2 and Corollary 3.2] for more details).

The partial derivative of index i $(1 \leqslant i \leqslant N)$ on $\Delta(A_y)$ is defined to be the mapping $\partial_i = \mathscr{G} \circ D_{y_i} \circ \mathscr{G}^{-1}$ (usual composition) of

$$\mathscr{D}^1(\Delta(A_y)) = \left\{\varphi \in \mathscr{C}(\Delta(A_y)) : \mathscr{G}^{-1}(\varphi) \in A_y^1\right\}$$

into $\mathscr{C}(\Delta(A_y))$, where $A_y^1 = \left\{\psi \in \mathscr{C}^1\left(\mathbb{R}_y^N\right) : \psi,\ D_{y_i}\psi \in A_y\ (1 \leqslant i \leqslant N)\right\}$, $D_{y_i} = \frac{\partial}{\partial y_i}$. Generally, we define the partial derivative of index i $(0 \leqslant i \leqslant N)$ on $\Delta(A)$ as the mapping $\partial_0 = \mathscr{G} \circ \frac{\partial}{\partial \tau} \circ \mathscr{G}^{-1}$, or $\partial_i = \mathscr{G} \circ D_{y_i} \circ \mathscr{G}^{-1}$ $(1 \leqslant i \leqslant N)$ of

$$\mathscr{D}^1(\Delta(A)) = \left\{\varphi \in \mathscr{C}(\Delta(A)) : \mathscr{G}^{-1}(\varphi) \in A^1\right\}$$

into $\mathscr{C}(\Delta(A))$ with $A^1 = \left\{\psi \in \mathscr{C}^1\left(\mathbb{R}_y^N \times \mathbb{R}_\tau\right) : \psi,\ \frac{\partial \psi}{\partial \tau} \text{ and } D_{y_i}\psi \in A\ (1 \leqslant i \leqslant N)\right\}$. Higher order derivatives can be defined analogously (see [17]). Now, let A^∞ be the space of functions $\psi \in \mathscr{C}^\infty\left(\mathbb{R}_y^N \times \mathbb{R}_\tau\right)$ such that

$$D_{(y,\tau)}^\alpha \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \dots \partial y_N^{\alpha_N} \partial \tau^{\alpha_0}} \in A$$

for every multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_N) \in \mathbb{N}^{N+1}$ $(|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_N)$, and let

$$\mathscr{D}(\Delta(A)) = \left\{\varphi \in \mathscr{C}(\Delta(A)) : \mathscr{G}^{-1}(\varphi) \in A^\infty\right\}.$$

Endowed with a suitable locally convex topology (see [17]), A^∞ (resp. $\mathscr{D}(\Delta(A))$) is a Fréchet space and further, $\mathscr{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathscr{D}(\Delta(A))$. We have the following fundamental result proved in [14] (see also [19, Proposition 2.24]).

PROPOSITION 2.9. *For any $\varphi \in \mathscr{D}^1(\Delta(A))$ we have*

$$\int_{\Delta(A)} \partial_i \varphi(s, s_0)\, d\beta(s, s_0) = 0 \qquad (0 \leqslant i \leqslant N).$$

By a distribution on $\Delta(A)$ is understood any continuous linear form on $\mathscr{D}(\Delta(A))$. The space of all distributions on $\Delta(A)$ is then the topological dual, $\mathscr{D}'(\Delta(A))$, of $\mathscr{D}(\Delta(A))$. We endow $\mathscr{D}'(\Delta(A))$ with the strong dual topology.

Let us note that A^∞ is dense in A. Indeed, we have $A = AP_{\mathscr{R}}\left(\mathbb{R}^{N+1}\right)$, where $\mathscr{R}$ is a countable subgroup of $\mathbb{R}^{N+1}$, and moreover, $T_{\mathscr{R}} = \{\gamma_k : k \in \mathscr{R}\} \subset A^\infty$ and we know that $T_{\mathscr{R}}$ is total in A. Thus, $\mathscr{D}(\Delta(A))$ is dense in $\mathscr{C}(\Delta(A))$. Consequently, we have

$L^p(\Delta(A)) \subset \mathscr{D}'(\Delta(A))$ $(1 \leqslant p \leqslant \infty)$ with continuous embedding. Further, we may define

$$H^1(\Delta(A)) = \left\{u \in L^2(\Delta(A)) : \partial_i u \in L^2(\Delta(A)) \ \ (0 \leqslant i \leqslant N)\right\},$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative is defined in the classical case). This is a Hilbert space with norm

$$\|u\|_{H^1(\Delta(A))} = \left(\|u\|^2_{L^2(\Delta(A))} + \sum_{i=0}^{N} \|\partial_i u\|^2_{L^2(\Delta(A))}\right)^{\frac{1}{2}} \quad \left(u \in H^1(\Delta(A))\right).$$

However, in practice the appropriate space is not $H^1(\Delta(A))$ but its closed subspace

$$H^1(\Delta(A))/\mathbb{C} = \left\{u \in H^1(\Delta(A)) : \int_{\Delta(A)} u(s)\, d\beta(s) = 0\right\}$$

equipped with the seminorm

$$\|u\|_{H^1(\Delta(A))/\mathbb{C}} = \left(\sum_{i=0}^{N} \|\partial_i u\|^2_{L^2(\Delta(A))}\right)^{\frac{1}{2}} \quad \left(u \in H^1(\Delta(A))/\mathbb{C}\right).$$

Unfortunately, the pre-Hilbert space $H^1(\Delta(A))/\mathbb{C}$ is in general nonseparated and noncomplete. We introduce the separated completion, $H^1_{\#}(\Delta(A))$, of $H^1(\Delta(A))/\mathbb{C}$, and the canonical mapping J of $H^1(\Delta(A))/\mathbb{C}$ into its separated completion. We have the following proposition:

PROPOSITION 2.10. *(i) J is linear.*
(ii) $J\left(H^1(\Delta(A))/\mathbb{C}\right)$ is dense in $H^1_{\#}(\Delta(A))$.
(iii) $\|J(v)\|_{H^1_{\#}(\Delta(A))} = \|v\|_{H^1(\Delta(A))/\mathbb{C}}$ for all $v \in H^1(\Delta(A))/\mathbb{C}$.
(iv) Let ∂_i $(0 \leqslant i \leqslant N)$ be considered as a mapping of $H^1(\Delta(A))/\mathbb{C}$ into $L^2(\Delta(A))$. There exists a unique continuous linear operator, still denoted by ∂_i, of $H^1_{\#}(\Delta(A))$ into $L^2(\Delta(A))$ such that

$$\partial_i J(v) = \partial_i v \qquad \text{for all } v \in H^1(\Delta(A))/\mathbb{C}.$$

Further,

$$\|v\|_{H^1_{\#}(\Delta(A))} = \left(\sum_{i=0}^{N} \|\partial_i v\|^2_{L^2(\Delta(A))}\right)^{\frac{1}{2}} \qquad \text{for all } v \in H^1_{\#}(\Delta(A)).$$

See [17] (and in particular Remark 2.4 and Proposition 2.6 there) for more details. As a consequence of Proposition 2.10 we have:

PROPOSITION 2.11. *(i) $J(\mathscr{D}(\Delta(A))/\mathbb{C})$ is dense in $H^1_{\#}(\Delta(A))$, where*

$$\mathscr{D}(\Delta(A))/\mathbb{C} = \left\{v \in \mathscr{D}(\Delta(A)) : \int_{\Delta(A)} v d\beta = 0\right\}.$$

(ii) $\int_{\Delta(A)} \partial_i v d\beta = 0$ $\quad (0 \leqslant i \leqslant N)$ for all $v \in H^1_{\#}(\Delta(A))$.

2.3. The Σ-convergence

Let us first introduce some basic notations. The letter E throughout will denote a family of real numbers $0<\varepsilon<1$ admitting 0 as an accumulation point. For example, E may be the whole interval $(0,1)$; E may also be an ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0<\varepsilon_n<1$ and $\varepsilon_n\to 0$ as $n\to\infty$. In the latter case E will be referred to as a *fundamental sequence.* On the other hand, let $Q=\Omega\times]0,T[$. For any real $0<\varepsilon<1$, we define u^ε as

$$u^\varepsilon(x,t)=u\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\ \ ((x,t)\in Q)$$

for $u\in L^1_{loc}\left(\mathbb{R}^N_y\times\mathbb{R}_\tau\right)$, as is customary in homogenization theory. More generally, for $u\in L^1_{loc}\left(Q\times\mathbb{R}^N_y\times\mathbb{R}_\tau\right)$, it is customary to put

$$u^\varepsilon(x,t)=u\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\ \ ((x,t)\in Q)$$

whenever the right-hand side makes sense (see, e.g., [14] and [16]). Now, let $1\leqslant p<+\infty$.

DEFINITION 2.5. A sequence $(u_\varepsilon)_{\varepsilon\in E}\subset L^p(Q)$ is said to:

(i) *weakly* Σ-converge in $L^p(Q)$ to some $u_0\in L^p(Q\times\Delta(A))=L^p(Q;L^p(\Delta(A)))$ if as $E\ni\varepsilon\to 0$,

$$\int_Q u_\varepsilon(x,t)\,\psi^\varepsilon(x,t)\,dxdt\to\int\int_{Q\times\Delta(A)}u_0(x,t,s,s_0)\,\widehat{\psi}(x,t,s,s_0)\,dxdtd\beta(s,s_0)\tag{2.4}$$

for all $\psi\in L^{p'}(Q;A)$ $\left(\frac{1}{p'}=1-\frac{1}{p}\right)$, where $\psi^\varepsilon(x,t)=\psi\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)$ and $\widehat{\psi}(x,t,\ldots)=\mathscr{G}(\psi(x,t,\ldots))$ a.e. in $(x,t)\in Q$;

(ii) *strongly* Σ-converge in $L^p(Q)$ to some $u_0\in L^p(Q\times\Delta(A))$ if the following property is verified:

$$\begin{cases}\textit{Given } \eta>0 \textit{ and } v\in L^p(Q;A) \textit{ with}\\ \|u_0-\widehat{v}\|_{L^p(Q\times\Delta(A))}\leqslant\frac{\eta}{2}, \textit{there is some } \alpha>0\\ \textit{such that } \|u_\varepsilon-v^\varepsilon\|_{L^p(Q)}\leqslant\eta \textit{ provided } E\ni\varepsilon\leqslant\alpha,\end{cases}\tag{2.5}$$

where $v^\varepsilon(x,t)=v\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)$ and $\widehat{v}(x,t,.,.)=\mathscr{G}(v(x,t,.,.))$ a.e. in $(x,t)\in Q$.

We will briefly express weak and strong Σ-convergence by writing $u_\varepsilon\to u_0$ in $L^p(Q)$-weak Σ and $u_\varepsilon\to u_0$ in $L^p(Q)$-strong Σ, respectively.

REMARK 2.2. The existence of such v's as in (ii) results from the density of $L^p(Q;\mathscr{C}(\Delta(A)))$ in $L^p(Q;L^p(\Delta(A)))$.

Let us now introduce some basic spaces. We denote by $\Xi^p\left(\mathbb{R}^N\right)$ the space of functions $u\in L^p_{loc}\left(\mathbb{R}^N_y\right)$ such that

$$\|u\|_{\Xi^p}=\sup_{0<\varepsilon\leqslant 1}\left(\int_{B_N}\left|u\left(\frac{x}{\varepsilon}\right)\right|^p dx\right)^{\frac{1}{p}}<\infty,$$

where B_N denotes the open unit ball in $\mathbb{R}^N$. Ξ^p is a complex vector space, and the mapping $u \to \|u\|_{\Xi^p}$, denoted by $\|.\|_{\Xi^p}$, is a norm on Ξ^p which makes it a Banach space.

We define $\mathfrak{X}^p_{\mathscr{R}_y}$ and $\mathfrak{X}^p_{\mathscr{R}}$ to be the closure of A_y and A in $\Xi^p\left(\mathbb{R}^N\right)$ and $\Xi^p\left(\mathbb{R}^{N+1}\right)$ respectively. We provide $\mathfrak{X}^p_{\mathscr{R}_y}$ (resp. $\mathfrak{X}^p_{\mathscr{R}}$) with the $\Xi^p\left(\mathbb{R}^N\right)$-norm (resp. $\Xi^p\left(\mathbb{R}^{N+1}\right)$-norm), which makes it a Banach space.

REMARK 2.3. Any function $u \in \mathfrak{X}^p_{\mathscr{R}_y}$ can be considered as a function in $\mathfrak{X}^p_{\mathscr{R}}$ which is independent of the variable τ. Indeed, let $u \in \mathfrak{X}^p_{\mathscr{R}_y}$ and $\eta > 0$. There exists a function $v \in A_y$ such that

$$\|u - v\|_{\Xi^p\left(\mathbb{R}^N\right)} \leqslant \frac{\eta}{2}, \text{ i.e., } \sup_{0<\varepsilon\leqslant 1}\left(\int_{B_N} |u^\varepsilon - v^\varepsilon|^p\, dy\right)^{\frac{1}{p}} \leqslant \frac{\eta}{2},$$

but $v = v \otimes 1 \in A_y \otimes A_\tau \subset A$ and

$$\int_{B_{N+1}} |u^\varepsilon - v^\varepsilon|^p\, dy d\tau \leqslant 2\int_{B_N} |u^\varepsilon - v^\varepsilon|^p\, dy \leqslant 2\,\|u - v\|_{\Xi^p\left(\mathbb{R}^N\right)}.$$

It follows from the preceding inequalities that $u \in \Xi^p\left(\mathbb{R}^{N+1}\right)$ and $\|u - v\|_{\Xi^p\left(\mathbb{R}^{N+1}\right)} \leqslant \eta$.

Let us also note that, if $\mathscr{R} = \mathscr{R}_y \times \mathscr{R}_\tau$ with $\mathscr{R}_y = \mathbb{Z}^N$ and $\mathscr{R}_\tau = \mathbb{Z}$ then $A = \mathscr{C}_{per}(Y \times Z)$, the space of all $Y \times Z$-periodic continuous complex functions on $\mathbb{R}^N_y \times \mathbb{R}_\tau$ (with $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$ and $Z = \left(-\frac{1}{2}, \frac{1}{2}\right)$), and we have $\mathfrak{X}^p_{\mathscr{R}} = L^p_{per}(Y \times Z)$ (see, e.g., [19, Remark 2.21]).

REMARK 2.4. It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ, then (2.4) holds for $\psi \in \mathscr{C}\left(\overline{Q}; \mathfrak{X}^{p'}_\infty\right)$, where $\mathfrak{X}^{p'}_\infty = \mathfrak{X}^{p'}_{\mathscr{R}} \cap L^\infty\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right)$. See [19, Proposition 3.7] for the proof.

Now, let us introduce the space $L^p_{AP,\mathscr{R}}\left(\mathbb{R}^N\right)$ $(1 \leqslant p < +\infty)$ for any countable subgroup $\mathscr{R}$ of $\mathbb{R}^N$. To begin, we note that the notion of a spectrum introduced in Subsection 2.2 extends naturally to $L^p_{AP}\left(\mathbb{R}^N\right)$ by virtue of Proposition 2.4. Furthermore, since $AP\left(\mathbb{R}^N\right)$ is dense in $L^p_{AP}\left(\mathbb{R}^N\right)$, and since each function in $AP\left(\mathbb{R}^N\right)$ has a countable spectrum, we see that $Sp(u)$ is countable for all $u \in L^p_{AP}\left(\mathbb{R}^N\right)$. We define

$$L^p_{AP,\mathscr{R}}\left(\mathbb{R}^N\right) = \left\{u \in L^p_{AP}\left(\mathbb{R}^N\right) : Sp(u) \subset \mathscr{R}\right\}.$$

We see that $L^p_{AP,\mathscr{R}}\left(\mathbb{R}^N\right) = \cap_{k\in\mathbb{R}^N\setminus\mathscr{R}} M_k^{-1}(\{0\})$, where M_k is the continuous linear form on $L^p_{AP}\left(\mathbb{R}^N\right)$ defined by $M_k(u) = M(\overline{\gamma}_k u)$ for any $u \in L^p_{AP}\left(\mathbb{R}^N\right)$. Thus, $L^p_{AP,\mathscr{R}}\left(\mathbb{R}^N\right)$ is a closed vector subspace of $L^p_{AP}\left(\mathbb{R}^N\right)$. Moreover, we have the following proposition proved in [14] (see also [20, Subsection 4.1, Problem II]).

PROPOSITION 2.12. *$L^p_{AP,\mathscr{R}}(\mathbb{R}^N)$ is the closure of $AP_{\mathscr{R}}(\mathbb{R}^N)$ in $(L^p,l^\infty)(\mathbb{R}^m)$ and $L^p_{AP,\mathscr{R}}(\mathbb{R}^N)$ is continuously embedded in $\mathfrak{X}^p_{\mathscr{R}}$ ($L^p_{AP,\mathscr{R}}(\mathbb{R}^N)$ equipped by the (L^p,l^∞)-norm), $\mathfrak{X}^p_{\mathscr{R}}$ being the closure of $AP_{\mathscr{R}}(\mathbb{R}^N)$ in $\Xi^p(\mathbb{R}^N)$.*

Instead of repeating here the main results underlying Σ-convergence theory for almost periodic structures, we find it more convenient to draw the reader's attention to a few references, see, e.g., [14], [15], [16], [17], [18], [19] and [20].

However, we recall below two fundamental results. First of all, let

$$\mathscr{Y}(0,T)=\left\{v\in L^2\left(0,T;H^1_0(\Omega;\mathbb{R})\right):v'\in L^2\left(0,T;H^{-1}(\Omega;\mathbb{R})\right)\right\}.$$

$\mathscr{Y}(0,T)$ is provided with the norm

$$\|v\|_{\mathscr{Y}(0,T)}=\left(\|v\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)}+\|v'\|^2_{L^2\left(0,T;H^{-1}(\Omega)\right)}\right)^{\frac{1}{2}}\qquad (v\in\mathscr{Y}(0,T))$$

which makes it a Hilbert space.

THEOREM 2.1. *Assume that $1<p<\infty$ and further E is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon\in E}$ be bounded in $L^p(Q)$. Then, a subsequence E' can be extracted from E such that $(u_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges in $L^p(Q)$.*

THEOREM 2.2. *Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $H^1(Q)$. Then, a subsequence E' can be extracted from E such that, as $E'\ni\varepsilon\to 0$,*

$$u_\varepsilon\to u_0 \text{ in } L^2(Q)\text{-weak } \Sigma,$$

$$\frac{\partial u_\varepsilon}{\partial x_j}\to\frac{\partial u_0}{\partial x_j}+\partial_j u_1 \text{ in } L^2(Q)\text{-weak } \Sigma\ (1\leqslant j\leqslant N)$$

and

$$\frac{\partial u_\varepsilon}{\partial t}\to\frac{\partial u_0}{\partial t}+\partial_0 u_1,$$

where $u_0\in H^1\left(Q;L^2(\Delta(A))\right)$ and $u_1\in L^2\left(Q;H^1_\#(\Delta(A))\right)$.

THEOREM 2.3. *Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $\mathscr{Y}(0,T)$. Then, a subsequence E' can be extracted from E such that, as $E'\ni\varepsilon\to 0$,*

$$u_\varepsilon\to u_0 \text{ in } \mathscr{Y}(0,T)\text{-weak},$$

$$u_\varepsilon\to u_0 \text{ in } L^2(Q)\text{-strong},$$

$$\frac{\partial u_\varepsilon}{\partial x_j}\to\frac{\partial u_0}{\partial x_j}+\partial_j u_1 \text{ in } L^2(Q)\text{-weak } \Sigma\ (1\leqslant j\leqslant N)$$

where $u_0\in\mathscr{Y}(0,T)$ and $u_1\in L^2\left(Q;L^2\left(\Delta(A_\tau);H^1_\#(\Delta(A_y))\right)\right)$.

The proof of Theorem 2.1 and Theorem 2.2 can be found in, e.g., [14], [18] and [19] whereas Theorem 2.3 has its proof in, e.g., [20] in a general deterministic setting.

3. A convergence result for (1.1)-(1.4)

In the sequel, we suppose that the square of the propagation speed of the light in the vacuum $\mathbf{c}$, and the potential $\mathbf{w}$ verify

$$\mathbf{c} \text{ and } \mathbf{w} \in L^2_{AP}\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right). \tag{3.1}$$

Moreover, we make the assumption that the initial values in (1.1)-(1.4) verify

$$f \in \mathscr{C}\left(\overline{\Omega}; L^\infty\left(\mathbb{R}^N_y\right)\right); \qquad f(x,.) \in L^2_{AP}\left(\mathbb{R}^N_y\right) \tag{3.2}$$

for all $x \in \overline{\Omega}$ and

$$g \in L^2_{AP}\left(\mathbb{R}^N_y\right). \tag{3.3}$$

Having made the assumptions (3.1)-(3.3), its follows that there exists a countable subgroup $\mathscr{R}^0$ of $\mathbb{R}^N_y \times \mathbb{R}_\tau$ containing $Sp(\mathbf{c})$ and $Sp(\mathbf{w})$. Further, in virtue of [14, Proposition 5.1] (see also [20, Proposition 4.1]) there exists a countable subgroup $\mathscr{R}'$ of $\mathbb{R}^N_y$ such that

$$g \in L^2_{AP,\mathscr{R}'}\left(\mathbb{R}^N_y\right) \text{ and } f(x,.) \in L^2_{AP,\mathscr{R}'}\left(\mathbb{R}^N_y\right)$$

for any $x \in \overline{\Omega}$. Let $\mathscr{R}^1$ be the subgroup of $\mathbb{R}^N_y \times \mathbb{R}_\tau$ spanned by $\mathscr{R}^0 \cup (\mathscr{R}' \times \{0\})$. We have $\mathscr{R}^1 \subset \mathscr{R}_y \times \mathscr{R}_\tau$ where $\mathscr{R}_y$ and $\mathscr{R}_\tau$ are respectively the projections of $\mathscr{R}^1$ on $\mathbb{R}^N_y$ and $\mathbb{R}_\tau$. Thus, in view of Proposition 2.12 we have

$$\mathbf{c},\ \mathbf{w} \in \mathfrak{X}^2_{\mathscr{R}} \tag{3.4}$$

with $\mathscr{R} = \mathscr{R}_y \times \mathscr{R}_\tau$;

$$g \in \mathfrak{X}^2_{\mathscr{R}_y}, \text{ and } f(x,.) \in \mathfrak{X}^2_{\mathscr{R}_y} \tag{3.5}$$

for all $x \in \overline{\Omega}$.

Before to state with some estimates of the solution to (1.1)-(1.4), let us recall the following regularity results due to Lions-Magenes [11, Chapitre 5, Théorème 2.1].

PROPOSITION 3.1. *Suppose that the initial data of (1.1)-(1.4) verify*

$$f \in \left(H^1_0(\Omega) \cap H^2(\Omega)\right) \otimes W^{2,\infty}\left(\mathbb{R}^N_y; \mathbb{R}\right) \text{ and } g \in W^{2,\infty}\left(\mathbb{R}^N_y; \mathbb{R}\right),$$

where $W^{2,\infty}\left(\mathbb{R}^N_y; \mathbb{R}\right)$ is the Sobolev space of functions in $L^\infty\left(\mathbb{R}^N_y; \mathbb{R}\right)$ with their derivatives of order $\leqslant 2$. Then, for any $\varepsilon > 0$, the solution of (1.1)-(1.4) verifies

$$u'_\varepsilon \in L^2\left(0,T; H^1_0(\Omega)\right) \text{ and } u''_\varepsilon \in L^2\left(0,T; L^2(\Omega)\right).$$

Let us state some estimates. We denote by $c(\Omega)$ the constant in the Poincaré inequality and we have the following proposition.

PROPOSITION 3.2. *Suppose that the hypotheses of Proposition 3.1 are satisfied and*

$$\alpha > c(\Omega)^2 \|\mathbf{w}\|_\infty, \tag{3.6}$$

α being the constant in (1.6). Then, for any $0<\varepsilon<1$, the solution u_ε of (1.1)-(1.4) verifies

$$\|u_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leqslant C \text{ and } \left\|u_\varepsilon'\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \leqslant C, \tag{3.7}$$

where $C>0$ is a constant independent of ε.

Proof. For all $u,v\in H_0^1(\Omega)$, we put

$$a_\varepsilon'(t;u,v)=\frac{d}{dt}a_\varepsilon(t;u,v) \qquad (t\in[0,T])$$

for any $\varepsilon>0$. Multiplying (1.1) by $u_\varepsilon'(t)$ and taking two times the real part of the obtained equation lead to

$$\frac{d}{dt}\left\|u_\varepsilon'(t)\right\|_{L^2(\Omega)}^2+a_\varepsilon\left(t;u_\varepsilon(t),u_\varepsilon'(t)\right)+a_\varepsilon\left(t;u_\varepsilon'(t),u_\varepsilon(t)\right)=0 \qquad (t\in[0,T]),$$

i.e.,

$$\frac{d}{dt}\left(\left\|u_\varepsilon'(t)\right\|_{L^2(\Omega)}^2+a_\varepsilon\left(t;u_\varepsilon(t),u_\varepsilon(t)\right)\right)=a_\varepsilon'\left(t;u_\varepsilon(t),u_\varepsilon(t)\right) \qquad (t\in[0,T]).$$

Integrating the preceding equality and using (1.6) lead to

$$\begin{aligned}&\|u_\varepsilon'(t)\|_{L^2(\Omega)}^2+\left(\alpha-c(\Omega)^2\|\mathbf{w}\|_\infty\right)\|u_\varepsilon(t)\|_{H_0^1(\Omega)}^2\\ &\leqslant \|g^\varepsilon\|_{L^2(\Omega)}^2+c_0\varepsilon^2\|f^\varepsilon\|_{H_0^1(\Omega)}^2+c_1\int_0^t\|u_\varepsilon(s)\|_{H_0^1(\Omega)}^2\,ds\end{aligned} \tag{3.8}$$

for all $t\in[0,T]$, where $c_0>0$ and $c_1>0$ are constants independent of ε. Let us note that $\|g^\varepsilon\|_{L^2(\Omega)}^2+c_0\varepsilon^2\|f^\varepsilon\|_{H_0^1(\Omega)}^2$ is uniformly bounded with respect to $0<\varepsilon<1$. Thus, there exists a constant $c_2>0$ such that

$$\|g^\varepsilon\|_{L^2(\Omega)}^2+c_0\varepsilon^2\|f^\varepsilon\|_{H_0^1(\Omega)}^2\leqslant c_2 \qquad (0<\varepsilon<1).$$

Therefore, in view of (3.6), it follows from (3.8) that there exists a constant $K>0$ such that

$$\left\|u_\varepsilon'(t)\right\|_{L^2(\Omega)}^2+\|u_\varepsilon(t)\|_{H_0^1(\Omega)}^2\leqslant Kc_2+Kc_1\int_0^t\left(\left\|u_\varepsilon'(s)\right\|_{L^2(\Omega)}^2+\|u_\varepsilon(s)\|_{H_0^1(\Omega)}^2\right)ds$$

for all $t\in[0,T]$. Thus, by the Gronwall Lemma the preceding inequality leads to

$$\left\|u_\varepsilon'(t)\right\|_{L^2(\Omega)}^2+\|u_\varepsilon(t)\|_{H_0^1(\Omega)}^2\leqslant Kc_2\exp(Kc_1t)$$

for all $t\in[0,T]$, and (3.7) follows.

Now, let us introduce some functions spaces.

We consider the space

$$\mathscr{W}(0,T)=\left\{v\in L^2\left(0,T;H_0^1(\Omega)\right):\frac{\partial v}{\partial t}\in L^2(Q)\right\}$$

which is a Hilbert space with the norm

$$\|v\|_{\mathscr{W}(0,T)} = \left(\|v\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)} + \left\| \frac{\partial v}{\partial t} \right\|^2_{L^2(Q)} \right)^{\frac{1}{2}} \qquad (v \in \mathscr{W}(0,T)).$$

Further, we set

$$\mathbb{F}^1_0 = \mathscr{W}(0,T) \times L^2\left(Q; H^1_{\#}\left(\Delta(A_y) \times \Delta(A_\tau)\right)\right)$$

provided with the norm

$$\|\mathbf{u}\|_{\mathbb{F}^1_0} = \left(\|u_0\|^2_{\mathscr{W}(0,T)} + \|u_1\|^2_{L^2\left(Q;H^1_{\#}\left(\Delta(A_y)\times\Delta(A_\tau)\right)\right)} \right)^{\frac{1}{2}} \qquad \left(\mathbf{u} = (u_0,u_1) \in \mathbb{F}^1_0\right),$$

which makes it Hilbert space. We consider also the space

$$\mathscr{F}^\infty_0 = \mathscr{D}(Q) \times \left[\mathscr{D}(Q) \otimes J\left(\mathscr{D}(\Delta(A))/\mathbb{C}\right)\right]$$

which is a dense subspace of $\mathbb{F}^1_0$. For $\mathbf{u} = (u_0,u_1)$ and $\mathbf{v} = (v_0,v_1) \in H^1_0(\Omega) \times L^2\left(\Omega; H^1_{\#}\left(\Delta(A_y)\times\Delta(A_\tau)\right)\right)$, we set

$$\mathfrak{a}(\mathbf{u},\mathbf{v}) = \sum_{i=1}^N \iiint_{\Omega\times\Delta(A)} \widehat{\mathbf{c}} \left(\frac{\partial u_0}{\partial x_i} + \partial_i u_1 \right) \left(\overline{\frac{\partial v_0}{\partial x_i}} + \overline{\partial_i v_1} \right) dx d\beta_y d\beta_\tau + M(\mathbf{w}) \int_\Omega u_0 \overline{v_0} dx,$$

with of course $M(\mathbf{w}) = \int\int_{\Delta(A_y)\times\Delta(A_\tau)} \widehat{\mathbf{w}} d\beta_y d\beta_\tau$. This defines a sesquilinear hermitian form on $\left[H^1_0(\Omega) \times L^2\left(\Omega; H^1_{\#}\left(\Delta(A_y)\times\Delta(A_\tau)\right)\right)\right]^2$ which is continuous. Further, we have the following result.

THEOREM 3.1. *For any $0 < \varepsilon < 1$, let u_ε be the unique solution to (1.1)-(1.4). Suppose that the hypotheses of Propositions 3.1 and 3.2 are satisfied. Then, given a fundamental sequence E, there exists a subsequence E' extracted from E and functions $u_0 \in \mathscr{Y}(0,T)$ and $u_1 \in L^2\left(Q; H^1_{\#}\left(\Delta(A_y)\times\Delta(A_\tau)\right)\right)$ such that as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } \mathscr{Y}(0,T)\text{-weak}, \tag{3.9}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \text{ in } L^2(Q)\text{-weak } \Sigma \quad (1 \leqslant j \leqslant N) \tag{3.10}$$

and

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} + \partial_0 u_1 \text{ in } L^2(Q)\text{-weak } \Sigma. \tag{3.11}$$

Further $\mathbf{u} = (u_0,u_1)$ verifies the variational equation

$$\begin{cases} \mathbf{u} = (u_0,u_1) \in \mathbb{F}^1_0,\ u_0(0) = 0 \text{ and } u_0'(0) = \widetilde{g}; \\ -\int_0^T (u_0'(t), v_0'(t))\, dt - \int\int_{Q\times\Delta(A)} \partial_0 u_1(x,t) \overline{\partial_0 v_1(x,t)} dx dt d\beta + \int_0^T \mathfrak{a}(\mathbf{u}(t),\mathbf{v}(t))\, dt = 0 \end{cases} \tag{3.12}$$

for all $\mathbf{v} = (v_0,v_1) \in \mathbb{F}^1_0$, where $\widetilde{g}(x,t) = \int_{\Delta(A)} \widehat{g}(x,t)\, d\beta \quad ((x,t) \in Q)$ ($(,)$ denotes the scalar product in $L^2(\Omega)$ as well as the duality pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$).

Proof. According to (3.7), the sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $H^1(Q)$ and $\mathscr{Y}(0,T)$. Hence, in virtue of Theorem 2.2 there exists a subsequence E' extracted from E and some vector function $\mathbf{u}=(u_0,u_1)$ with $u_0\in H^1\big(Q;L^2(\Delta(A))\big)$ and $u_1\in L^2\big(Q;H^1_\#(\Delta(A))\big)$ such that

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-weak } \Sigma \tag{3.13}$$

and (3.10)-(3.11) hold when $E'\ni\varepsilon\to 0$. Moreover, by Theorem 2.3 we see that the subsequence E' can be extracted such that

$$u_\varepsilon \to w_0 \text{ in } \mathscr{Y}(0,T)\text{-weak}, \tag{3.14}$$

$$u_\varepsilon \to w_0 \text{ in } L^2(Q)\text{-strong} \tag{3.15}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial w_0}{\partial x_j} + \partial_j w_1 \text{ in } L^2(Q)\text{-weak } \Sigma \quad (1\leqslant j\leqslant N) \tag{3.16}$$

as $E'\ni\varepsilon\to 0$, where $w_1\in L^2\left(Q;L^2\left(\Delta(A_\tau);H^1_\#(\Delta(A_y))\right)\right)$. By (3.13)-(3.16) we have $u_0=w_0$ and $\partial_j w_1=\partial_j u_1 \quad (1\leqslant j\leqslant N)$, thus (3.9) holds when $E'\ni\varepsilon\to 0$. On the other hand (3.11) implies $\frac{\partial u_0}{\partial t}\in L^2(Q)$, thus $\mathbf{u}=(u_0,u_1)\in\mathbb{F}^1_0$. The theorem is certainly proved if we can show that $\mathbf{u}$ verifies (3.12). We begin by verifying that $u_0(0)=0$ (it is worth recalling that u_0 may be viewed as a continuous mapping of $[0,T]$ into $L^2(\Omega)$).

Let $v\in H^1_0(\Omega)$, and let $\varphi\in\mathscr{C}^1([0,T])$ with $\varphi(T)=0$. By an integration by parts, we have,

$$\int_0^T \left(u'_\varepsilon(t),v\right)\varphi(t)\,dt+\int_0^T (u_\varepsilon(t),v)\,\varphi'(t)\,dt=-(u_\varepsilon(0),v)\,\varphi(0)=-\varepsilon\,(f^\varepsilon,v)\,\varphi(0).$$

In view of (3.9), (3.5), Remarks 2.3 and 2.4, we pass to the limit in the preceding equality as $E'\ni\varepsilon\to 0$. We obtain

$$\int_0^T \left(u'_0(t),v\right)\varphi(t)\,dt+\int_0^T (u_0(t),v)\,\varphi'(t)\,dt=0.$$

Since φ and v are arbitrary, we see that $u_0(0)=0$. Further, we have

$$\int_0^T \left(u'_\varepsilon(t),v\right)\varphi'(t)\,dt+\int_0^T \left(u''_\varepsilon(t),v\right)\varphi(t)\,dt=-\left(u'_\varepsilon(0),v\right)\varphi(0)=-(g^\varepsilon,v)\,\varphi(0).$$

Let us mention that by using (3.7), it is easy to see that the sequence $(u''_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. Thus, using (3.5), Remark 2.4 and the same argument as above we pass to the limit in preceding equality as $E'\ni\varepsilon\to 0$ (E' being well chosen). We obtain

$$\int_0^T \left(u'_0(t),v\right)\varphi'(t)\,dt+\int_0^T \left(u''_0(t),v\right)\varphi(t)\,dt=-\varphi(0)\int_\Omega \widetilde{g}\overline{v}dx.$$

As φ and v are arbitrary, one has $u'_0(0)=\widetilde{g}$. Now, let us establish the variational equation in (3.12). Fix any arbitrary two functions

$$\psi_0\in\mathscr{D}(Q) \text{ and } \psi_1\in\mathscr{D}(Q)\otimes(A^\infty/\mathbb{C}),$$

and let

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon, \text{ i.e., } \psi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) \text{ for all } (x,t) \in Q,$$

where $\varepsilon > 0$ is arbitrary. By (1.1), one as

$$-\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial \overline{\psi}_\varepsilon}{\partial t}dxdt + \int_0^T a_\varepsilon(t;u_\varepsilon(t),\psi_\varepsilon(t))\,dt = 0. \tag{3.17}$$

The aim is to pass to the limit in (3.17) as $E' \ni \varepsilon \to 0$. First, we have

$$\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial \overline{\psi}_\varepsilon}{\partial t}dxdt = \int_Q \frac{\partial u_\varepsilon}{\partial t}\left(\frac{\partial \overline{\psi}_0}{\partial t} + \varepsilon\left(\frac{\partial \overline{\psi}_1}{\partial t}\right)^\varepsilon + \left(\frac{\partial \overline{\psi}_1}{\partial \tau}\right)^\varepsilon\right)dxdt.$$

Thus, in view of (3.9) and (3.11) (and using Definition 2.5), we have,

$$\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial \overline{\psi}_\varepsilon}{\partial t}dxdt \to \int_Q \frac{\partial u_0}{\partial t}\frac{\partial \overline{\psi}_0}{\partial t}dxdt + \int\int_{Q\times\Delta(A)} \partial_0 u_1 \overline{\partial_0 \widehat{\psi}_1}\,dxdtd\beta$$

as $E' \ni \varepsilon \to 0$, since

$$\int_Q \left(\int_{\Delta(A)} \overline{\partial_0 \widehat{\psi}_1}\,d\beta\right)\frac{\partial u_0}{\partial t}dxdt = 0$$

in virtue of Proposition 2.9.
Next, we have

$$\int_0^T a^\varepsilon(t;u_\varepsilon(t),\psi_\varepsilon(t))\,dt \to \int_0^T \mathfrak{a}(\mathbf{u}(t),\phi(t))\,dt$$

as $E' \ni \varepsilon \to 0$, where $\phi = (\psi_0, J\circ\widehat{\psi}_1)$. Indeed, $\psi_\varepsilon \to \psi_0$ in $L^2(Q)$-strong and $\frac{\partial \psi_\varepsilon}{\partial x_j} \to \frac{\partial \psi_0}{\partial x_j} + \partial_j\widehat{\psi}_1$ in $L^2(Q)$-strong Σ as $\varepsilon \to 0$. Further the sequences $(\psi_\varepsilon)_{\varepsilon>0}$ and $\left(\frac{\partial \psi_\varepsilon}{\partial x_j}\right)_{\varepsilon>0}$ are bounded in $L^\infty(Q)$. Thus using (3.4), Remark 2.4 and [19, Corollary 3.19] one achieves the result in virtue of (3.10) and (3.15). Hence, passing to the limit in (3.17) as $E' \ni \varepsilon \to 0$ leads to

$$-\int_Q \frac{\partial u_0}{\partial t}\frac{\partial \overline{\psi}_0}{\partial t}dxdt - \int\int_{Q\times\Delta(A)} \partial_0 u_1 \overline{\partial_0 \widehat{\psi}_1}\,dxdtd\beta + \int_0^T \mathfrak{a}(\mathbf{u}(t),\phi(t))\,dt = 0 \tag{3.18}$$

for all $\phi = (\psi_0, J\circ\widehat{\psi}_1) \in \mathscr{F}_0^\infty$. Moreover, since $\mathscr{F}_0^\infty$ is a dense subspace of $\mathbb{F}_0^1$, by (3.18) we see that $\mathbf{u} = (u_0,u_1)$ verifies (3.12). The theorem is proved.

For further needs, we wish to give a simple representation of the function u_1 in Theorem 3.1. For this purpose, let us introduce the form $\widehat{\mathfrak{a}}$ on $H_\#^1(\Delta(A)) \times H_\#^1(\Delta(A))$ defined by

$$\widehat{\mathfrak{a}}(w,v) = \sum_{j=1}^N \int_{\Delta(A)} \widehat{\mathfrak{c}}\partial_j w \overline{\partial_j v}\,d\beta$$

for all $w,\ v \in H^1_{\#}(\Delta(A))$. The sesquilinear form $\widehat{\mathfrak{a}}$ is continuous and hermitian. Next, for any index l with $1 \leqslant l \leqslant N$, we consider the variational problem

$$\begin{cases} \chi^l \in H^1_{\#}(\Delta(A)), \\ -\int_{\Delta(A)} \partial_0 \chi^l \partial_0 v d\beta + \widehat{\mathfrak{a}}(\chi^l, v) = \int_{\Delta(A)} \widehat{\mathbf{c}} \partial_l v d\beta \quad \text{for all } v \in H^1_{\#}(\Delta(A)). \end{cases} \tag{3.19}$$

In the sequel we suppose that (3.19) admits a solution and we set

$$z(x,t) = -\sum_{j=1}^{N} \chi^j \frac{\partial u_0}{\partial x_j}(x,t) \qquad ((x,t) \in Q).$$

The function z belongs to $L^2\left(Q; H^1_{\#}(\Delta(A))\right)$. Further, z verifies

$$-\int_{\Delta(A)} \partial_0 z(x,t)\, \partial_0 v d\beta + \widehat{\mathfrak{a}}(z(x,t), v) = -\sum_{l=1}^{N} \frac{\partial u_0}{\partial x_l}(x,t) \int_{\Delta(A)} \widehat{\mathbf{c}} \partial_l v d\beta \quad \text{for all } v \in H^1_{\#}(\Delta(A)) \tag{3.20}$$

$((x,t) \in Q)$. Indeed, multiplying the equality in (3.19) by $\frac{\partial u_0}{\partial x_l}(x,t)$ and taking the sum over $1 \leqslant l \leqslant N$ of the obtained equation lead to (3.20).

3.1. The macroscopic homogenized equation

Our aim here is to derive the initial boundary value problem for u_0. To begin, for $1 \leqslant i, j \leqslant N$, let

$$q_{ij} = \delta_{ij} \int_{\Delta(A)} \widehat{\mathbf{c}} d\beta - \int_{\Delta(A)} \widehat{\mathbf{c}} \partial_i \chi^j d\beta,$$

δ_{ij} being the Krönecker symbol, and let $\widetilde{\mathbf{w}} = \int_{\Delta(A)} \widehat{\mathbf{w}} d\beta$. To the coefficients q_{ij} we attach the differential operator $\mathscr{Q}$ on Q mapping $\mathscr{D}'(Q)$ into $\mathscr{D}'(Q)$ ($\mathscr{D}'(Q)$ being the usual space of complex distributions on Q) as

$$\mathscr{Q} u = -\sum_{i,j=1}^{N} q_{ij} \frac{\partial^2 u}{\partial x_j \partial x_i} \text{ for all } u \in \mathscr{D}'(Q).$$

We consider the following initial boundary value problem:

$$\frac{\partial^2 u_0}{\partial t^2} + \mathscr{Q} u_0 + \widetilde{\mathbf{w}} u_0 = 0 \text{ in } Q = \Omega \times]0,T[, \tag{3.21}$$

$$u_0 = 0 \text{ on } \partial\Omega \times]0,T[, \tag{3.22}$$

$$u_0(0) = 0 \text{ in } \Omega, \tag{3.23}$$

$$\frac{\partial u_0}{\partial t}(0) = \widetilde{g} \text{ in } \Omega. \tag{3.24}$$

The initial boundary value problem (3.21)-(3.24) is the so-called macroscopic homogenized equation.

THEOREM 3.2. *Suppose the hypotheses of Propositions 3.1 and 3.2 are satisfied. Suppose further that (3.20) admits at most one solution. Then, for any fundamental sequence E, there exists a subsequence E' extracted from E such that as $E' \ni \varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $\mathscr{Y}(0,T)$-weak ($u_\varepsilon \in \mathscr{Y}(0,T)$ being defined by (1.1)-(1.4)), where u_0 is a weak solution to (3.21)-(3.24) in $\mathscr{Y}(0,T)$.*

Proof. As in the proof of Theorem 3.1, from any fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (3.9)-(3.11), and further (3.12) holds. In (3.12), choosing a test function $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1$ such that $v_0 = 0$ and $v_1(x,t) = \varphi(x,t)\, v$ in $(x,t) \in Q$, where $\varphi \in \mathscr{D}(Q)$ and $v \in H_\#^1(\Delta(A))$, we see that u_1 verifies (3.20) since φ is arbitrary. Thus, in virtue of the unicity of solution to (3.20), one has

$$u_1(x,t) = -\sum_{j=1}^{N} \chi^j \frac{\partial u_0}{\partial x_j}(x,t) \quad ((x,t) \in Q). \tag{3.25}$$

Now, substituting (3.25) in (3.18) and then choosing therein the ϕ's such that $\psi_1 = 0$, a simple computation yields (3.21) with (3.22)-(3.24), of course. Hence the theorem follows.

A concluding remark. In our study, we come up against the lack of unicity of the solutions of both the global homogenized equation (3.12) and the local or microscopic problem (3.19) as well as the equation (3.20) verified by the corrector term. Those results would have been fundamental in the proof of the macroscopic homogenized problem (3.21)-(3.23). Without those unicity results, it would be a difficult task, even to derive the macroscopic homogenized equation (3.21) since the corrector term u_1 verifying (3.20) is not certainly of the form (3.25).

Acknowledgement. The author would like to thank the anonymous referee for his useful remarks and suggestions.

REFERENCES

[1] A. BENSOUSSAN, J.L. LIONS AND G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, North-Holland, 1978.
[2] A.S. BESICOVITCH, *Almost Periodic Functions*, Cambridge at the University Press, Dover Publications, Inc. 1954.
[3] N. BOURBAKI, *Intégration*, Chap. 1-4, Hermaan, Paris, 1966.
[4] N. BOURBAKI, *Topologie générale*, Chap. 5-10, Hermaan, Paris, 1974.
[5] S. BRAHIM-OTSMANE, G.A. FRANCFORT AND F. MURAT, *Correctors for the homogenization of the wave and heat equations*, J. Math. Pures Appl.,71 (1992), p. 197–231.
[6] D. CIORANESCU, P. DONATO, *An Introduction to Homogenization*, Oxford lecture series in Mathematics and Its Applications; 17 (1999).
[7] J. J. F. FOURNIER, J. STEWART, *Amalgams of L^p and l^q*, Bull. Amer. Math. Soc. 13 (1985), p. 1–21.
[8] A. GUICHARDET, *Analyse Harmonique commutative*. Dunod, Paris, 1968.
[9] R. LARSEN, *Banach Algebras*, Marcel Dekker, New York, 1973.
[10] J. L. LIONS, *Contrôle Optimal de Systèmes Gouvernés par les Equations aux Dérivées Partielles*, Dunod Gauthier-Villars, Paris 1968.
[11] J.L. LIONS, E. MAGENES, *Problèmes aux Limites non homogènes et Applications 2*, Dunod, Paris 1968.

[12] D. LUKKASSEN, G. NGUETSENG AND P. WALL, *Two-scale convergence*, Int. J. Pure and Appl. Math, **2** (2002), 35–86.
[13] G. NGUETSENG, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., **20** (1989), 608–623.
[14] G. NGUETSENG, *Almost periodic homogenization: Asymptotic analysis of a second order elliptic equation*, (Publ. math. LAN 01). Univ. Yde I 2000.
[15] G. NGUETSENG, *Deterministic homogenization of a semilinear elliptic partial differential equation of order* $2m$, Maths, Reports, **8 (58)** (2006), 167–195.
[16] G. NGUETSENG, *Sigma-convergence of parabolic differential operators*, Multiscale problems in biomathematics, mechanics, physics and numerics, 93–132, Gakuto Internat. Ser. Math. Sci. Appl., 31, Tokyo, 2009.
[17] G. NGUETSENG, *Homogenization Structures and applications I*, Zeit. Anal. Anwend. **22** (2003) 73–107.
[18] G. NGUETSENG AND H. NNANG, *Homogenization of Nonlinear monotone operators beyond the periodic setting*, Electronic Journal of Differential Equations, Vol. 2003 (2003), No. 36, pp. 1–24.
[19] G. NGUETSENG AND N. SVANSTEDT, Σ*-convergence*, Banach J. Math. Anal. 5 (2011), No. 1, 101–135.
[20] G. NGUETSENG AND J.L. WOUKENG, *Deterministic homogenization of parabolic monotone operators with time dependent coefficients*, Electronic Journal of Differential Equations, Vol. 2004 (2004), No. 82, pp. 1–23.

(Received June 3, 2019)

Lazarus Signing
Department of Mathematics and Computer Science
University of Ngaoundere
P.O.Box 454 Ngaoundere, Cameroon
e-mail: lsigning@yahoo.fr