

## STABILITY OF NONAUTONOMOUS IMPULSIVE EVOLUTION SYSTEM ON TIME SCALE

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*Abstract.* The main theme of this article is to discuss the existence, uniqueness and  $\beta$ -Ulam type stability for nonautonomous impulsive differential systems on time scale by applying fixed point method. The major components to proof the results are the Grönwall inequality on time scale, abstract Grönwall lemma and Picard operator. Some suppositions are made for achieving our results. At last, the main result is validated by the example specified in this paper.

### 1. Introduction

In the real world, there are many phenomena which are subjected during their development to the short-term external affects. The duration of these external affects are negligible than the total duration of the observed phenomena. Therefore, we can suppose that these external influences are actually in form of impulses. Now to investigate these abrupt changes, impulsive differential equations play key rule for modeling the physical real world problems. Such type of impulsive differential equations has argued concerned to different application, including biological system such as blood flow, heart beat and impulse rate, population dynamics, radio physics, electric technology, metallurgy, pharmacokinetics, viscoelastic, electrodynamics, mathematical economy, theoretical physics, chemical engineering technology and control theory etc. [4, 5, 17, 37].

Ulam [35, 36], in 1940 asked a question: “How we relate an approximate homomorphism from a group  $G_1$  to a metric group  $G_2$  by an exact homomorphism?”. Hyers [11] partially gave answer to this question for Banach spaces ( $\mathcal{B.S}'s$ ) and later, this theory was given the name Hyers–Ulam (HU) stability. Further Rassias [21] worked on it and generalized Hyers result which is known as Hyers–Ulam–Rassias (HUR) stability. Analysis of these stability concepts provide fruitful benefits in different applications, like numerical analysis and control theory. For more study, we refer the readers to see [12, 13, 14, 15, 18, 19, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42, 45, 46, 47, 49, 50].

To the best of our knowledge, there are several papers dealing with Ulam’s type stability of impulsive evolution equations. Impulsive evolution equation are also suitable to describe problem of population dynamics, theoretical physics, biological system, biotechnology process, mathematical economy and so on. Recently, Yu *et al.* [44]

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proved  $\beta$ -HUR stability for non-autonomous impulsive evolution equation for both compact and unbounded interval.

Hilger [10] in 1988, presented the theory of time scale to unify the continuous and discrete calculus. For details on time scales, see [1, 2, 3, 6, 7, 8, 9, 16, 20, 30, 31, 32, 33, 34, 43, 48] with delay and finite impulses.

Inspired by the work done in [44], we prove the existence and uniqueness (EU) and stability of solution of nonlinear impulsive evolution equation of nonautonomous impulsive differential systems with instantaneous impulses,

$$\begin{cases} \mathbb{U}^\Delta(v) = M(v)\mathbb{U}(v) + B(v)\omega(v) + E(v)\zeta(v) + \mathcal{F}(v, \mathbb{U}(v), \omega(v), \zeta(v)), \\ v \in \mathbb{T}_S^* = \mathbb{T}_S \setminus \{v_1, v_2, \dots, v_m\}, \\ \mathbb{U}(v_k^+) - \mathbb{U}(v_k^-) = I_k(\mathbb{U}(v_k^-)), \quad k = 1, 2, \dots, m, \\ \mathbb{U}(v_0) = \mathbb{U}_0, \end{cases} \tag{1.1}$$

and with noninstantaneous impulses of the form,

$$\begin{cases} \mathbb{U}^\Delta(v) = M(v)\mathbb{U}(v) + B(v)\omega(v) + E(v)\zeta(v) + F(v, \mathbb{U}(v), \omega(v), \zeta(v)), \\ v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, \\ \mathbb{U}(v) = \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}(\mu)) \Delta\mu, \quad \mu \in (v_i, \mu_i] \cap \mathbb{T}_S, \quad i = 1, 2, \dots, m, \\ \mathbb{U}(v_0) = \mathbb{U}_0, \end{cases} \tag{1.2}$$

where  $\alpha \in (0, 1)$ ,  $M(v)$ ,  $B(v)$  and  $E(v)$  are  $m \times m$  regressive square matrices (piecewise continuous),  $\mathbb{T}_S^0 := [v_0, v_f]_{\mathbb{T}_S}$ ,  $v_0 = \mu_0 < v_1 < \mu_1 < v_2 < \dots < v_m < \mu_m < v_{m+1} = v_f$  are pre-fixed numbers and  $\omega, \zeta : \mathbb{T}_S \rightarrow \mathbb{R}$ ,  $\mathcal{F} : \mathbb{T}_S^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F : (\mu_i, v_{i+1}] \cap \mathbb{T}_S \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i = 0, 1, 2, \dots, m$ ,  $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g_i : (v_i, \mu_i] \cap \mathbb{T}_S \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i = 1, 2, \dots, m$ , are continuous functions. Also  $\mathbb{U}(v_k^-) = \lim_{v \rightarrow 0^+} \mathbb{U}(v_k - v)$  and  $\mathbb{U}(v_k^+) = \lim_{v \rightarrow 0^+} \mathbb{U}(v_k + v)$  are left and right side limits of  $\mathbb{U}(v)$  at  $v_k$  respectively.

### 2. Preliminaires

Any non-empty arbitrary closed subset of real numbers is called time scale denoted by  $\mathbb{T}_S$ .  $\varpi : \mathbb{T}_S \rightarrow \mathbb{T}_S$  and  $\rho : \mathbb{T}_S \rightarrow \mathbb{T}_S$  are respectively the forward and backward jump operators defined as:

$$\varpi(s) = \inf\{v \in \mathbb{T}_S : v > s\}, \quad \rho(s) = \sup\{v \in \mathbb{T}_S : v < s\}.$$

The graininess operator  $\eta : \mathbb{T}_S \rightarrow [0, \infty)$  defined as  $\eta(s) = \varpi(s) - s$ ,  $s \in \mathbb{T}_S$ , is used to find the distance between two consecutive points. For a time scale, the derived form is denoted by  $\mathbb{T}_S^z$  and is defined as:

$$\mathbb{T}_S^z = \begin{cases} \mathbb{T}_S \setminus (\rho(\sup \mathbb{T}_S), \sup \mathbb{T}_S], & \text{if } \sup \mathbb{T}_S < \infty, \\ \mathbb{T}_S, & \text{if } \sup \mathbb{T}_S = \infty. \end{cases}$$

The function  $\mathcal{H} : \mathbb{T}_S \rightarrow \mathbb{R}$  is called right-dense (rd) continues if it is continuous at every right- dense point on  $\mathbb{T}_S$  and if its left-sided limit exists at every left-dense point on  $\mathbb{T}_S$ . The function  $\mathcal{H} : \mathbb{T}_S \rightarrow \mathbb{R}$  is said to be regressive (respectively positively regressive) if  $1 + \eta(t)\mathcal{H}(t) \neq 0$ , (respectively  $1 + \eta(t)\mathcal{H}(t) > 0$ ) for all  $t \in \mathbb{T}_S^\zeta$ . The collection of all rd–continuous regressive functions (respectively rd–continuous positively regressive functions) is represented by  $\mathcal{R}_g(\mathbb{T}_S)$  (resp.  $\mathcal{R}_g(\mathbb{T}_S)^+$ ). For the function  $\mathcal{Y} : \mathbb{T}_S \rightarrow \mathbb{R}$  defined as

$$\mathcal{Y}^\Delta(\nu) = \lim_{\nu \rightarrow \mu, \nu \neq \sigma(\nu)} \frac{\mathcal{Y}(\sigma(\nu)) - \mathcal{Y}(\mu)}{\sigma(\nu) - \mu}, \nu \in \mathbb{T}_S^\zeta,$$

$$\int_a^b \mathcal{Y}(\nu)\Delta t = y(b) - y(a), \forall a, b \in \mathbb{T}_S,$$

where  $y^\Delta = \mathcal{Y}$  on  $\mathbb{T}_S^\zeta$ , represents respectively delta derivative and  $\Delta$ –integral.

$\Phi_M(\nu, \nu_0)$  which is the fundamental matrix represents the solution for the equation  $\mathbb{U}^\Delta(\nu) = M(\nu)\mathbb{U}(\nu), \mathbb{U}(\nu_0) = \mathbb{U}_0, \nu \in \mathbb{T}_S^0$ .

REMARK 1. Throughout this paper, we consider that  $\mathbb{T} \not\subseteq \mathbb{Z}$ , where  $\mathbb{T}$  is a time scale and  $\mathbb{Z}$  is the set of integers. Also, the impulses  $\mathbb{V}(\mu_k)$  on the isolated points are assumed to be zero. Also for all constants depend upon  $\beta$ , we use  $\beta$  with subscript and superscript.

### 3. Basic Concepts and Remarks

Let  $\mathbb{C}(\mathbb{T}_S^0, \mathbb{R}^n)$  (resp.  $\mathbb{P}\mathbb{C}(\mathbb{T}_S^0, \mathbb{R}^n)$ ) be  $\mathcal{B}\mathcal{S}$  of piecewise continuous functions with norm  $\|\mathbb{U}\|_\infty = \sup_{\nu \in \mathbb{T}_S^0} \|\mathbb{U}(\nu)\|$ . We denote  $\mathbb{P}\mathbb{C}^1(\mathbb{T}_S^0, \mathbb{R}^n) = \{\mathbb{U} \in \mathbb{P}\mathbb{C}(\mathbb{T}_S^0, \mathbb{R}^n) : \mathbb{U}^\Delta \in \mathbb{P}\mathbb{C}(\mathbb{T}_S^0, \mathbb{R}^n)\}$ , the  $\mathcal{B}\mathcal{S} \|\mathbb{U}\|_1 = \max\{\|\mathbb{U}\|_\infty, \|\mathbb{U}^\Delta\|_\infty\}$ . Here  $\|x\| = \sum_{i=1}^n |x_i|$  for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Consider  $V$  to be a vector space over some field  $K$ . A function  $\|\cdot\|_\beta : \rightarrow [0, \infty)$  is called  $\beta$ –norm if (i)  $\|\mathbb{U}\|_\beta = 0$  if and only if  $\mathbb{U} = 0$ , (ii)  $\|\gamma\mathbb{U}\|_\beta = |\gamma|\|\mathbb{U}\|$  for each  $\gamma \in K$  and  $\mathbb{U} \in V$  (iii)  $\|\mathbb{U} + y\|_\beta \leq \|\mathbb{U}\|_\beta + \|y\|_\beta$ . Then  $(V, \|\cdot\|_\beta)$  is known as  $\beta$ –norm space. Our space will be  $\mathbb{P}\beta$ – $\mathcal{B}\mathcal{S}$  with norm  $\|\mathbb{U}\|_{\mathbb{P}\beta} = \sup_{\nu \in \mathbb{T}_S^0} \|\mathbb{U}(\nu)\|^\beta$ , where  $\nu \in I = [\nu_0, \nu_f] \cap \mathbb{T}_S$  and  $0 < \beta < 1$ . To define  $\mathbb{P}\beta$ – $\mathcal{B}\mathcal{S}$ , we consider the space  $\mathbb{P}\mathbb{C}(D, \mathbb{R}^n)$  and choose another interval  $\nu \in D = [\nu_0, \nu_f] \cap \mathbb{T}_S, \nu \neq \nu_k, k = 1, 2, \dots, m$ . Consider,

$$\left\{ \begin{aligned} & \left\| \mathbb{V}^\Delta(\mu) - M(\mu)\mathbb{V}(\mu) - B(\mu)\omega(\mu) - E(\mu)\zeta(\mu) - \mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu)) \right\| \leq \varepsilon; \\ & \mu \in \mathbb{T}_S', \\ & \left\| \Delta\mathbb{V}(\mu_k) - I_k(\mathbb{V}(\mu_k^-)) \right\| \leq \varepsilon, k = 1, 2, \dots, m, \end{aligned} \right.$$

(3.1)

$$\left\{ \begin{aligned} & \left\| \mathbb{V}^\Delta(\mu) - M(\mu)\mathbb{V}(\mu) - B(\mu)\omega(\mu) - E(\mu)\zeta(\mu) - \mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu)) \right\| \leq \psi(\mu); \\ & \mu \in \mathbb{T}_S', \\ & \left\| \Delta\mathbb{V}(\mu_k) - I_k(\mathbb{V}(\mu_k^-)) \right\| \leq \kappa, k = 1, 2, \dots, m, \end{aligned} \right. \tag{3.2}$$

$$\left\{ \begin{aligned} & \left\| \mathbb{V}^\Delta(\mu) - M(\mu)\mathbb{V}(\mu) - B(\mu)\omega(\mu) - E(\mu)\zeta(\mu) - \mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu)) \right\| \leq \varepsilon, \\ & \mu \in (\mu_i, \nu_{i+1}] \cap \mathbb{T}_S, i = 0, 1, \dots, m, \\ & \left\| \mathbb{V}(\mu) - \frac{1}{\Gamma(\alpha)} \int_{\mu_i}^\mu (\mu - \nu)^{\alpha-1} g_i(\nu, \mathbb{V}(\nu)) \Delta\nu \right\| \leq \varepsilon, \mu \in (\nu_i, \mu_i] \cap \mathbb{T}_S, \\ & i = 1, 2, \dots, m, \end{aligned} \right. \tag{3.3}$$

$$\left\{ \begin{aligned} & \left\| \mathbb{V}^\Delta(\mu) - M(\mu)\mathbb{V}(\mu) - B(\mu)\omega(\mu) - E(\mu)\zeta(\mu) - \mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu)) \right\| \leq \psi(\mu), \\ & \mu \in (\mu_i, \nu_{i+1}] \cap \mathbb{T}_S, i = 0, 1, \dots, m, \\ & \left\| \mathbb{V}(\mu) - \frac{1}{\Gamma(\alpha)} \int_{\mu_i}^\mu (\mu - \nu)^{\alpha-1} g_i(\nu, \mathbb{V}(\nu)) \Delta\nu \right\| \leq \kappa, \mu \in (\nu_i, \mu_i] \cap \mathbb{T}_S, \\ & i = 1, 2, \dots, m, \end{aligned} \right. \tag{3.4}$$

where  $\psi : \mathbb{T}_S^0 \rightarrow \mathbb{R}^+$  represents rd-continuous and increasing.

DEFINITION 1. Eq. (1.1) is called  $\beta$ -HU stable on  $\mathbb{T}_S^0$  if for every  $\psi \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  satisfying (3.1), there exists solution  $\psi_0 \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  of (1.1) such that  $\|\mathbb{U}_0(\mu) - \mathbb{U}(\mu)\|^\beta \leq \mathcal{C}_\beta \varepsilon_\beta$ ,  $\mathcal{C}_\beta > 0$ , is true for all  $\mu \in \mathbb{T}_S^0$ .

DEFINITION 2. Eq. (1.1) is known as  $\beta$ -HUR stable on  $\mathbb{T}_S^0$  if for every  $\mathbb{U} \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  satisfying (3.2), there exists  $\mathbb{U}_0 \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  of (1.1) with  $\|\mathbb{U}_0(\mu) - \mathbb{U}(\mu)\|^\beta \leq \mathcal{C}_\beta \psi^\beta(\mu)$ ,  $\mathcal{C}_\beta > 0$ , is true for all  $\mu \in \mathbb{T}_S^0$ .

DEFINITION 3. Eq. (1.2) is known as  $\beta$ -HU stable, if for every  $\varepsilon > 0$  and  $\mathbb{U} \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  satisfying (3.3), there exist  $\psi \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  of (1.2) such that  $\|\psi(\nu) - \mathbb{U}(\nu)\|^\beta \leq \mathcal{C}_\beta \varepsilon_\beta$ ,  $\mathcal{C}_\beta > 0$ , is true for all  $\nu \in D$ .

DEFINITION 4. Eq. (1.2) is known  $\beta$ -HUR stable, if for every  $(\rho, \kappa) \in \mathbb{P}\mathbb{C}(D, \mathbb{R}^+)$   $\times \mathbb{R} \geq 0$  and for each  $\mathbb{U} \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  satisfying (3.4), there exist  $\mathbb{U}_1 \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  of (1.2) such that the inequality  $\|\mathbb{U}(\nu) - \mathbb{U}_1(\nu)\|^\beta \leq \mathcal{C}_\beta \psi^\beta(\nu)$ ,  $\mathcal{C}_\beta > 0$ , is true for all  $\nu \in D^0$ .

LEMMA 1. [16] Let  $\omega \in \mathbb{T}_S^+$ ,  $z, \nu \in \mathcal{R}_g(\mathbb{T}_S^+)$ ,  $p \in \mathcal{R}_g((\mathbb{T}_S^+)^+)$  and  $c, \nu_k \in \mathbb{R}^+$ ,  $k = 1, 2, \dots$ , then

$$z(\nu) \leq c + \int_\omega^\nu p(\mu)z(\mu)\Delta\mu + \sum_{\omega < \nu_k < \nu} \nu_k z(\nu_k),$$

implies

$$z(v) \leq c \prod_{\omega < v_k < v} (1 + v_k) e_p(v, \omega), \quad v \geq \omega.$$

REMARK 2. A function  $V \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  satisfies (3.1) if and only if there exist  $h \in \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^n)$  such that a sequence  $h_k$  with  $\|h(v)\| \leq \varepsilon, \forall v \in \mathbb{T}_S^0, \|h_k\| \leq \varepsilon, \forall k = 1, 2, \dots, m$ , and;

$$\begin{cases} V^\Delta(v) = M(v)V(v) + B(v)\omega(v) + E(v)\zeta(\mu) + \mathcal{F}(v, V(v), \omega(v), \zeta(v)) \\ + h(v), \quad v(v_0) = v_0, \quad v \in \mathbb{T}_S', \\ V(v) = V(v_k^+) - V(v_k^-) = I_k(V(v_k^-)) + h_k. \end{cases}$$

LEMMA 2. Each solution  $V \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  for inequality (3.1) further hold :

$$\begin{cases} \left\| \left\| V(v) - \Phi_M(v, v_0)V_0 - \sum_{j=1}^m I_j(V(v_j^-)) \right. \right. \\ \left. - \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu - \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \right. \\ \left. - \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, V(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \right\| \leq \delta\varepsilon, \end{cases}$$

for  $v \in (v_k, v_{k+1}] \subset \mathbb{T}_S^0, \|\Phi_M(v, \varpi(\mu))\| \leq C$  and  $\delta = (m + C(v_h - v_0))$ .

Proof. If  $V \in \mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  satisfies (3.1), then by Remark 2, we have

$$\begin{cases} V^\Delta(v) = M(v)V(v) + B(v)\omega(v) + E(v)\zeta(v) + \mathcal{F}(v, V(v), \omega(v), \zeta(v)) \\ + h(v), \quad V(v_0) = V_0, \quad v \in \mathbb{T}_S', \\ V(v_k) = I_k(V(v_k^-)) + h_k, \quad k = 1, 2, \dots, m. \end{cases}$$

Then

$$\begin{aligned} V(v) &= \Phi_M(v, v_0)V_0 + \sum_{j=1}^m I_j(V(v_j^-)) + \sum_{i=1}^m h_i + \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu \\ &\quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \\ &\quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, V(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \\ &\quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))h(\mu)\Delta\mu. \end{aligned}$$

So,

$$\begin{aligned} & \left\| \mathbb{V}(v) - \Phi_M(v, v_0)\mathbb{V}_0 - \sum_{j=1}^m I_j(\mathbb{V}(v_j^-)) - \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu \right. \\ & \quad \left. - \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu - \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \right\| \\ & \leq \int_{v_0}^t \|\Phi_M(v, \varpi(\mu))\| \|f(\mu)\| \Delta\mu + \sum_{i=1}^m \|h_i\| \\ & \leq \delta\varepsilon. \end{aligned}$$

Similar approach for inequality (3.2).  $\square$

REMARK 3. A function  $\mathbb{V} \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  satisfies inequality (3.3) (resp. inequality (3.4)) if and only if there exist a finite sequence  $\{h_k : k = 1, 2, \dots, m\} \subset \mathbb{R}^n$  and a function  $h \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  such that  $\|h_i\| \leq \varepsilon$  (resp.  $\|h_i\| \leq \kappa$ ) for every  $i = 1, 2, \dots, m$  and  $\|h(v)\| \leq \varepsilon$  for all  $v \in D$  and

$$\begin{cases} \mathbb{V}^\Delta(v) = M(v)\mathbb{V}(v) + B(v)\omega(v) + E(v)\zeta(v) + F(v, \mathbb{V}(v), \omega(v), \zeta(v)) + h(v), \\ \mathbb{V}(v_0) = \mathbb{V}_0, v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, i = 0, 1, \dots, m, \\ \mathbb{V}(v) = \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu + h_i, v \in (v_i, \mu_i] \cap \mathbb{T}_S, i = 1, 2, \dots, m. \end{cases}$$

LEMMA 3. If  $\mathbb{V} \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  satisfies inequality (3.3) (resp. inequality (3.4)), then the following inequalities

$$\begin{cases} \left\| \mathbb{V}(v) - \Phi_M(v, v_0)\mathbb{V}_0 - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu \right. \\ \quad \left. - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \right. \\ \quad \left. - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \right\| \\ \leq (Cv_f - C\mu_i + m)\varepsilon, v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, i = 1, 2, \dots, m, \\ \|\mathbb{V}(v) - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu\| \leq m\varepsilon, \text{ (resp. } m\kappa), \\ v \in (v_i, \mu_i] \cap \mathbb{T}_S, i = 1, 2, \dots, m. \end{cases}$$

are true.

Proof. If  $\mathbb{V} \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$  satisfies (3.3), so by Remark 3,

$$\begin{cases} \mathbb{V}^\Delta(v) = M(v)\mathbb{V}(v) + B(v)\omega(v) + E(v)\zeta(v) + F(v, \mathbb{V}(v), \omega(v), \zeta(v)) + h(v), \\ v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, i = 0, 1, \dots, m, \\ \mathbb{V}(v) = \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu + h_i, v \in (v_i, \mu_i] \cap \mathbb{T}_S, i = 1, 2, \dots, m. \end{cases} \tag{3.5}$$

Clearly for (3.5), the solution is given as

$$\begin{aligned} \mathbb{V}(v) &= \Phi_M(v, v_0)\mathbb{V}(v_0) + \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) \left( F((\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu)) + h(\mu)) \right) \Delta\mu \\ &\quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) B(\mu)\omega(\mu)\Delta\mu + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) E(\mu)\zeta(\mu)\Delta\mu \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu + \sum_{i=1}^m h_i, \quad v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, \quad i = 1, 2, \end{aligned}$$

For  $v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, i = 1, 2, \dots, m$ , we get

$$\begin{aligned} &\left\| \mathbb{V}(v) - \Phi_M(v, v_0)\mathbb{V}(v_0) - \int_{v_0}^v \Phi_M(v, \varpi(\mu)) B(\mu)\omega(\mu)\Delta\mu \right. \\ &\quad \left. - \int_{v_0}^v \Phi_M(v, \varpi(\mu)) E(\mu)\zeta(\mu)\Delta\mu - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) F(\mu, \mathbb{V}(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu \right\| \\ &\leq \int_{\mu_i}^v \|\Phi_M(v, \varpi(\mu))\| \|h(\mu)\| \Delta\mu + \sum_{i=1}^m \|h_i\| \\ &\leq (Cv_f - C\mu_i + m)\varepsilon. \end{aligned}$$

As above, we see

$$\begin{aligned} &\left\| \mathbb{V}(v) - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{V}(\mu))\Delta\mu \right\| \leq m\varepsilon, \\ &v \in (v_i, \mu_i] \cap \mathbb{T}_S, \quad i = 1, 2, \dots, m. \end{aligned}$$

For inequality (3.4), we use the similar approach.  $\square$

### 4. Analysis of Equation (1.1)

In this part, we are going to establish the EU and stability results for the solution of Eq. (1.1). For this we use some conditions to overcome the difficulties of the proof. These conditions are:

(C1)  $\mathcal{F} : \mathbb{T}_S^0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the Lipschitz condition

$$\left\| \mathcal{F}(v, \mu_1, \mu_2, \mu_3) - \mathcal{F}(v, v_1, v_2, v_3) \right\| \leq \sum_{i=1}^3 L\|\mu_i - v_i\|,$$

$L > 0, \forall v \in \mathbb{T}_S^0$  and  $\mu_i, v_i \in \mathbb{R}^n, i \in \{1, 2, 3\}$ ;

(C2)  $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies  $\|I_k(\mu_1) - I_k(v_2)\| \leq \sum_{i=1}^2 M_i\|\mu_i - v_i\|, M_i > 0, \forall i \in \{1, 2, \dots, m\}$  and  $\mu_1, v_2 \in \mathbb{R}^n$ ;

(C3)  $\left( \sum_{j=1}^m M_j + CL(v_f - v_0) \right) < 1$  and  $\left( \sum_{j=1}^m M_{j\beta} + C_\beta L_\beta(v_f - v_0) \right) < 1$ :

(C4)  $\psi : \mathbb{T}_S^0 \rightarrow \mathbb{R}^+$  is increasing such that

$$\int_{v_0}^v \psi(\mu) \Delta\mu \leq \varepsilon \psi(v), \varepsilon > 0.$$

**THEOREM 1.** *If (C1) – (C4) satisfies, then Eq. (1.1) has only one solution in  $\mathbb{P}_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$ .*

*Proof.* Define an operator  $\Xi : \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^n) \rightarrow \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^n)$  by

$$(\Xi \mathbb{U})(v) = \begin{cases} \Phi_M(v, v_0) \mathbb{U}_0 + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) B(\mu) \omega(\mu) \Delta\mu \\ + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) E(\mu) \zeta(\mu) \Delta\mu \\ + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) \mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) \Delta\mu, \quad v \in (v_0, v_1], \\ \sum_{j=1}^i I_j(\mathbb{U}(v_j^-)) + \Phi_M(v, v_0) \mathbb{U}_0 + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) B(\mu) \omega(\mu) \Delta\mu \\ + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) E(\mu) \zeta(\mu) \Delta\mu \\ + \int_{v_0}^v \Phi_M(v, \varpi(\mu)) \mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) \Delta\mu, \quad v \in (v_i, v_{i+1}], \\ i = 1, \dots, m. \end{cases}$$

For  $v \in (v_i, v_{i+1}]$  and  $\mathbb{U}_1, \mathbb{U}_2 \in \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^n)$ , simple calculation shows that

$$\begin{aligned} & \|(\Xi \mathbb{U}_1)(v) - (\Xi \mathbb{U}_2)(v)\| \leq \sum_{j=1}^i \|I_j(\mathbb{U}_1(v_j^-)) - I_j(\mathbb{U}_2(v_j^-))\| \\ & + \left\| \int_{v_0}^v \Phi_M(v, \varpi(\mu)) \left( \mathcal{F}(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) \right. \right. \\ & \left. \left. - \mathcal{F}(\mu, \mathbb{U}_2(\mu), \omega(\mu), \zeta(\mu)) \right) \Delta\mu \right\| \\ & \leq \sum_{j=1}^i M_j \|\mathbb{U}_1(v_j^-) - \mathbb{U}_2(v_j^-)\| + \int_{v_0}^v \|\Phi_M(v, \varpi(\mu))\| \left\| \mathcal{F}(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) \right. \\ & \left. - \mathcal{F}(\mu, \mathbb{U}_2(\mu), \omega(\mu), \zeta(\mu)) \right\| \Delta\mu \\ & \leq \sum_{j=1}^i M_j \|\mathbb{U}_1(v_j^-) - \mathbb{U}_2(v_j^-)\| + \int_{v_0}^v CL \|\mathbb{U}_1(\mu) - \mathbb{U}_2(\mu)\| \Delta\mu \\ & \leq \sum_{j=1}^i M_j \sup_{v \in \mathbb{T}_S^0} \|\mathbb{U}_1(v) - \mathbb{U}_2(v)\| + \int_{v_0}^v CL \sup_{v \in \mathbb{T}_S^0} \|\mathbb{U}_1(v) - \mathbb{U}_2(v)\| \Delta\mu \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{j=1}^i M_j \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty + \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty \int_{v_0}^v CL \Delta \mu \\
 &= \sum_{j=1}^i M_j \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty + \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty CL(v - v_0) \\
 &\leq \sum_{j=1}^i M_j \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty + \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty CL(v_f - v_0) \\
 &\leq \|\mathbb{U}_1 - \mathbb{U}_2\|_\infty \left( \sum_{j=1}^m M_j + CL(v_f - v_0) \right).
 \end{aligned}$$

From (C<sub>3</sub>),  $\Xi$  is strictly contractive and a Picard operator (P $\mathbb{O}$ ) with only one FP which is the only one solution of (1.1) in  $P_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$ .  $\square$

**THEOREM 2.** *If conditions (C1) – (C4) hold, then Eq. (1.1) has  $\beta$ -HU stability on  $\mathbb{T}_S^0$ .*

*Proof.* Suppose  $\mathbb{U}_1 \in P_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  satisfies (3.1). The solution  $\mathbb{U} \in P_C^1(\mathbb{T}_S^0, \mathbb{R}^n)$  of Eq. (1.1) is given by

$$\mathbb{U}(v) = \begin{cases} \Phi_M(v, v_0)\mathbb{U}_0 + \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu \\ \quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \\ \quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu))\Delta\mu, \quad v \in (v_0, v_1], \\ \sum_{j=1}^i I(\mathbb{U}(v_j^-)) + \Phi_M(v, v_0)\mathbb{U}_0 \\ \quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\omega(\mu)\Delta\mu + \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \\ \quad + \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu))\Delta\mu, \\ v \in (v_i, v_{i+1}], \quad i = 1, \dots, m. \end{cases}$$

For  $v \in (v_m, v_{m+1}]$ , using Lemma 2, we have

$$\begin{aligned}
 &\|\mathbb{U}(v) - \mathbb{U}_1(v)\|^\beta \\
 &\leq \left\| \mathbb{U}_1(v) - \Phi_M(v, v_0)\mathbb{U}_0 - \sum_{j=1}^m I(\mathbb{U}_1(v_j^-)) - \int_{v_0}^v \Phi_M(v, \varpi(\mu))B(\mu)\mathbb{U}_1(\mu)\Delta\mu \right. \\
 &\quad - \int_{v_0}^v \Phi_M(v, \varpi(\mu))E(\mu)\zeta(\mu)\Delta\mu \\
 &\quad \left. - \int_{v_0}^v \Phi_M(v, \varpi(\mu))\mathcal{F}(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu))\Delta\mu \right\|^\beta
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m \left\| I(\mathbb{U}(v_j^-)) - I(\mathbb{U}_1(v_j^-)) \right\|^\beta \\
 & + \left\| \int_{v_0}^v \Phi_M(v, \varpi(\mu)) \left( \mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) - \mathcal{F}(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) \right) \Delta\mu \right\|^\beta \\
 \leq & (\delta\varepsilon)^\beta + \sum_{j=1}^m M_j^\beta \left\| \mathbb{U}(v_j^-) - \mathbb{U}_1(v_j^-) \right\|^\beta + \int_{v_0}^v \|\Phi_M(v, \varpi(\mu))\|^\beta \\
 & \times \left\| \left( \mathcal{F}(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) - \mathcal{F}(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) \right) \right\|^\beta \Delta\mu \\
 \leq & \delta^\beta \varepsilon^\beta + \sum_{j=1}^m M_{j\beta} \left\| \mathbb{U}(v_j^-) - \mathbb{U}_1(v_j^-) \right\|^\beta + \int_{v_0}^v C^\beta L^\beta \|\mathbb{U}(\mu) - \mathbb{U}_1(\mu)\|^\beta \Delta\mu \\
 \leq & \delta^\beta \varepsilon^\beta + \sum_{j=1}^m M_j^\beta \left\| \mathbb{U}(v_j^-) - \mathbb{U}_1(v_j^-) \right\|^\beta + \int_{v_0}^v C^\beta L^\beta \|\mathbb{U}(\mu) - \mathbb{U}_1(\mu)\|^\beta \Delta\mu.
 \end{aligned}$$

Further define an operator  $T : \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^+) \rightarrow \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^+)$  as

$$(Tw)(v) = \left\{ \delta^\beta \varepsilon^\beta + \sum_{j=1}^i M_{j\beta} w(v_j^-) + \int_{v_0}^v C_\beta L_\beta w(\mu) \Delta\mu, v \in (v_i, v_{i+1}], i = 1, \dots, m. \right. \tag{4.1}$$

For  $v \in (v_m, v_{m+1}]$  and  $w_1, w_2 \in \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^+)$ , we have,

$$\begin{aligned}
 \|(Tw_1)(v) - (Tw_2)(v)\|^\beta & \leq \sum_{j=1}^m M_{j\beta} \|w_1(v_j^-) - w_2(v_j^-)\|^\beta \\
 & + \int_{v_0}^v C_\beta L_\beta \|w_1(\mu) - w_2(\mu)\|^\beta \Delta\mu \\
 & \leq \sum_{j=1}^m M_{j\beta} \|w_1 - w_2\|_\infty^\beta + \|w_1 - w_2\|_\infty^\beta C_\beta L_\beta (v_f - v_0) \\
 & \leq \|w_1 - w_2\|_\infty^\beta \left( \sum_{j=1}^m M_{j\beta} + C_\beta L_\beta (v_f - v_0) \right).
 \end{aligned}$$

Now clearly from condition (C4), the operator is contractive on  $\mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^+)$ . Also  $T$  is PO with only one FP  $w^* \in \mathbb{P}_C(\mathbb{T}_S^0, \mathbb{R}^+)$  i.e.

$$w^*(v) = \delta_\beta \varepsilon_\beta + \sum_{j=1}^m M_{j\beta} w^*(v_j^-) + \int_{v_0}^v CLw^*(\mu) \Delta\mu.$$

Using Grönwall’s inequality,

$$w^*(v) = \delta_\beta \varepsilon_\beta \prod_{v_0 < v_j < v} \left( 1 + M_{j\beta} \right) e_p(v, v_0)$$

where  $P = CL$ . Select  $w(v) = \|\omega(v) - \zeta(v)\|^\beta$  and  $w(v) \leq (Tw)(v)$  from (4.1), by abstract Grönwall(AG) lemma [27], we get

$$\|\omega_1(v) - \omega_2(v)\|^\beta \leq \delta_\beta \varepsilon_\beta \prod_{v_0 < v_j < v} \left(1 + M_j \beta\right) e_p(v, v_0). \quad \square$$

**THEOREM 3.** *If (C1) – (C4) hold, then Eq. (1.1) has  $\beta$ -HUR stable on  $\mathbb{T}_S^0$ .*

### 5. Analysis of Equation (1.2)

In this part, we prove the EU and stability results for the solution of Eq. (1.2). To prove our desired results, we assume the following conditions:

( $\mathcal{A}_1$ )  $F : (\mu_i, v_{i+1}] \cap \mathbb{T}_S \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous with the Lipschitz condition:  $\|F(v, \mu_1, \mu_2, \mu_3) - F(v, v_1, v_2, v_3)\| \leq \sum_{k=1}^3 L \|\mu_k - v_k\|$ , for some  $L > 0$ ,  $v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S$ ,  $i = 0, 1, \dots, m$  and  $\mu_k, v_k \in \mathbb{R}^n$ ,  $k \in \{1, 2, 3\}$ ;

( $\mathcal{A}_2$ )  $g_i : (v_i, \mu_i] \cap \mathbb{T}_S \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies  $\|g_i(\mu, \mu_1) - g_i(\mu, \mu_2)\| \leq M_k \|\mu_1 - \mu_2\|$ ,  $M_k > 0$ ,  $\forall k \in \{1, 2, \dots, m\}$ ,  $\mu \in (v_i, \mu_i] \cap \mathbb{T}_S$  and  $\mu_1, \mu_2 \in \mathbb{R}^n$ ;

( $\mathcal{A}_3$ )  $\left(\frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C' \Delta \mu + CL(v_f - \mu_i)\right) < 1$  and

$\left(\frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C' \beta \Delta \mu + C_\beta L \beta\right) < 1$ ,  $i = 1, 2, \dots, m$ ;

( $\mathcal{A}_4$ )  $\psi \in \mathbb{PC}(J, \mathbb{R}^+)$  is increasing with

$$\int_{v_0}^v \psi(\mu) \Delta \mu \leq \varepsilon \psi(v), \varepsilon > 0.$$

**THEOREM 4.** *If conditions ( $\mathcal{A}_1$ ) – ( $\mathcal{A}_3$ ) hold, then Eq. (1.2) has precisely a unique solution in  $\mathbb{PC}^1(D, \mathbb{R}^n)$ .*

*Proof.* Define an operator  $\Xi : \mathbb{PC}^1(D, \mathbb{R}^n) \rightarrow \mathbb{PC}^1(D, \mathbb{R}^n)$  by

$$(\Xi U)(v) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, U(\mu)) \Delta \mu, & v \in (v_i, \mu_i] \cap \mathbb{T}_S, i = 1, 2, \dots, m, \\ \Phi_M(v, v_0) U_0 + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, U(\mu)) \Delta \mu \\ + \int_{v_i}^v \phi_M(v, \varpi(\mu)) B(\mu) \omega(\mu) \Delta \mu + \int_{v_i}^v \phi_M(v, \varpi(\mu)) E(\mu) \zeta(\mu) \Delta \mu \\ + \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) F(\mu, U(\mu), \omega(\mu), \zeta(\mu)) \Delta \mu, \\ v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, i = 1, 2, \dots, m. \end{cases}$$

For any  $\mathbb{U}_1, \mathbb{U}_2 \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$ ,  $v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S$ ,  $i = 1, 2, \dots, m$ , we have

$$\begin{aligned}
 & \|(\Xi\mathbb{U}_1)(v) - (\Xi\mathbb{U}_2)(v)\| \\
 & \leq \left\| \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}_1(\mu)) \Delta\mu - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}_2(\mu)) \Delta\mu \right\| \\
 & \quad + \int_{\mu_i}^v \|\Phi_M(v, \varpi(\mu))\| \|F(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) - F(\mu, \mathbb{U}_2(\mu), \omega(\mu), \zeta(\mu))\| \Delta\mu \\
 & \leq \left\| \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} (g_i(\mu, \mathbb{U}_1(\mu)) - g_i(\mu, \mathbb{U}_2(\mu))) \Delta\mu \right\| \\
 & \quad + \int_{\mu_i}^v CL \|\mathbb{U}_1(\mu) - \mathbb{U}_2(\mu)\| \|\Delta\mu \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} \|g_i(\mu, \mathbb{U}_1(\mu)) - g_i(\mu, \mathbb{U}_2(\mu))\| \Delta\mu \\
 & \quad + \int_{\mu_i}^v CL \|\mathbb{U}_1(\mu) - \mathbb{U}_2(\mu)\| \|\Delta\mu \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C' \|\mathbb{U}_1(\mu) - \mathbb{U}_2(\mu)\| \Delta\mu + \int_{\mu_i}^v CL \|\mathbb{U}_1(\mu) - \mathbb{U}_2(\mu)\| \|\Delta\mu \\
 & \leq \left\| \mathbb{U}_1 - \mathbb{U}_2 \right\|_{\infty} \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C' \Delta\mu + \left\| \mathbb{U}_1 - \mathbb{U}_2 \right\|_{\infty} CL(v - \mu_i) \\
 & \leq \left( \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C' \Delta\mu + CL(v_f - \mu_i) \right) \left\| \mathbb{U}_1 - \mathbb{U}_2 \right\|_{\infty}.
 \end{aligned}$$

Clearly from condition  $(\mathcal{A}_3)$ ,  $\Xi$  is contractive and thus a  $\mathbb{P}\mathbb{O}$  on  $\mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$ . So it has only one  $\mathbb{F}\mathbb{P}$  and this  $\mathbb{F}\mathbb{P}$  is actually the only one solution of Eq. (1.2).  $\square$

**THEOREM 5.** *The Eq. (1.2) has  $\beta$ -HU stability on  $D$  if  $(\mathcal{A}_1) - (\mathcal{A}_3)$  hold.*

*Proof.* Let (3.3) has a solution  $\mathbb{U}_1 \in \mathbb{P}\mathbb{C}^1(D, \mathbb{R}^n)$ . The only one solution of Eq. (1.2) is

$$\mathbb{U}(v) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}(\mu)) \Delta\mu, & v \in (v_i, \mu_i] \cap \mathbb{T}_S, \quad i = 1, 2, \dots, m, \\ \Phi_M(v, v_0) \mathbb{U}_0 + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}(\mu)) \Delta\mu \\ \quad + \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) B(\mu) \omega(\mu) \Delta(\mu) + \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) E(\mu) \zeta(\mu) \Delta(\mu) \\ \quad + \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) F(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) \Delta\mu, & v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S, \\ i = 1, 2, \dots, m. \end{cases}$$

We examine that for all  $v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S$ ,  $i = 1, 2, \dots, m$ , applying Lemma 3, we get

$$\begin{aligned} & \| \mathbb{U}_1(v) - \mathbb{U}(v) \|^\beta \\ \leq & \left\| \mathbb{U}_1(v) - \Phi_M(v, v_0) \mathbb{U}_0 - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) B(\mu) \omega(\mu) \Delta\mu \right. \\ & - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) E(\mu) \zeta(\mu) \Delta\mu \\ & - \int_{\mu_i}^v \Phi_M(v, \varpi(\mu)) F(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) \Delta\mu \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}_1(\mu)) \Delta\mu \right\|^\beta \\ & + \left\| \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}_1(\mu)) \Delta\mu - \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} g_i(\mu, \mathbb{U}(\mu)) \Delta\mu \right\|^\beta \\ & + \int_{\mu_i}^v \| \Phi_M(v, \varpi(\mu)) \|^\beta \| F(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) - F(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) \|^\beta \Delta\mu \\ \leq & \left( (m + Cv_f - C\mu_i) \varepsilon \right)^\beta + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} \| g_i(\mu, \mathbb{U}_1(\mu)) - g_i(\mu, \mathbb{U}(\mu)) \|^\beta \Delta\mu \\ & + \int_{\mu_i}^v \| \Phi_M(v, \varpi(\mu)) \|^\beta \left\| F(\mu, \mathbb{U}_1(\mu), \omega(\mu), \zeta(\mu)) - F(\mu, \mathbb{U}(\mu), \omega(\mu), \zeta(\mu)) \right\|^\beta \Delta\mu \\ \leq & (m + Cv_f - C\mu_i)^\beta \varepsilon^\beta + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta \| \mathbb{U}_1(\mu) - \mathbb{U}(\mu) \|^\beta \Delta\mu \\ & + \int_{\mu_i}^v C^\beta L^\beta \| \mathbb{U}_1(\mu) - \mathbb{U}(\mu) \|^\beta \Delta\mu \\ \leq & (m + Cv_f - C\mu_i)^\beta \varepsilon^\beta + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta \| \mathbb{U}_1(\mu) - \mathbb{U}(\mu) \|^\beta \Delta\mu \\ & + \int_{\mu_i}^v C_\beta L_\beta \| \mathbb{U}_1(\mu) - \mathbb{U}(\mu) \|^\beta \Delta\mu. \end{aligned}$$

Consider the operator  $T : \mathbb{PC}(D, \mathbb{R}^+) \rightarrow \mathbb{PC}(D, \mathbb{R}^+)$  given below

$$(Tg)(v) = (m + Cv_f - C\mu_i)^\beta \varepsilon^\beta + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta g(\mu) \Delta\mu + C_\beta L_\beta \int_{\mu_i}^v g(\mu) \Delta\mu.$$

For any  $g_1, g_2 \in \mathbb{PC}(D, \mathbb{R}^+)$ ,  $v \in (\mu_i, v_{i+1}] \cap \mathbb{T}_S$ ,  $i = 1, 2, \dots, m$  and by using same process as in Theorem 4, we get

$$\| (Tg_1)(v) - (Tg_2)(v) \|^\beta \leq \left( \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta \Delta\mu + C_\beta L_\beta \right) \| g_1 - g_2 \|_\infty^\beta.$$

So from condition  $(\mathcal{A}_3)$ ,  $T$  is  $\mathbb{PO}$  with unique  $\mathbb{FP}$   $g^* \in \mathbb{PC}(D, \mathbb{R}^+)$  i.e.

$$g^*(v) = (m + Cv_f - C\mu_i)^\beta \varepsilon^\beta + \frac{1}{\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta g(\mu) \Delta\mu + C_\beta L_\beta \int_{\mu_i}^v g(\mu) \Delta\mu$$

By using Lemma 1, we achieve that

$$g^*(v) \leq (m + Cv_f - C\mu_i)\beta \varepsilon_\beta \prod_{\mu_i < \mu < v} \left( 1 + \frac{1}{m\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta \Delta\mu \right) e_q(v, \mu_i),$$

where  $q = C_\beta L_\beta$ . If we set  $g = \|\mathbb{U}_1(v) - \mathbb{U}(v)\|^\beta$ , then by applying AG lemma [27], we get

$$\begin{aligned} & \|\mathbb{U}_1(v) - \mathbb{U}(v)\|^\beta \\ & \leq (m + Cv_f - C\mu_i)\beta \varepsilon_\beta \prod_{\mu_i < \mu < v} \left( 1 + \frac{1}{m\Gamma(\alpha)} \int_{v_i}^v (v - \mu)^{\alpha-1} C'^\beta \Delta\mu \right) e_q(v, \mu_i). \quad \square \end{aligned}$$

**THEOREM 6.** *The Eq. (1.2) has  $\beta$ -HUR stability on  $D$  if  $(\mathcal{A}_1) - (\mathcal{A}_4)$  hold.*

**EXAMPLE 1.** Consider the dynamic equation

$$\begin{cases} \mathbb{U}^\Delta(v) = \frac{1}{t-1} \mathbb{U}(v) + e_p(v, \varpi(\mathbb{U}(v))) + \omega(v) + \zeta(v), & v \in [0, 2]_{\mathbb{T}_S} \setminus \{1\}, \\ \Delta\mathbb{U}(v_k) = \mathbb{U}(v_k^-) + I_k(v, \mathbb{U}(v_k^-)), & k = 1, \end{cases} \quad (5.1)$$

and its related inequality

$$\begin{cases} \left| \mathbb{U}_1^\Delta(v) - \frac{1}{t-1} \mathbb{U}_1(v) - e_p(v, \varpi(\mathbb{U}_1(v))) - \omega(v) - \zeta(v) \right| \leq 1, & v \in [0, 2]_{\mathbb{T}_S} \setminus \{1\}, \\ \left| \mathbb{U}_1(v_k^+) - \mathbb{U}_1(v_k^-) - I_k(v, \mathbb{U}_1(v_k^-)) \right| \leq 1, & k = 1. \end{cases} \quad (5.2)$$

By setting  $\mathbb{T}_S' = [0, 2]_{\mathbb{T}_S} \setminus \{1\}$ ,  $v_1 = 1$  and  $p(v) = \frac{1}{t^2-1}$ . Denote  $F(v, \mathbb{U}(v), \omega(v), \zeta(v)) = e_p(v, \varpi(\mathbb{U}(v))) + \omega(v) + \zeta(v)$  where  $\omega(v)$  and  $\zeta(v)$  are control functions for  $v \in \mathbb{T}_S'$  and put  $\varepsilon_\beta = 1$ . If  $\mathbb{U}_1 \in \mathbb{PC}^1([0]_{\mathbb{T}_S}, \mathbb{R})$  satisfies the inequality (5.2), then there exist  $f \in \mathbb{PC}^1([0, 2]_{\mathbb{T}_S}, \mathbb{R})$  and  $f_0 \in \mathbb{R}$  such that  $|f(v)| \leq 1$  for  $v \in \mathbb{T}_S'$  and  $|f_0| \leq 1$ . So we have

$$\begin{cases} \mathbb{U}_1^\Delta(v) = \frac{1}{t-1} \mathbb{U}_1(v) + e_p(v, \varpi(\mathbb{U}_1(v))) + \omega(v) + \zeta(v) + f(v), & v \in \mathbb{T}_S', \\ \mathbb{U}_1(v_k^+) - \mathbb{U}_1(v_k^-) = I_k(v, \mathbb{U}_1(v_k^-)) + f_0, & k = 1. \end{cases}$$

So the solution of (5.1) is

$$\begin{aligned} \mathbb{U}(v) &= I_1(\mathbb{U}(v_1^-)) + e_p(v, 0) \\ &+ \int_0^v e_p(v, \varpi(\mu)) \left( e_p(\mu, \varpi(\mathbb{U}_1(\mu))) + \omega(\mu) + \zeta(\mu) \right) \Delta\mu. \end{aligned}$$

According to our theoretical results, we showed unique solution for Eq. (5.1) in  $\mathbb{PC}^1([0]_{\mathbb{T}_S}, \mathbb{R})$  and is  $\beta$ -HU stable on  $\mathbb{T}_S'$ .

## Conclusion

In this paper, we have proved  $\beta$ -HU stability of equations (1.1) and (1.2) by using FP method, AG lemma and Lemma 1. The concept of  $\beta$ -HU stability is very important when the exact solution is very tiresome.

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