

\mathcal{C} -SYMMETRIC SECOND ORDER DIFFERENTIAL OPERATORS WITH LARGE LEADING COEFFICIENT

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Abstract. We continue the spectral analysis of Sturm-Liouville operators with complex coefficients. By means of asymptotic integration the Titchmarsh-Weyl m -function is determined without the nesting circle analysis. With it the resolvent is constructed. The primary case is that of a dominant leading coefficient, but Euler type cases are also considered. This leads to resolvents that are compact and even Hilbert-Schmidt.

1. Introduction

This paper is a continuation of our earlier work on second order linear differential operators with complex coefficients. Here we consider the singular second order operator

$$L[y] = \frac{1}{w} [-(py)'] + qy, \quad x \in I, \quad (1.1)$$

where $I = [a, \infty)$ or $I = (0, a]$ with one singular endpoint or $I = (0, \infty)$ or $I = (-\infty, \infty)$ with two singular endpoints. The coefficient w is continuous and positive and $p = p_1 + ip_2 \neq 0$, $q = q_1 + iq_2$ are complex valued. It is assumed that p is continuously differentiable and that q is locally Lebesgue integrable. Further conditions on the functions q and p will be made to obtain asymptotic solutions of $L[y] = zy$. The analysis is similar, though different, to that of section 2.7 of the book by Eastham [11]. Eastham is primarily concerned with asymptotic solutions, and he only applies his results to the calculation of deficiency indices. We carry the spectral theory further calculating not only deficiency indices, but also resolvents and determine if they are compact or even Hilbert-Schmidt.

Following this approach one diagonalizes the systems version of (1.1). For asymptotic integration write (1.1) in systems form

$$Y' = \begin{bmatrix} 0 & 1/p \\ q - zw & 0 \end{bmatrix} Y, \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}. \quad (1.2)$$

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In section 2 we utilize asymptotic theorems from [18] where p is always the dominant term. To handle the case of Euler type operators the system (1.2) we diagonalize it in section 6 with a transformation of the form

$$T = \begin{bmatrix} 1 & 1 \\ (pq)^{1/2} & -(pq)^{-1/2} \end{bmatrix} \quad \text{and} \quad TZ = Y,$$

to obtain the system

$$Z' = \left(\begin{bmatrix} (q/p)^{1/2} & 0 \\ 0 & -(q/p)^{1/2} \end{bmatrix} - r \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) Z, \quad r = \frac{(pq)'}{4(pq)}. \tag{1.3}$$

In [5, 7] the coefficient q was the dominant term in solving $L[y] = zy$ asymptotically. Here we take a complementary case where p is the dominant term, and we concentrate mainly on the case $(q/p)^{1/2} = o((pq)'/(pq))$. To illustrate this complementary, let $I = [1, \infty)$, $p(x) = p_0x^\alpha$, $w(x) = x^\beta$, $q(x) = q_0x^\gamma$. Then the results of [5] apply if $\gamma > \beta \geq \alpha - 2$, the results of [7] apply if $\gamma > \beta$ and $\gamma > \alpha - 2$. The results in section 2 apply if $\alpha > \beta + 2$, $\alpha > \gamma + 2$. It should be noted that the second matrix in (1.3) is of rank one. Hence we expect a solution y of (1.1) with $y(x) \sim 1$ in this case.

The second case, where both summands in (1.3) are comparable, the Euler case, will be discussed in section 6. For the example above, the Euler like case at infinity is $\alpha = \beta + 2$ and $\gamma \leq \beta$, or $\alpha = \gamma + 2$ and $\gamma \geq \beta$. A representative theorem of this case, although with z dependent hypotheses, is Theorem 2.6.1 of Eastham [11, p. 75]. The asymptotic solutions of an Euler like case at a finite singularity can often be found by applying Frobenius theory as in the Bessel equation.

It turns out in the theory developed here that the singular endpoints can be either of the limit point or limit circle type which is in contrast to the limit point type only of [5, 7]. Further the resolvent in the application of Theorems 2.1 and 2.2 is always of Hilbert-Schmidt type. In section 6 Euler type equations are studied where the resolvent may not be of Hilbert-Schmidt type. There are a number of applications given of Theorems 2.1 and 2.2 especially when the singular point is finite. Several examples are given in section 2.

The study of spectral theory of differential operators with complex coefficients has a long but sporadic history. The text by Glazman [16] contains a number of results. A more recent text that deals with complex coefficients is that of Edmunds and Evans [12]. Earlier Sims [31] extended the nesting circle analysis for the Titchmarsh-Weyl m -function to $L[y] = -y'' + q(x)y$ with q complex valued, but with a sign restriction on $\text{Im } q$ which implies the numerical range of the minimal operator lies in a half-plane. The method of Sims was further developed by Brown, McCormack, Evans, and Plum [8], and later extended to \mathcal{J} -symmetric Hamiltonian systems by Brown, Evans, and Plum [9] and Muzzolini [25]. A characterization of the boundary conditions for \mathcal{J} -symmetric extensions of the minimal operator have been given by Knowles [21] and Race [30]. The point spectra are studied by Knowles and Race in [22], and the location of the essential spectrum has been investigated by Race [27, 29]. Race has also studied the Titchmarsh-Weyl m -function for second order equations [28]. A survey article on the general theory of complex symmetric operators has been given

by Garcia, Prodan, and Putinar [15]. \mathcal{C} -symmetric Hamiltonian systems with almost constant coefficients have been studied by Behncke and Hinton in [6]. The results there for the essential spectrum somewhat parallel those of selfadjoint systems and equations [3, 4].

We mention some notation for the operator (1.1). It will act in the weighted Hilbert space $\mathcal{L}_w^2(I)$. The norm and inner product will be denoted by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$, respectively.

We now give some definitions and quote some basic results. With only the conditions w is continuous and $1/p, q$ are locally Lebesgue integrable, the differential expression L determines a maximal operator T and an unclosed minimal operator T_0' defined by the action of L on the domain,

$$D(T) = \{y \in \mathcal{L}_w^2[a, \infty) : y, py' \in AC_{loc} \text{ and } L[y] \in \mathcal{L}_w^2(I)\}$$

and

$$D(T_0') = \{y \in D(T) : y \text{ has compact support in the interior of } I\}$$

where AC_{loc} means locally absolutely continuous. The operator T is closed and T_0' has a closure T_0 and both are densely defined. For a discussion of these properties we refer to the paper of Knowles [21].

The formal adjoint of L is given by

$$L^+[y] = \frac{1}{w} [(-\bar{p}y')' + \bar{q}y], \tag{1.4}$$

and we define the maximal operator T^+ and minimal operator T_0^+ for L^+ analogous to those for L . We have the adjoint relations Goldberg [17, p. 130] or Kauffman, Read, and Zettl [20, p. 14].

$$T_0^* = T^+, \quad T = T_0^{+*}, \quad T_0 = T^{+*}, \quad T^* = T_0^+.$$

For a closed, densely defined operator S in a Hilbert space, the regularity field, $\Pi(S)$, is defined by

$$\Pi(S) = \{z \in \mathbb{C} : \|(S - z)(x)\| \geq k_z \|x\|, x \in D(S), \text{ for some } k_z > 0.\}$$

By definition \mathcal{C} -symmetric, respectively, \mathcal{C} -selfadjoint means for S ,

$$S \subseteq \mathcal{C}S^*\mathcal{C}, \quad \text{respectively,} \quad S = \mathcal{C}S^*\mathcal{C}.$$

The type of \mathcal{C} -symmetry we use is conjugation, i.e., $\mathcal{J}(y) = \bar{y}$. Note T_0 is \mathcal{C} -symmetric as $T_0 \subseteq \mathcal{J}T_0^*\mathcal{J} = \mathcal{J}T^+\mathcal{J}$. In general a conjugation map \mathcal{C} on a Hilbert space is one that is conjugate linear, involutive, and isometric. Thus our \mathcal{C} -symmetry is \mathcal{J} -symmetry. If S has a compact resolvent, then S has no eigenvalues of infinite algebraic multiplicity.

The resolvent set $\rho(S)$ of S is the set of all z in $\Pi(S)$ such that the range of $S - z$ is H . The spectrum $\sigma(S)$ of S is the complement of $\rho(S)$. The set $\sigma(S)$ is the union three sets: the eigenvalues of S , $\sigma_p(S)$, the residual spectrum $\sigma_r(S)$ which is the set of values of $z \notin \sigma_p(S)$ for which the range of $S - z$ is closed but $\neq H$ (a \mathcal{C} -selfadjoint operator

has no residual spectrum), and finally, the essential spectrum of S , $\sigma_{ess}(S)$ which is the set of z such that the range of $S - z$ is not closed. Glazman [16, p. 9] proves that this is equivalent (when there are no eigenvalues of infinite geometric multiplicity) to there being a singular sequence for z , i.e., a bounded noncompact sequence $\{f_n\}$ such that $(S - z)(f_n) \rightarrow 0$ as $n \rightarrow \infty$. In general then, $\sigma(S) = \sigma_p(S) \cup \sigma_r(S) \cup \sigma_{ess}(S)$ and $\sigma(S) = \sigma_p(S) \cup \sigma_{ess}(S)$ if S is a \mathcal{C} -selfadjoint operator. Let $\mathcal{N}(S)$, respectively, $\mathcal{R}(S)$, denote the nullspace and the range of S . Then we have the well known relations, Kato [19, p. 267],

$$\mathcal{N}(T^+ - \bar{z}) = (\mathcal{R}(T_0 - z))^\perp, \mathcal{N}(T - z) = (\mathcal{R}(T_0^+ - \bar{z}))^\perp.$$

The conjugation map $y \rightarrow \bar{y}$ shows that $\dim \mathcal{N}(T^+ - \bar{z}) = \dim \mathcal{N}(T - z)$.

Recall that the numerical range $N(S)$ of a linear operator S acting in a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ is defined by

$$N(S) = \{\langle Sf, f \rangle : f \in D(S), \|f\| = 1\}.$$

In the general case studied here the numerical range of T_0 may be \mathbb{C} .

For $z \notin N(T_0)$, we have from Kato [19, p. 268] that $T_0 - z$ has a closed range, nullity $\text{nul } T_0 - z = 0$, and the defect of $T_0 - z$ is constant on each connected component of $\overline{N(T_0)}^c$. Thus one has $\sigma_{ess}(T_0) \subseteq N(T_0)$.

If $z \notin \sigma_{ess}(T_0)$ and $\sigma_p(T_0) = \emptyset$, then by the closed graph theorem, $z \in \Pi(T_0)$; hence

$$\mathbb{C} = \Pi(T_0) \cup \sigma_{ess}(T_0), \quad \sigma_{ess}(T_0) \cap \Pi(T_0) = \emptyset. \tag{1.5}$$

Further, for the one singular endpoint case, $\sigma_p(T_0) = \emptyset$ since $y \in D(T_0)$ implies $y(a) = y'(a) = 0$. Thus (1.5) holds in this case. Race [27] has proved in general in the limit circle one singular endpoint case that $\sigma_{ess}(T_0) = \emptyset$ and thus $\Pi(T_0) = \mathbb{C}$. This result has been extended by Niessen [26] to operators of order $2n$.

Define

$$s = \dim (D(T)/D(T_0)). \tag{1.6}$$

In the one singular endpoint case $s \geq 2$ since one can construct compactly supported independent functions y_1, y_2 in $D(T)/D(T_0)$ with initial values given by $y_1(a) = 1, (py_1)'(a) = 0, y_2(a) = 0, (py_2)'(a) = 1$. Further it follows in the one singular endpoint case from Kauffman, Read, and Zettl [20, p. 16], that when $T_0 - z$ has a closed range,

$$s = \text{nul } (T - z) + \text{nul } (T^+ - \bar{z}) = 2 \text{nul } (T - z). \tag{1.7}$$

From these we get in the one singular endpoint case and for all $z \notin \sigma_{ess}(T_0)$,

$$\text{def } (T_0 - z) := \dim (\mathcal{R}(T_0 - z))^\perp \geq 1, \quad \text{and also } \text{def } (T_0^+ - \bar{z}) \geq 1. \tag{1.8}$$

For a \mathcal{C} -symmetric operator these defect numbers are independent of $z \notin \sigma_{ess}(T_0)$ [21], and we refer to them as $\text{def } T_0$ and $\text{def } T_0^+$. Following the notation of the self-adjoint case where a limiting circle analysis is used, we will call a singular endpoint *limit point* if $\text{def } T_0 = 1$ and *limit circle* if $\text{def } T_0 = 2$.

Under the hypotheses of Theorem 2.1 or 2.2 plus an additional mild hypothesis, it will follow in all four cases of I that T_0 has an extension whose resolvent is Hilbert-Schmidt. Thus $\sigma_{ess}(T_0) = \emptyset$ here. Hence $T_0 - z$ has a closed range for all $z \in \mathbb{C}$, and we also have that $T - z, T_0^+ - \bar{z}, T^+ - \bar{z}$ all have a closed range Goldberg [17, p. 130] or Kauffman, Read, and Zettl [20, p. 15]. The results here then imply that all the resolvents for (1.1) are Hilbert-Schmidt. This is in contrast to [5, 6, 7], where the singular endpoints are always limit point, and the resolvents are not always Hilbert-Schmidt.

EXAMPLE 1.1. McLeod [24] has given the example of the equation

$$-y'' - 2i \exp(2(1+i)x)y = zy, \quad 0 \leq x < \infty,$$

whose solutions can be expressed in terms of Bessel functions, and no nontrivial solution is in $\mathcal{L}^2([0, \infty))$. If $z \notin \sigma_{ess}(T_0) = \sigma_{ess}(T)$ for some $z \in \mathbb{C}$, then (1.7) yields that $\text{nul}(T - z) \geq 1$ which is a contradiction. Thus $\sigma_{ess}(T_0) = \mathbb{C}$ which also implies $N(T_0) = \mathbb{C}$ as $\sigma_{ess}(T_0) \subseteq N(T_0)$.

With nonselfadjoint problems the possibility of eigenvalues of infinite algebraic multiplicity occurs. This may happen even with simple boundary value problems. The examples in [23, p. 85] and Coddington and Levinson [10, p. 300] for the operator $L[y] = -y''$ on a compact interval show that some extensions of the minimal operator T_0 have every $z \in \mathbb{C}$ as an eigenvalue while other extensions have no eigenvalues. Of course the selfadjoint extensions of T_0 for $L[y] = -y''$ have a countable set of eigenvalues. Since the minimal operator T_0 is bounded below by zero in these examples, they show that even a two dimensional extension of a bounded below symmetric operator may not be bounded below in contrast to the selfadjoint extensions. Clearly an operator cannot have eigenvalues outside its numerical range. Using this property it is proved in [7] that a \mathbb{C} -symmetric operator generated by (1.1) on a compact interval with separated boundary conditions has no eigenvalue of infinite algebraic multiplicity if either the real part or the imaginary part of p does not vanish on the interval. This was extended to $I = [a, \infty)$ in the limit point case with the possible exception of a single initial point a and a single boundary condition. Thus the occurrence of an eigenvalue of infinite algebraic multiplicity is an anomaly in the singular case. However, it is an open problem if it ever occurs, and we must allow for it.

2. Asymptotic theorems

The system (1.3) is not yet in Levinson form so more diagonalizations are needed. The conditions for this to be successful may be rather complex. However, we recall a special case of an asymptotic theorem from [18]. We say a nonvanishing complex valued function Ω on $[a, \infty)$ is *essentially decreasing* (ED) if for some number $M > 0$,

$$\left| \Omega(x)/\Omega(s) \right| \leq M \text{ for } a \leq s \leq x \leq \infty \text{ and } \lim_{x \rightarrow \infty} \Omega(x) = 0,$$

and we say Ω is *essentially increasing* (EI) if for some number $M > 0$,

$$|\Omega(x)/\Omega(s)| \leq M \text{ for } a \leq x \leq s \leq \infty.$$

The conditions on Ω are just the dichotomy conditions of the Levinson asymptotic theorem [10, p. 92] or Eastham [11, p. 8]. Theorem 2.1 below, which follows from Theorem 1 of [18], is similar to Theorem 2.7.1 of [11] where the ρ there is the Ω here. Our theorem allows some oscillatory behavior of q not covered by Theorem 2.7.1. As applied to the case of power coefficients in Example 2.2 below, the two theorems give the same results with the choice $\Omega = \rho = [p(x)w(x)]^{1/2}$. By strengthening the hypotheses, Eastham obtains more precise asymptotic results in other theorems in section 2.7 of [11].

To apply Theorem 1 of [18] to $L[y] = zy$, we choose in the notation of [18]: $n = 2$, $p_0 = q - zw$, $p_1 = 0$, $r_1 = p$, $r_2 = 1$, $\Omega_1 = \Omega$, and $\Omega_2 = 1$. Then following the application as discussed in section 3 of [18], we have the following asymptotic theorem given by equation (3.2) of [18].

THEOREM 2.1. *Assume $I = [a, \infty)$ and there is a function Ω on $[a, \infty)$ that is (ED) or (EI), and that*

$$\int_a^\infty \left| \frac{\Omega}{p} \right| ds < \infty, \quad Q_0(x) := \int_x^\infty \left(\frac{q - zw}{\Omega} \right) ds \text{ exists and } \int_a^\infty \left| \frac{\Omega'}{\Omega} Q_0 \right| ds < \infty. \quad (2.1)$$

Then there are solutions y_1, y_2 of $L[y] = zy$ for $I = [a, \infty)$ such that

$$y_1(x) \rightarrow 1, \frac{(py_1')(x)}{\Omega(x)} \rightarrow 0 \text{ as } x \rightarrow \infty \text{ and } (py_2')(x) \rightarrow 1, y_2(x)\Omega(x) \rightarrow 0 \text{ as } x \rightarrow \infty. \quad (2.2)$$

In our applications of Theorem 2.1, we want the hypotheses to be independent of the spectral parameter z . This is accomplished by replacing (2.1) by

$$\int_a^\infty \left| \frac{\Omega}{p} \right| ds < \infty, \quad \int_a^\infty \frac{w}{|\Omega|} ds < \infty \text{ and } \int_a^\infty \left| \frac{\Omega'}{\Omega} \right| \left(\int_x^\infty \frac{w}{|\Omega|} ds \right) dx < \infty, \quad (2.3)$$

and

$$Q_0(x) := \int_x^\infty \frac{q}{\Omega} ds \text{ exists and } \int_a^\infty \left| \frac{\Omega'}{\Omega} Q_0 \right| ds < \infty. \quad (2.4)$$

Two choices of Ω which are useful are:

$$\Omega(x) = \frac{p(x)}{x^{1+\varepsilon}} \text{ for some } \varepsilon > 0, \quad (2.5)$$

and

$$\Omega(x) = (p(x)w(x))^{1/2}. \quad (2.6)$$

EXAMPLE 2.1. In the case $p(x) = x^\alpha$ and $w(x) = x^\beta$, both (2.5) and (2.6) imply (2.3) if $\text{Re}(\alpha) - \beta > 2$. In the case $p(x) = x^\alpha(\ln x)^\beta$ and $w(x) = x^{\alpha-2}(\ln x)^\gamma$, both $\Omega = x^{\alpha-1}(\ln x)^{\beta-\delta}$ and (2.6) imply (2.3) if $\text{Re}(\beta) - \gamma > 2$ with $\delta > 1$ satisfying $\text{Re}(\beta) - \gamma > 1 + \delta$.

Another approach to examples like these is a change of variable which is discussed in section 6.

For a discussion of boundary conditions, we will use the Lagrange form or Wronskian defined by

$$[f, g] := (pg')f - (pf')g = \det \begin{bmatrix} f & g \\ pf' & pg' \end{bmatrix} \tag{2.7}$$

for differentiable functions f and g . Note that in the work of Knowles and Race, g is replaced by \bar{g} in the definition of $[\cdot, \cdot]$ on the right hand side of (2.7). In the case of Theorem 2.1 we have

$$[y_1, y_2] := (py_2')y_1 - (py_1')y_2 \equiv 1 \tag{2.8}$$

since the Wronskian is constant for solutions of $L[y] = zy$.

For the construction of the resolvents we need to know sufficient conditions for $y_1, y_2 \in \mathcal{L}_w^2([a, \infty))$. For y_1 , it is clear that

$$y_1 \in \mathcal{L}_w^2([a, \infty)) \Leftrightarrow \int_a^\infty w ds < \infty. \tag{2.9}$$

For y_2 , we see that $|y_2| = o(|pw|^{-1/2})$ in case of (2.5) and that $|y_2(x)| = o(x^{1+\varepsilon}/|p(x)|)$ in case of (2.6). This leads to in case of (2.5),

$$\int_a^\infty \frac{1}{|p|} dx < \infty \Rightarrow y_2 \in \mathcal{L}_w^2([a, \infty)), \tag{2.10}$$

and in case of (2.6)

$$\int_a^\infty \frac{wx^{2+2\varepsilon}}{|p|^2} dx < \infty \Rightarrow y_2 \in \mathcal{L}_w^2([a, \infty)), \tag{2.11}$$

EXAMPLE 2.2. Assume $p(x) = x^\alpha$ and $w(x) = x^\beta$ with the condition $\text{Re}(\alpha) - \beta > 2$ holding so Theorem 2.1 applies. Then (2.9) holds if $\beta < -1$, (2.10) holds if $\text{Re} \alpha > 1$, and (2.11) holds if $\beta < 2 \text{Re} \alpha - 3$ with $\varepsilon > 0$ satisfying $\beta + 2\varepsilon < 2 \text{Re} \alpha - 3$.

The next example shows that the term q can be large if highly oscillatory.

EXAMPLE 2.3. Again assume $p(x) = x^\alpha$ and $w(x) = 1$ with the condition $\text{Re}(\alpha) > 2$ holding so Theorem 2.1 applies. With Ω as in (2.5) the conditions on q are

$$Q_0(x) := \int_x^\infty q(s)s^{1+\varepsilon}/p(s) ds \text{ exists and } \int_a^\infty |Q_0(s)/s| ds < \infty. \tag{2.12}$$

For $q(x) = cx^\delta \sin x^\gamma$, $c \in \mathbb{C}$, $\gamma > 0$, calculations show that (2.12) holds if $\gamma + \text{Re} \alpha > 2 + \delta$.

In case p is real, we can apply L'Hospital's rule to (2.2) to obtain

$$\lim_{x \rightarrow \infty} \frac{y_2(x)}{\int_x^\infty \frac{1}{p(s)} ds} = -1 \text{ if } \int_a^\infty \frac{1}{p(s)} ds < \infty, y_2(x) = o(1),$$

or

$$\lim_{x \rightarrow \infty} \frac{y_2(x)}{\int_a^x \frac{1}{p(s)} ds} = 1 \text{ if } \int_a^\infty \frac{1}{p(s)} ds = \infty,$$

With p complex and further conditions on p , Eastham [11, p. 79] has shown that the second of these two asymptotic formulae for y_2 will hold. Alternately, one might apply transformations of variables to reduce to the case $p = w = 1$. This incidentally shows that “a large leading coefficient” is not a unitary invariant.

Many equations in applications are defined for intervals $(0, a]$. For a singular point at 0 define

$$\tilde{L}[u] = \frac{1}{w}[-(pu')' + qu], \quad 0 < t \leq a, \quad \cdot = d/dt. \tag{2.13}$$

Now transform the point 0 to ∞ by defining $y(x) = u(t), x = 1/t$. Then computations show that

$$\tilde{L}[u] = L[y] = \frac{1}{W}[-(Py')' + Qy], \quad W(x) = \frac{w(t)}{x^2}, \quad Q(x) = \frac{q(t)}{x^2}, \quad P(x) = x^2 p(t) \tag{2.14}$$

with $' = d/dx$. Note that the map $u \rightarrow y$ above is unitary as

$$\int_0^a w(t)|u(t)|^2 dt = \int_{1/a}^\infty W(x)|y(x)|^2 dx.$$

For more general change of independent and dependent variables see [1] or [2].

We now apply Theorem 2.1 to (2.14) to obtain

THEOREM 2.2. *Assume there is a function $\tilde{\Omega}$ on $(0, a]$ so that $\Omega(x) := \tilde{\Omega}(1/x)$ is (ED) or (EI), and that*

$$\int_0^a \left| \frac{\tilde{\Omega}}{p} \right| dt < \infty, \quad \int_0^a \frac{w}{|\tilde{\Omega}|} dt < \infty, \quad \text{and} \quad \int_0^a \left| \frac{\tilde{\Omega}'}{\tilde{\Omega}} \right| \left(\int_0^t \frac{w}{|\tilde{\Omega}|} ds \right) dt < \infty, \tag{2.15}$$

and

$$Q_0(t) := \int_0^t \frac{q}{\tilde{\Omega}} ds \text{ exists and } \int_0^a \left| \frac{\tilde{\Omega}'}{\tilde{\Omega}} Q_0 \right| ds < \infty. \tag{2.16}$$

Then there are solutions u_1, u_2 of $\tilde{L}[u] = zu$ for $I = (0, a]$ such that

$$u_1(t) \rightarrow 1, \frac{(pu'_1)(t)}{\tilde{\Omega}(t)} \rightarrow 0 \text{ as } t \rightarrow 0 \text{ and } (pu'_2)(t) \rightarrow 1, u_2(t)\tilde{\Omega}(t) \rightarrow 0 \text{ as } t \rightarrow 0. \tag{2.17}$$

As with Theorem 2.1 two choices of $\tilde{\Omega}$ are useful:

$$\tilde{\Omega}(t) = \frac{p(t)}{t^{1-\varepsilon}} \text{ for some } \varepsilon > 0, \tag{2.18}$$

and

$$\tilde{\Omega}(t) = (p(t)w(t))^{1/2}. \tag{2.19}$$

As with $I = [a, \infty)$ we can give sufficient conditions for $u_1, u_2 \in \mathcal{L}_w^2((0, a])$. For u_1 , it is clear that

$$u_1 \in \mathcal{L}_w^2((0, a]) \Leftrightarrow \int_0^a w dt < \infty. \tag{2.20}$$

For u_2 , we see that $|u_2| = o(|pw|^{-1/2})$ in case of (2.19) and that $|u_2(x)| = o(t^{1-\varepsilon}/|p(t)|)$ in case of (2.18). This leads to in case of (2.19),

$$\int_0^a \frac{1}{|p|} dt < \infty \Rightarrow u_2 \in \mathcal{L}_w^2([a, \infty)), \tag{2.21}$$

and in case of (2.18)

$$\int_0^a \frac{wt^{2-2\varepsilon}}{|p|^2} dt < \infty \Rightarrow u_2 \in \mathcal{L}_w^2((0, a]). \tag{2.22}$$

REMARK 2.3. Theorem 2.2 applies to a number of equations of physical interest. If the singular point is not at zero, say at $x = e$, then the shift of coordinate $s = x - e$ allows to apply the results above.

Everitt [13, p. 271–331] has made a catalogue of fifty-two real Sturm-Liouville equations which occur in mathematical physics and in tests for numerical eigenvalue solvers. In these examples he classifies such properties as limit point, limit circle, oscillatory or not, discrete spectrum, and essential spectrum. We consider some of these examples and show how Theorem 2.2 also gives the classification of limit point or limit circle at a finite singular point. Nonoscillatory always holds with real coefficients when Theorem 2.1 or 2.2 applies since one solution is asymptotic to one. While these examples have real coefficients, the asymptotic solutions given are valid if q is replaced by cq , $c \in \mathbb{C}$, or a perturbation term is added which satisfies the conditions of Theorem 2.1 or 2.2. We use in these examples $' = d/dt$.

EXAMPLE 2.4. Fuel cell equation (Number 55). This differential equation is

$$-(ty'(t))' - t^3y(t) = zty(t), \quad t \in (0, b]. \tag{2.23}$$

The choice $\tilde{\Omega}(t) = [p(t)w(t)]^{1/2} = t$ (same as $\varepsilon = 1$ in (2.18)) in Theorem 2.2 yields solutions u_1, u_2 with the asymptotic behavior

$$u_1(t) = 1 + o(1), \quad u_2(t) = \ln(t)[1 + o(1)] \text{ as } t \rightarrow 0+.$$

These two solutions are both in $\mathcal{L}_w^2((0, b])$ which shows that zero is of limit circle type.

EXAMPLE 2.5. Laplace tidal wave equation (Number 45). This differential equation is

$$-(t^{-1}y'(t))' + (kt^{-2} + k^2t^{-1})y(t) = zy(t), \quad t \in (0, \infty). \tag{2.24}$$

The choice $\tilde{\Omega}(t) = p(t)/t^{1-\varepsilon} = t^{-3/2}$ with $\varepsilon = 1/2$ in Theorem 2.2 yields solutions u_1, u_2 with the asymptotic behavior

$$u_1(t) = 1 + o(1), \quad u_2(t) = \frac{t^2}{2}[1 + o(1)] \text{ as } t \rightarrow 0+.$$

These two solutions are both in $\mathcal{L}_w^2((0, 1])$ which shows that zero is of limit circle type. At infinity Theorem 2.1 does not apply, but the asymptotic behavior can be obtained by applying Theorem 2.3.1 of Eastham [11, p. 57] yielding limit point at infinity.

EXAMPLE 2.6. Boyd equation (Number 35). This differential equation is

$$-y''(t) - t^{-1}y(t) = zy(t), \quad t \in (-\infty, 0) \cup (0, \infty). \quad (2.25)$$

The choice $\tilde{\Omega}(t) = t^{-1/2}$ ($\varepsilon = 1/2$ in (2.18)) in Theorem 2.2 yields solutions u_1, u_2 with the asymptotic behavior

$$u_1(t) = 1 + o(1), \quad u_2(t) = t[1 + o(1)] \quad \text{as } t \rightarrow 0+.$$

Similar calculations hold as $t \rightarrow 0-$. These two solutions are both in $\mathcal{L}_w^2((0, b])$ which shows that $0 \pm$ is of limit circle type. Again Theorem 2.1 does not apply at $\pm\infty$, but the above mentioned theorem of Eastham can be used to show limit point at $\pm\infty$.

Our next example is one that has been studied a lot.

EXAMPLE 2.7. Latzko equation (Number 46). This differential equation is

$$-((1-t^7)y'(t))' = zt^7y(t), \quad t \in (0, 1]. \quad (2.26)$$

This equation has the singular point at $t = 1$ and a translation can shift it to zero. Calculations similar to the above examples yields solutions u_1, u_2 with the asymptotic behavior

$$u_1(t) = 1 + o(1), \quad u_2(t) = \left[\int_0^t (1-s^7)^{-1} ds \right] [1 + o(1)] \quad \text{as } t \rightarrow 1-.$$

Using L'Hopital's rule we find that $u_2(t) = \ln(1-t) \left[\frac{1}{7} + o(1) \right]$. These two solutions are both in $\mathcal{L}_w^2((0, 1])$ which shows that $t = 1$ is of limit circle type.

Our last example is a classical one.

EXAMPLE 2.8. Laguerre equation (Number 27). This differential equation is

$$-(t^{\alpha+1} \exp(-t)y'(t))' = zt^\alpha \exp(-t)y(t), \quad t \in (0, \infty), \quad \alpha \in (-\infty, \infty). \quad (2.27)$$

The choice $\tilde{\Omega}(t) = [p(t)w(t)]^{1/2} = t^{\alpha+1/2}$ in Theorem 2.2 yields solutions u_1, u_2 with the asymptotic behavior: $u_1(t) = 1 + o(1)$ as $t \rightarrow 0+$, and for u_2 , we calculate that as $t \rightarrow 0+$,

$$\alpha < 0 \Rightarrow u_2(t) = \frac{-1}{\alpha t^\alpha} [1 + o(1)], \quad \alpha = 0 \Rightarrow u_2(t) = \ln(t) [1 + o(1)], \quad \text{and}$$

$$\alpha > 0 \Rightarrow u_2(t) = \frac{-t^{-\alpha} - 1}{\alpha} [1 + o(1)].$$

We see that $u_1 \in \mathcal{L}_w^2((0, 1]) \Leftrightarrow \alpha > -1$, and $u_2 \in \mathcal{L}_w^2((0, 1]) \Leftrightarrow \alpha < 1$, Thus we have that zero is of limit circle type if $|\alpha| < 1$ and limit point type otherwise. Actually zero is a regular point if $-1 < \alpha < 0$. Asymptotic behavior at infinity is obtained by first applying a Kummer-Liouville transformation, see example number 2.8. The theory of [7] applies to give limit point at infinity with no essential spectrum generated from the singular point there. Theorem 4.1 below gives that $t = 0$ is a Hilbert-Schmidt type singular point so by the decomposition principle the Laguerre operator has only discrete spectrum as is well known.

3. The operator $R(z)$

First we describe a general inversion operator $R(z)$ and then apply it to the four cases of I . In case I has two singular endpoints let a be an interior point of I , and define $I_1 = (-\infty, a] \cap I$ and $I_2 = [a, \infty) \cap I$. Suppose $y_3(x, z), y_4(x, z)$ are solutions of $L[y] = zy$ such that

$$y_3|_{I_1} \in \mathcal{L}_w^2(I_1), \quad y_4|_{I_2} \in \mathcal{L}_w^2(I_2), \quad [y_3, y_4] \equiv 1. \tag{3.1}$$

Define the kernel function $K(x, s, z)$ on $I \times I$ by

$$K(x, s, z) = \begin{cases} y_4(x, z)y_3(s, z), & \text{if } s \leq x, \\ y_3(x, z)y_4(s, z), & \text{if } s > x. \end{cases} \tag{3.2}$$

Throughout the remainder of the paper let c be the left endpoint of I , and let d be the right endpoint of I . Define the operator $R(z)$ on $\mathcal{L}_w^2(I)$ by

$$\begin{aligned} (R(z)f)(x) &= \int_c^d K(x, s, z)w(s)f(s) ds \\ &= \int_c^x y_4(x, z)y_3(s, z)w(s)f(s) ds + \int_x^d y_3(x, z)y_4(s, z)w(s)f(s) ds. \end{aligned} \tag{3.3}$$

By choice of y_3, y_4 , $R(z)$ is defined on $\mathcal{L}_w^2(I)$, and a computation shows that $y := R(z)f$ satisfies $(L - z)[y] = f$; thus $R(z)$ is one-to-one. When $R(z)$ is Hilbert-Schmidt, $R(z)$ will be a compact map into $\mathcal{L}_w^2(I)$. This occurs when, using $K(x, s, z) = K(s, x, z)$, and

$$\begin{aligned} \int_c^d \int_c^d |K(x, s, z)|^2 w(s)w(x) ds dx &= 2 \int_c^d \int_x^d |y_3(x, z)y_4(s, z)|^2 w(s)w(x) ds dx \\ &= 2 \int_c^d \int_c^x |y_4(x, z)y_3(s, z)|^2 w(s)w(x) ds dx < \infty. \end{aligned} \tag{3.4}$$

THEOREM 3.1. *Assume $R(z)$ is a Hilbert-Schmidt operator on $\mathcal{L}_w^2(I)$ for all z (see equations (3.9)–(3.12) below). Then $(L - z)R(z)f = f$ for all $f \in \mathcal{L}_w^2(I)$ so $\mathcal{R}(R(z)) \subseteq D(T)$. For all four cases of I , the operator $T - z$ is a Fredholm operator and has the closed range $\mathcal{L}_w^2(I)$ for all $z \in \mathbb{C}$, and $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T_0) = \emptyset$.*

Proof. Since $R(z)$ is a Hilbert-Schmidt operator on $\mathcal{L}_w^2(I)$, it follows that $R(z)f \in \mathcal{L}_w^2(I)$ and $\mathcal{R}(R(z)) \subseteq D(T)$. As noted above $(L - z)R(z)f = f$ for all $f \in \mathcal{L}_w^2(I)$ so that $T - z$ is onto $\mathcal{L}_w^2(I)$ so that $T - z$ has a closed range. Thus $\sigma_{\text{ess}}(T) = \emptyset$. Since the null spaces of $T - z$ and $T^+ - \bar{z}$ are finite dimensional, this implies $T - z$ is Fredholm. Since T is a finite dimensional extension of T_0 , they have the same essential spectrum. \square

We now turn to the boundary conditions that members of $\mathcal{R}(R(z))$ satisfy. Calculations show that

$$[y_3, Rf](x) = [y_3, y_4](x) \int_c^x y_3(s, z)w(s)f(s) ds = \int_c^x y_3(s, z)w(s)f(s) ds \tag{3.5}$$

and

$$[y_4, Rf](x) = -[y_3, y_4](x) \int_x^d y_4(s, z)w(s)f(s) ds = - \int_x^d y_4(s, z)w(s)f(s) ds \tag{3.6}$$

Thus

$$[y_3, Rf](c) = [y_4, Rf](d) = 0. \tag{3.7}$$

As noted by Knowles [21], for all $y, v \in D(T)$,

$$[y, v](c) = \lim_{x \rightarrow c} [y, v](x), \quad [y, v](d) = \lim_{x \rightarrow d} [y, v](x)$$

exist. At a LP endpoint the limit is zero.

Whenever $\mathcal{R}(R(z)) \subseteq D(T)$ we now prove that the functions in the range of $R(z)$ satisfy the right boundary conditions:

$$\mathcal{R}(R(z)) = \{y \in D(T) : [y, y_3](c) = [y, y_4](d) = 0\}. \tag{3.8}$$

The equation (3.7) shows the inclusion one way. For the other way suppose $y \in D(T)$ and $[y, y_3](c) = [y, y_4](d) = 0$. Let $f = (T - z)(y)$, $\tilde{y} = R(z)f$, $\hat{y} = \tilde{y} - y$. Then $(T - z)(\hat{y}) = f - f = 0$ so that $\hat{y} = c_1 y_3 + c_2 y_4$ for some constants c_1, c_2 . Thus

$$[\hat{y}, y_3](c) = c_1 [y_3, y_3](c) + c_2 [y_4, y_3](c) = c_1 \cdot 0 + c_2 \cdot 1 = c_2.$$

On the other hand,

$$[\hat{y}, y_3](c) = [\tilde{y}, y_3](c) - [y, y_3](c) = 0$$

by choice of y, \tilde{y} . Then $c_2 = 0$. Similarly $c_1 = 0$ by considering $[\hat{y}, y_4](d)$.

When $R(z)$ is Hilbert-Schmidt equation (3.8) implies $D(T_0) \subseteq \mathcal{R}(R(z))$ and that $R(z)^{-1} + z$ is an extension of T_0 .

We will give some sufficient conditions for $R(z)$ to be Hilbert-Schmidt (H-S). Here we consider only the one singular endpoint case. The two endpoint case is studied in section 5. It is clear that if each endpoint is regular or limit circle, then (3.4) holds since

$$\int_c^d \int_c^d |y_4(x, z)y_3(s, z)|^2 w(s)w(x) ds dx < \infty \Rightarrow R(z) \text{ is H-S}.$$

Now consider the case of one singular point when Theorems 2.1 and 2.2 apply. First let $I = [a, \infty)$. Since $y_1 = O(1)$ and $y_2 = o(\Omega^{-1})$ we have, substituting into the second part of (3.4),

$$\int_a^\infty w ds = \infty, y_3 = y_1, y_4 = y_2, \int_a^\infty \left(\int_a^x w ds \right) w(x) |\Omega(x)^{-2}| dx < \infty \Rightarrow R(z) \text{ is H-S.} \tag{3.9}$$

and, substituting into the first part of (3.4),

$$\int_a^\infty w ds < \infty, y_3 = y_2, y_4 = y_1, \int_a^\infty \left(\int_x^\infty w ds \right) w(x) |\Omega(x)^{-2}| dx < \infty \Rightarrow R(z) \text{ is H-S.} \tag{3.10}$$

In the case of $I = (0, a]$, the conditions become

$$\int_0^a w ds = \infty, y_3 = u_1, y_4 = u_2, \int_0^a \left(\int_t^a w ds \right) w(t) |\tilde{\Omega}(t)^{-2}| dt < \infty \Rightarrow R(z) \text{ is H-S.} \tag{3.11}$$

and

$$\int_0^a w ds < \infty, y_3 = u_2, y_4 = u_1, \int_0^a \left(\int_0^t w ds \right) w(t) |\tilde{\Omega}(t)^{-2}| dt < \infty \Rightarrow R(z) \text{ is H-S.} \tag{3.12}$$

EXAMPLE 3.1. Again we assume $p(x) = x^\alpha$ and $w(x) = x^\beta$ with the condition $(\text{Re } \alpha) - \beta > 2$ holding so Theorem 2.1 applies. Choose $\Omega = (pw)^{1/2}$. Then (3.10) holds if $\beta < -1$ and (3.9) holds if $\beta \geq -1$. Thus with power coefficients, the asymptotic hypothesis also implies Hilbert-Schmidt.

Note that limit circle cases are included in this example. While the conditions above are sufficient for $R(z)$ to be Hilbert-Schmidt, it is not clear that the conditions of Theorem 2.1, resp. 2.2, alone are sufficient for Hilbert-Schmidt. However, those of Theorem 2.1 are sufficient in the case of Example 3.1.

4. Operators with separated boundary conditions

First we define a special basis for solutions of $L[y] = zy$. In case $I = (0, a]$ or $I = [a, \infty)$ choose the base point at a . For the two singular endpoint case, choose the base point at an interior point a of I . Let $\alpha \in \mathbb{C}$, and define solutions $\theta_\alpha, \phi_\alpha$ of $L[y] = zy$ by the initial conditions

$$\begin{bmatrix} \theta_\alpha & \phi_\alpha \\ p\theta'_\alpha & p\phi'_\alpha \end{bmatrix} (a, z) = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix}. \tag{4.1}$$

The boundary conditions to be imposed for $y \in D(T)$ are as follows.

$$\text{At a regular point } a : (\cos \alpha)y(a) + (\sin \alpha)(py')(a) = 0. \tag{4.2}$$

Note that ϕ_α satisfies (4.2) and $[\phi_\alpha, \theta_\alpha] \equiv 1$.

$$\text{At a singular limit point endpoint: No boundary conditions} \tag{4.3}$$

At a limit circle endpoint with $b = c$ or $b = d$: $(\cos \beta)[y, u](b) + (\sin \beta)[y, v](b) = 0$. (4.4)

Here u, v is a typical fundamental set of solutions of $L[y] = 0$ with $[u, v] \equiv 1$; more generally we need only require $u, v \in D(T)$ with $[u, v] \equiv 1$. We are following the parametrization of limit circle boundary conditions given by Fulton [14].

Operators are defined by restricting the domain of the maximal operator T . For $I = (0, a]$ or $I = [a, \infty)$ and the singular point is limit point,

$$D(T_\alpha) = \{y \in D(T) : (4.2) \text{ holds}\}. \tag{4.5}$$

For $I = (0, a]$ or $I = [a, \infty)$ and the singular point is limit circle,

$$D(T_{\alpha\beta}) = \{y \in D(T) : (4.2) \text{ and } (4.4) \text{ hold}\}. \tag{4.6}$$

For the two singular endpoint case we have four cases: $T_{pp}, T_{pc}, T_{cp}, T_{cc}$ where p means limit point and c means limit circle. At a limit point endpoint we impose a boundary condition of type (4.3) and at a limit circle endpoint we impose a boundary condition of type (4.4). For the remainder of this section we consider only T_α and $T_{\alpha\beta}$. The two singular endpoint operators are considered in section 5.

To define the resolvent operators for T_α and $T_{\alpha\beta}$, we need to first define the Titchmarsh-Weyl function at a singular point.

First consider the case where the singular point is in the limit point case. For $z \in \rho(T_\alpha)$ we have from (1.8) that

$$1 = \text{def}(T_0 - z) = \dim \mathcal{N}(T - z).$$

Let $\psi_\alpha(\cdot, z) = c_1 \theta_\alpha(\cdot, z) + c_2 \phi_\alpha(\cdot, z)$ be the unique $\mathcal{L}_w^2(I)$ solution of $L[y] = zy$ normalized as follows. We have $c_1 \neq 0$ since $\phi_\alpha(\cdot, z) \notin \mathcal{L}_w^2(I)$ as $z \in \rho(T_\alpha)$ and ϕ_α satisfies (4.2). Thus take $c_1 = 1$. Then c_2 is uniquely determined and we define it as m_α . Thus m_α is defined on $\rho(T_\alpha)$, and

$$\psi_\alpha(\cdot, z) = \theta_\alpha(\cdot, z) + m_\alpha(z)\phi_\alpha(\cdot, z) \in \mathcal{L}_w^2(I).$$

For the limit circle case we require, when z is not an eigenvalue of $T_{\alpha\beta}$ or equivalently when ϕ_α does not satisfy (4.4), that

$$\psi_{\alpha\beta}(\cdot, z) := \theta_\alpha(\cdot, z) + m_{\alpha\beta}(z)\phi_\alpha(\cdot, z)$$

satisfy (4.4). A calculation then gives

$$m_{\alpha\beta}(z) = -\frac{\cos \beta [\theta_\alpha, u](b) + \sin \beta [\theta_\alpha, v](b)}{\cos \beta [\phi_\alpha, u](b) + \sin \beta [\phi_\alpha, v](b)}. \tag{4.7}$$

We are now able to give resolvent formulae for T_α and $T_{\alpha\beta}$ by specifying y_3, y_4 in (3.2). We assume (3.9)–(3.10) hold for $I = [a, \infty)$, and (3.11)–(3.12) hold for $I = (0, a]$ so we need only check that the boundary conditions in (3.8) are the same as those which define T_α and $T_{\alpha\beta}$.

First we choose $I = [a, \infty)$. Chose $y_3 = \phi_\alpha$ and $y_4 = \psi_\alpha$ in the limit point case and $y_4 = \psi_{\alpha\beta}$ in the limit circle case. In the limit circle case it is clear that (3.1) holds. In the limit point case ψ_α is a multiple of the $\mathcal{L}_w^2(I)$ solution, y_2 for (3.9) and y_1 for (3.10) so (3.1) holds. Then

$$\begin{aligned} [y, y_3](a) &= [y, \phi_\alpha](a) = [(p\phi'_\alpha)y - (py')\phi_\alpha](a) \\ &= -(\cos \alpha)y(a) - (\sin \alpha)(py')(a) = 0 \end{aligned}$$

which is equivalent to (4.2). In the limit point case there is nothing further to prove as $y_4 = \psi_\alpha \in \mathcal{L}_w^2(I)$.

Suppose now the limit circle case holds. We must prove $[y, \psi_{\alpha\beta}](\infty) = 0$ is equivalent to (4.4). This equivalence establishes that the boundary condition $[y, \psi_{\alpha\beta}](\infty) = 0$ is independent of z . For this we need a modified form of an algebraic identity given by Fulton. It is: For y, g in the domain of $[\cdot, \cdot]$,

$$[y, g] = \det \begin{bmatrix} [y, v] & [g, v] \\ [y, u] & [g, u] \end{bmatrix} \tag{4.8}$$

which may be verified by a direct calculation.

Suppose now (4.4) holds for some $y \in D(T)$. Then

$$[y, \psi_{\alpha\beta}] = \det A, \quad A = \begin{bmatrix} [y, v] & [\psi_{\alpha\beta}, v] \\ [y, u] & [\psi_{\alpha\beta}, u] \end{bmatrix} \tag{4.9}$$

and

$$[\sin \beta, \cos \beta]A(\infty) = [0, 0] \Rightarrow \det A(\infty) = 0 \Rightarrow [y, \psi_{\alpha\beta}](\infty) = 0.$$

Conversely suppose $[y, \psi_{\alpha\beta}](\infty) = 0$ for some $y \in D(T)$. Then $\det A(\infty) = 0$ and thus the columns of $A(\infty)$ are linearly dependent. First we show the second column of $A(\infty)$ is not the zero vector. If it is then

$$[\theta_\alpha, v](\infty) + m_{\alpha\beta}[\phi_\alpha, v](\infty) = 0, \quad [\theta_\alpha, u](\infty) + m_{\alpha\beta}[\phi_\alpha, u](\infty) = 0.$$

This implies

$$\det B = 0, \quad B = \begin{bmatrix} [\theta_\alpha, v] & [\phi_\alpha, v] \\ [\theta_\alpha, u] & [\phi_\alpha, u] \end{bmatrix} (\infty).$$

However, as with the algebraic identity (4.8), $\det B = [\theta_\alpha, \phi_\alpha](\infty) = 1$. Thus the first column of $A(\infty)$ is a multiple of the second and hence satisfies (4.4).

Thus under the hypotheses of $z \in \rho(T_\alpha)$, resp., $z \in \rho(T_{\alpha\beta})$, the resolvent of T_α , resp., $T_{\alpha\beta}$, is given by (3.3) where

$$K(x, s, z) = \begin{cases} \psi_\alpha(x, z)\phi_\alpha(s, z), & \text{resp., } \psi_{\alpha\beta}(x, z)\phi_\alpha(s, z), & \text{if } s \leq x, \\ \phi_\alpha(x, z)\psi_\alpha(s, z), & \text{resp., } \phi_\alpha(x, z)\psi_{\alpha\beta}(s, z), & \text{if } s > x. \end{cases} \tag{4.10}$$

For the case $I = (0, a]$, we choose $y_4 = \phi_\alpha$ and $y_3 = \psi_\alpha$ in the limit point case and $y_3 = \psi_{\alpha\beta}$ in the limit circle case. Then the proofs are similar to the $I = [a, \infty)$ case and are omitted.

THEOREM 4.1. *Assume the hypotheses of Theorems 2.1 and 2.2. Assume that (3.9)–(3.10) hold for $I = [a, \infty)$, and that (3.11)–(3.12) hold for $I = (0, a]$. Then T_α and $T_{\alpha\beta}$ are \mathcal{C} -selfadjoint operators and have Hilbert-Schmidt resolvents.*

Proof. That T_α , resp., $T_{\alpha\beta}$, is a \mathcal{C} -selfadjoint operator follows from the theory of Knowles [21] and Race [30] which connects the number of boundary conditions for a \mathcal{C} -selfadjoint extension of T_0 with the defect number $\text{def } T_0$. The construction of the resolvents for T_α and $T_{\alpha\beta}$ shows that Theorem 3.1 applies to give Hilbert-Schmidt resolvents. \square

REMARK 4.2. For the construction of the resolvent of T_α , resp., $T_{\alpha\beta}$, it was necessary to have the existence of one z which is not an eigenvalue so the assumption of no eigenvalue of infinite algebraic multiplicity is implicit for these boundary conditions.

The functions m_α and $m_{\alpha\beta}$ have certain analytic properties which we now state.

THEOREM 4.3. *Assume $I = (0, a]$ or $I = [a, \infty)$ and the hypotheses of Theorem 4.1 hold. Then m_α , resp., $m_{\alpha\beta}$, are meromorphic on \mathbb{C} , and its poles are the eigenvalues of T_α , resp., $T_{\alpha\beta}$.*

Proof. For the limit point case, i.e., m_α , the proof can be found as Theorem 10 of [5]. The analyticity of $m_{\alpha\beta}$ on $\rho(T_{\alpha\beta})$ also follows as that of T_α in [5] yielding in fact

$$m_{\alpha\beta}(z) = - \int_I w \psi_{\alpha\beta}(\cdot, z)^2 dx, \quad \cdot = d/dz.$$

Since $m_{\alpha\beta}$ is analytic on $\rho(T_{\alpha\beta})$, the only possible singular points of $m_{\alpha\beta}$ are the eigenvalues of $T_{\alpha\beta}$. Suppose z_0 is an eigenvalue of $T_{\alpha\beta}$. Let $\gamma = \alpha - \pi/2$. Then by (4.1), we have $\phi_\gamma = -\theta_\alpha$, $\theta_\gamma = \phi_\alpha$. Using these in (4.7) gives $m_{\alpha\beta} = -1/m_{\alpha\gamma}$. Thus either $m_{\alpha\beta}$ has a pole at z_0 (if $m_{\alpha\gamma}(z_0) = 0$) or a removable singularity (if $m_{\alpha\gamma}(z_0) \neq 0$). Since z_0 is an eigenvalue of $T_{\alpha\beta}$, the function ϕ_α is an eigenfunction. Since θ_α satisfies an independent boundary condition at a , we see that θ_α does not satisfy the boundary condition (4.4) at z_0 . Thus the numerator of (4.7) is not zero, and the denominator is zero. This shows $m_{\alpha\beta}(z_0)$ has a pole at z_0 . \square

EXAMPLE 4.1. Assume $p(x) = 1$ and with w satisfying $\int_1^\infty x^2 w(x) dx < \infty$ on $[1, \infty)$ and w satisfying the conditions of Theorem 2.1, e.g., $w(x) = x^{-4}$. Two independent solutions of $L[y] = -y''/w = 0$ are $u(x) = 1$, $v(x) = x$. Now $[y, u] = -y'$ and $[y, v] = y - xy'$. Thus the boundary condition (4.4) is

$$-(\cos \beta)y'(\infty) + (\sin \beta)(y - xy')(\infty) = 0,$$

and we know the limits exist since $y, u, v \in D(T)$.

5. Two singular endpoints with separated boundary conditions

The standard method in this case is the decomposition principle. To apply Theorem 2.1 at $-\infty$, we make the change of variable $g(x) = y(-x)$. Then

$$L[y] = zy \Leftrightarrow \hat{L}[g] = \frac{1}{\hat{w}} [(-\hat{p}g')' + \hat{q}g] = zg, \quad -\infty < x < \infty,$$

where $\hat{p}(x) = p(-x)$, $\hat{q}(x) = q(-x)$, $\hat{w}(x) = w(-x)$. We say Theorem 2.1 holds for L at $-\infty$ if it holds for \hat{L} at ∞ .

In [21] Knowles has characterized maximal \mathcal{J} -selfadjoint extensions and their boundary conditions. An important lemma of [21] is:

LEMMA 5.1. *A linear manifold \mathcal{D}' of $\mathcal{L}_w^2(I)$ is the domain of definition of a \mathcal{J} -selfadjoint extension of T_0 if and only if \mathcal{D}' satisfies the following conditions.*

- (i) $D(T_0) \subseteq \mathcal{D}' \subseteq D(T)$;
- (ii) For any two functions y, g in \mathcal{D}' , the relation $[y, g](d) - [y, g](c) = 0$ holds;
- (iii) Every function $g \in D(T)$ which satisfies $[y, g](d) - [y, g](c) = 0$ for all $y \in \mathcal{D}'$ belongs to \mathcal{D}' .

THEOREM 5.1. *Suppose $I = (0, \infty)$ or $I = (-\infty, \infty)$ and that the conditions of Theorem 4.1 hold at each singular endpoint. Then each of $T_{pp}, T_{pc}, T_{cp}, T_{cc}$ is a \mathcal{C} -selfadjoint operator with empty essential spectrum.*

Proof. By Theorem 4.1 all one singular endpoint problems have empty essential spectrum. By the decomposition principle, see Glazman [16, p. 101], the two singular endpoint problem will also have empty essential spectrum.

To show the operators are \mathcal{C} -selfadjoint operators we use the lemma above and consider the separate cases.

(i) T_{pp} : Since $[y, g](b) = 0$ for all $y, g \in D(T)$ at a limit point endpoint b , we see that by choosing $\mathcal{D}' = D(T_0)$ in Lemma 5.1 it follows that T_0 is \mathcal{C} -selfadjoint and $D(T_0) = D(T)$ or $T_0 = T$.

(ii) T_{pc} : Define \mathcal{D}' by

$$\mathcal{D}' = \{y \in D(T) : (4.4) \text{ holds}\}.$$

Recall (4.4) $\Leftrightarrow [y, \psi_{\alpha\beta}](\infty) = 0$. Then $\mathcal{D}' = D(T_{pc})$. Let now $y, g \in \mathcal{D}'$. By (4.8) we have $[y, g](\infty) = 0$ so condition (ii) of Lemma 5.1 holds. To show condition (iii) of Lemma 5.1 holds, suppose $g \in D(T)$ is such that $[y, g](\infty) = 0$ for all $y \in \mathcal{D}'$. By applying the patching Lemma 4.2 of Knowles [21], we can truncate smoothly $\psi_{\alpha\beta}$ so the truncated function $\hat{\psi}_{\alpha\beta}$ has support in $[a, \infty)$. Then choosing $y = \hat{\psi}_{\alpha\beta}$, we have $y \in \mathcal{D}'$ and $[y, g](\infty) = 0 = [\psi_{\alpha\beta}, g](\infty)$. Thus $g \in \mathcal{D}'$.

(iii) The cases of T_{pc}, T_{cc} are similar to that of T_{pc} and are omitted. For the case of T_{cc} we may use different β 's at the two endpoints. \square

For the remainder of this section T_1 denotes any one of the operators $T_{pp}, T_{pc}, T_{cp}, T_{cc}$.

For $z \in \rho(T_1)$, define the functions $m_{\pm}(z)$ and functions ψ_{\pm} as follows: let $m_+ = m_{\alpha}$, $\alpha = 0$, if the right endpoint is limit point, and let $m_+ = m_{\alpha\beta}$ if the right endpoint is limit circle. Similarly define m_- for the left endpoint.

Let $\psi_+(\cdot, z) = \theta_{\alpha}(\cdot, z) + m_+(z)\phi_{\alpha}(\cdot, z)$ and let $\psi_-(\cdot, z) = \theta_{\alpha}(\cdot, z) + m_-(z)\phi_{\alpha}(\cdot, z)$. As proved in section 4, the functions m_{\pm} are meromorphic functions under the hypotheses of Theorem 4.3.

We define a Green's function $K(x, y, z)$, for $z \in \rho(T_1)$, by choosing for $R(z)$ in (3.3): $y_3 = \psi_-$, $y_4 = \psi_+ / (m_+ - m_-)$. (below it is noted that the meromorphic function $m_+(z) - m_-(z)$ is analytic on $z \in \rho(T_1)$) Then

$$K(x, y, z) = \begin{cases} -\frac{\psi_+(x, z)\psi_-(y, z)}{m_+(z) - m_-(z)}, & y \leq x, \\ -\frac{\psi_-(x, z)\psi_+(y, z)}{m_+(z) - m_-(z)}, & x < y. \end{cases} \tag{5.1}$$

Then with $M =: (m_+ - m_-)^{-1}$, $c =$ left endpoint of I ,

$$\begin{aligned} (R(z)f)(x) &= \int_c^{\infty} K(x, y, z)w(y)f(y) dy \\ &= \int_c^x -M(z)\psi_+(x, z)\psi_-(y, z)w(y)f(y) dy - \int_x^{\infty} M(z)\psi_-(x, z)\psi_+(y, z)w(y)f(y) dy. \end{aligned} \tag{5.2}$$

THEOREM 5.2. *Assume the hypotheses of Theorem 5.1 hold, and the one singular endpoints operators are Hilbert-Schmidt. Then the resolvent operator for T_1 is given by (5.2) and $R(z)$ is Hilbert-Schmidt.*

Proof. As in the proof of Theorem 3.1, it will follow that $R(z)$ is the resolvent operator for T_1 after we prove that $R(z)$ is Hilbert-Schmidt. Here we follow the proof of Theorem 7.4 of [7]. The operator $R(z)$ of (5.2) can be written as the sum of two operators $R_1(z), R_2(z)$ where $R_1(z)$, respectively, $R_2(z)$, acts on functions f with support in $(c, a]$, respectively, $[a, \infty)$. For $R_2(z)$, we have

$$(R_2(z)f)(x) = \begin{cases} \int_a^{\infty} -M\psi_+(x, z)\psi_-(y, z)w(y)f(y) dy = C(f)\psi_+(x, z), & x < a, \\ \int_a^x -M\psi_+(x, z)\psi_-(y, z)w(y)f(y) dy \\ \qquad - \int_x^{\infty} M\psi_-(x, z)\psi_+(y, z)w(y)f(y) dy, & x \geq a, \end{cases} \tag{5.3}$$

where $C(f) := -\int_a^{\infty} M\psi_-(y, z)w(y)f(y) dy$. Since

$$(R_2(z)f)(a) = -M\psi_-(a, z) \int_a^{\infty} \psi_+(y, z)w(y)f(y) dy,$$

$$(R_2(z)f)'(a) = -M\psi_-'(a, z) \int_a^{\infty} \psi_+(y, z)w(y)f(y) dy,$$

we see that the $x \geq a$ part of $R_2(z)$ is an extension of the minimal operator for the interval $[a, \infty)$ with boundary condition

$$y(a)\psi'_-(a, z) - y'(a)\psi_-(a, z) = 0,$$

and is hence a Hilbert-Schmidt operator by Theorem 4.1. The operator $C(f)$ is a bounded rank one operator and is thus Hilbert-Schmidt. It follows then that $R_2(z)$ is a Hilbert-Schmidt operator. Similarly $R_1(z)$ is Hilbert-Schmidt making $R(z)$ Hilbert-Schmidt. \square

The proof of Theorem 16 in [5] also applies to give the following result.

THEOREM 5.3. *Assume the hypotheses of Theorem 5.1 hold. Then the eigenvalues of T_1 are given by:*

$$\text{Spectrum } T_1 = \mathcal{S}_1 \cup \mathcal{S}_2,$$

where

$$\mathcal{S}_1 = \{z : m_+, m_- \text{ are analytic at } z \text{ and } m_-(z) = m_+(z)\},$$

and

$$\mathcal{S}_2 = \{z : m_+, m_- \text{ each have a pole at } z\}.$$

6. Asymptotic integration of Euler type and nondominant potentials

We will now present another approach to the problem with nondominant potentials. This method is more general for nonoscillatory potentials and avoids the use of the auxiliary function Ω of section 2 or the function ρ of section 2.7 of Eastham [11]. However the hypotheses are more complicated than those of section 2. Unlike the theorems of section 2 which generally imply a Hilbert-Schmidt resolvent, the spectrum here may contain nonempty essential spectrum. Even with compact resolvent it may fail to be Hilbert-Schmidt. For simplicity here we only consider the case $I = [a, \infty)$.

First we give a simple example to show that essential spectrum may now appear.

EXAMPLE 6.1. On $[1, \infty)$ let

$$L[y] := \frac{1}{x^{\alpha-2}}[-(x^\alpha y)'] = zy, \quad \alpha, z \text{ real}. \tag{6.1}$$

This Euler equation has solutions of the form x^δ , and simple computations show that the nontrivial solutions are oscillatory and not in $\mathcal{L}_w^2([1, \infty))$ if $z > (\alpha - 1)^2/4$. For $z < (\alpha - 1)^2/4$ the solutions are nonoscillatory and there is one linearly independent solution in $\mathcal{L}_w^2([1, \infty))$. Thus the oscillation constant is $(\alpha - 1)^2/4$, and the essential spectrum is $[(\alpha - 1)^2/4, \infty)$ [32, p. 220]. Below $(\alpha - 1)^2/4$ a selfadjoint extension of the minimal operator T_0 may have at most one eigenvalue, and no \mathcal{C} -symmetric extension of T_0 has a Hilbert-Schmidt or even compact resolvent.

Another example with explicit solutions on $[0, \infty)$ is

$$L[y] := \frac{1}{e^{\alpha x}}[-(e^{\alpha x} y)'] + M e^{\alpha x} y = zy, \quad \alpha \text{ real}. \tag{6.2}$$

This equation has the solutions $y_{\pm}(x) = e^{\lambda_{\pm}x}$ where

$$\lambda_{\pm} = \frac{1}{2} [-\alpha \pm (\alpha^2 + 4(m-z))^{1/2}].$$

Neither solution y_{\pm} is in $\mathcal{L}_w^2([0, \infty)$ for $\alpha^2 + 4(M-z) \in (-\infty, 0)$ so again no \mathcal{C} -symmetric extension of T_0 has a compact resolvent.

On the other hand, the equation

$$L[y] := -(x^3y')' - (8x/9)y = zy, \tag{6.3}$$

has the two $\mathcal{L}^2([1, \infty))$ solutions $x^{-4/3}, x^{-2/3}$ for $z = 0$ and hence two linearly independent $\mathcal{L}^2([1, \infty))$ solutions for all $z \in \mathbb{C}$, and is thus limit circle at infinity. Hence \mathcal{C} -symmetric extensions of T_0 are Hilbert-Schmidt.

Again the starting point is

$$L[y] = \frac{1}{w} [- (py')' + qy] = zy, \quad x \in I. \tag{6.4}$$

For the coefficients we assume that

$$p \neq 0, q = q_1 + q_2 + q_3 \quad w = w_1 + w_2, w_1, w_2 > 0, \tag{6.5}$$

where p, q_1, w_1 are twice differentiable, q_2, w_2 are once differentiable with

$$q_2, w = o(q_1), q'_2, w'_2 = o(q'_1), q'_2, w'_2 \in \mathcal{L}^2([a, \infty)), \tag{6.6}$$

and

$$\frac{q''_1}{1+|q|}, \frac{w''_1}{1+|q|} \in \mathcal{L}_0([a, \infty)), \quad \frac{q'_2}{1+|q|}, \frac{w'_2}{1+|q|} \in \mathcal{L}^2([a, \infty)), \tag{6.7}$$

where \mathcal{L}^p_0 stands for all p -integrable functions vanishing at infinity. Further assumptions will be stated later. In the remainder we will call $f = f_1 + f_2$ the smooth part of f if f_1 is absolutely continuous.

For asymptotic integration write (6.4) in systems form

$$Y' = \begin{bmatrix} 0 & 1/p \\ q-zw & 0 \end{bmatrix} Y, \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \tag{6.8}$$

and diagonalizing the smooth part of (6.8) with q replaced by $\tilde{q} = q_1 + q_2 - zw$, we get with

$$T = \begin{bmatrix} 1 & 1 \\ (p\tilde{q})^{1/2} & -(p\tilde{q})^{-1/2} \end{bmatrix} \quad \text{and} \quad TZ = Y,$$

the system

$$Z' = \left(\begin{bmatrix} (\tilde{q}/p)^{1/2} & 0 \\ 0 & -(\tilde{q}/p)^{1/2} \end{bmatrix} - \frac{(p\tilde{q})'}{4(p\tilde{q})} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) Z. \tag{6.9}$$

For (6.9) to be equivalent to (6.8) in the sense of asymptotic solutions we need that

$$T^{-1} \begin{bmatrix} 0 & 0 \\ q_3 & 0 \end{bmatrix} T$$

be integrable, i.e.,

$$q_3/(p\tilde{q})^{1/2} \in \mathcal{L}([a, \infty)). \tag{6.10}$$

The potential $q = q_1 + q_2 - zw$ is nondominant if $(\tilde{q}/p)^{1/2}$ is comparable or dominated by $(p\tilde{q})'/(p\tilde{q})$. In this case we assume for some constant k and function ϕ that

$$(\tilde{q}/p)^{1/2} = \frac{-(p\tilde{q})'}{4(p\tilde{q})}(k + \phi), \quad \phi = o(1), (\phi')^2, \phi'' \in \mathcal{L}([a, \infty)), k \neq \pm i. \tag{6.11}$$

With this we can write

$$Z' = -\frac{(p\tilde{q})'}{4(p\tilde{q})} \begin{bmatrix} 1 + (k + \phi) & -1 \\ -1 & 1 - (k + \phi) \end{bmatrix} Z. \tag{6.12}$$

In the remaining calculations we will substitute $\tilde{k} = k + \phi$. The eigenvalues of the matrix in (6.12) are $\lambda_{\pm} = 1 \pm (1 + \tilde{k}^2)^{1/2}$ and with the matrix

$$T_1 = \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix}, \quad b = \tilde{k} - (1 + \tilde{k}^2)^{1/2} = k - (1 + k^2)^{1/2} + o(1),$$

one gets with $T_1W = Z$ that

$$\begin{aligned} W' &= \left(-\frac{(p\tilde{q})'}{4(p\tilde{q})} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} - T_1^{-1}T_1' \right) W \\ &= \left(-\frac{(p\tilde{q})'}{4(p\tilde{q})} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} - \frac{1}{(1 + b^2)} \begin{bmatrix} bb' & -b' \\ b' & bb' \end{bmatrix} \right) W. \end{aligned} \tag{6.13}$$

This systems matrix is not yet in Levinson form. So a further diagonalization is necessary. However, by (6.11) the off-diagonal element $b' = O(\phi')$ is square integrable so that a further diagonalization will most add integrable terms to the diagonal. Thus the eigenvalues of the system (6.13) are essentially

$$\mu_{\pm} = \left(\frac{-(p\tilde{q})'\lambda_{\pm}}{4(p\tilde{q})} - \frac{bb'}{1 + b^2} \right). \tag{6.14}$$

It remains to check on the influence of the terms q_2 and zw on the result. It follows from (6.5) that these terms have at all levels a $(1 + o(1))$, respectively, $\mathcal{L}([a, \infty))$ effect on the problem. Moreover, $\phi = \phi(x, z)$ will be analytic in z . The Euler case arises if $k \neq 0$ as in Eastham [11], section 2.6. Expanding the expression $(1 + \tilde{k}^2)^{1/2}$ gives

$$\begin{aligned} [1 + (k + \phi)^2]^{1/2} &= (1 + k^2)^{1/2} \left[1 + \frac{2k\phi + \phi^2}{2(1 + k^2)} + \dots \right] \\ &:= (1 + k^2)^{1/2} + \Phi. \end{aligned}$$

Thus the eigenvalues in (6.13) have the form

$$\begin{aligned} \mu_{\pm} &= \left(\frac{-(p\tilde{q})'}{4(p\tilde{q})} \left[1 \pm ((1+k^2)^{1/2} + \Phi) \right] - \frac{bb'}{1+b^2} \right), \\ \Phi &= \frac{k\phi}{(1+k^2)^{1/2}} + O(\phi^2), \quad b' = o(1). \end{aligned} \tag{6.15}$$

Expanding b with respect to ϕ and w shows that $bb' \in \mathcal{L}_0^2([a, \infty))$ by (6.11). Thus the dichotomy condition holds if, modulo $\mathcal{L}([a, \infty))$ terms, where k is as in (6.11),

$$\operatorname{Re} \left(\frac{((1+k^2)^{1/2} + \Phi)(p\tilde{q})'}{(p\tilde{q})} \right) \text{ is of one sign in } [a, \infty). \tag{6.16}$$

When $(p\tilde{q})'\Phi/(p\tilde{q}) \in \mathcal{L}([a, \infty))$, the Φ term can be dropped in (6.16).

With this we get the solutions of (6.13),

$$W_{\pm} = [e_{\pm} + o(1)] \exp \left(\int_a^x \mu_{\pm}(t, z) dt \right) \quad \text{where } e_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.17}$$

The leading part can easily be evaluated and gives $(pq)^{-[1 \pm (1+k^2)^{1/2}]/4}$.

Thus transforming back through $Y = TT_1W$ yields solutions y_{\pm} of (6.4),

$$y_{\pm} = [1 + o(1)](p\tilde{q})^{-[1 \pm (1+k^2)^{1/2}]/4} (1+b^2)^{-1/2} \exp \left(\int_a^x \left(\frac{\mp(p\tilde{q})'}{4(p\tilde{q})} \Phi \right) dt \right). \tag{6.18}$$

Hence we have the following theorem.

THEOREM 6.1. *Under the conditions (6.5), (6.6), (6.10), (6.11), and (6.16), the equation (6.4) has a pair of independent solutions given by (6.18).*

Eastham [11, Theorem 2.6.1] gives a similar result.

REMARK 6.2. In the important case where $(p\tilde{q})'\Phi/(p\tilde{q}) \in \mathcal{L}([a, \infty))$, a renormalization yields, under the hypotheses of Theorem 6.1, solutions of (6.4) of the form

$$y_{\pm} = [1 + o(1)](p\tilde{q})^{-[1 \pm (1+k^2)^{1/2}]/4}. \tag{6.19}$$

In this situation we have

$$y_{\pm} \in \mathcal{L}_w^2([a, \infty)) \Leftrightarrow \int_a^{\infty} |(p\tilde{q})^{-[1 \pm (1+k^2)^{1/2}]/2}| w dx < \infty. \tag{6.20}$$

EXAMPLE 6.2. In (6.4) let

$$p(x) = x^{\alpha}, \quad q(x) = Mx^{\alpha-2}, \quad M \neq 0, \quad w(x) = x^{\delta}, \quad \alpha - 2 > \delta, \quad \alpha \neq 1,$$

where $M \in \mathbb{C}$. It follows that

$$k + \phi = \frac{-4(M - zs)^{3/2}}{M(2\alpha - 2) - z(\alpha + \delta)s}, \quad s := x^{\delta+2-\alpha},$$

so that

$$k = \frac{-2M^{1/2}}{\alpha - 1}, \quad \phi = O(s), \quad (p\bar{q})'\Phi/(p\bar{q}) \in \mathcal{L}([a, \infty)). \tag{6.21}$$

By (6.19) there are solutions of (6.4) of the form

$$y_{\pm} = [1 + o(1)] \left[x^{2\alpha-2} (M - zx^{\delta+2-\alpha}) \right]^{\frac{-[1 \pm (1+k^2)^{1/2}]}{4}} = [1 + o(1)] M_{\pm} x^{r_{\pm}}, \tag{6.22}$$

where

$$M_{\pm} = M^{\frac{-[1 \pm (1+k^2)^{1/2}]}{4}}, \quad r_{\pm} = \frac{-(\alpha - 1) \mp [(\alpha - 1)^2 + 4M]^{1/2}}{2}.$$

For M real and $M < -(\alpha - 1)^2/4$, then a computation using $\alpha - 2 > \delta$ shows that $y_{\pm} \in \mathcal{L}_w^2([a, \infty))$ so the resolvent is Hilbert-Schmidt.

For $M \notin (-\infty, -(\alpha - 1)^2/4]$, write r_{\pm} in (6.22) as

$$r_{\pm} = \frac{1}{2} [-(\alpha - 1) \mp (\eta + i\mu)], \quad \eta > 0.$$

Then again $y_{\pm} \in \mathcal{L}_w^2([a, \infty))$ for $\eta < \alpha - 2 - \delta$.

For $\eta \geq \alpha - 2 - \delta$, $y_+ \in \mathcal{L}_w^2([a, \infty))$ and $y_- \notin \mathcal{L}_w^2([a, \infty))$. In (3.2) choose $y_4 = y_+$, $y_3 = y_-$. Then since $\text{Re}(r_+ + r_-) = 1 - \alpha$,

$$\begin{aligned} \int_a^x |K(x, s)|^2 w(s) ds &= \int_a^x |y_4(x)y_3(s)|^2 w(s) ds \\ &= O(x^{2\text{Re } r_+} \int_a^x s^{2\text{Re } r_- + \delta} ds) = O(x^{2\text{Re } r_+ + 2\text{Re } r_- + \delta + 1}) \\ &= O(x^{\delta + 3 - 2\alpha}). \end{aligned} \tag{6.23}$$

Since $\alpha - 2 - \delta > 0$, we have that $2\delta + 3 - 2\alpha < -1$ and

$$\int_a^{\infty} \int_a^x |K(x, s)|^2 w(s)w(x) ds dx = O\left(\int_a^{\infty} x^{2\delta + 3 - 2\alpha} dx \right) < \infty,$$

and thus by (3.4) the resolvent is Hilbert-Schmidt in the $M \notin (-\infty, -(\alpha - 1)^2/4]$ case as well.

A more extensive example is:

EXAMPLE 6.3. For $\alpha \geq -1, \beta \geq \gamma > \delta$, let

$$p(x) = p_0 x^{\alpha+2} (\ln x)^{\beta}, \quad q(x) = q_0 x^{\alpha} (\ln x)^{\gamma}, \quad w(x) = x^{\alpha} (\ln x)^{\delta}, \tag{6.24}$$

where $p_0, q_0 \in \mathbb{C}$. Then

$$k + \phi = -4[1 + o(1)](q_0/p_0)^{1/2} \frac{(\ln x)^{(\gamma-\beta)/2}}{(2\alpha + 2)}$$

Exact expressions for $z = 0$ are given with $\beta = \gamma$,

$$k = -4 \frac{(q_0/p_0)^{1/2}}{2\alpha + 2}, \quad \phi(x) = 4 \frac{(q_0/p_0)^{1/2}}{2\alpha + 2} \left(1 - \frac{1}{1 + \left(\frac{\beta + \gamma}{2\alpha + 2}\right)(\ln x)^{-1}} \right),$$

and with $\gamma < \beta$,

$$k = 0, \quad \phi(x) = -4(q_0/p_0)^{1/2} \frac{(\ln x)^{(\gamma - \beta)/2}}{(2\alpha + 2) + (\beta + \gamma)(\ln x)^{-1}}.$$

In general, the expressions for y_{\pm} are quite complicated so we examine a simpler special case. Let in (2.26) $\delta = 0$, $0 < \gamma < \beta - 1$, $\alpha > -1$, $\gamma + \beta > 1$. Then

$$(p\tilde{q}) = p_0 x^{2\alpha + 2} [q_0 (\ln x)^{\gamma + \beta} - z (\ln x)^{\beta}], \quad \frac{(p\tilde{q})'}{(p\tilde{q})} = O\left(\frac{1}{x}\right).$$

This gives

$$\begin{aligned} \phi &= O((\ln x)^{(\gamma - \beta)/2}), \quad \Phi = O(\phi^2) = O((\ln x)^{\gamma - \beta}), \\ \frac{(p\tilde{q})'}{(p\tilde{q})} \Phi &= O((\ln x)^{(\gamma - \beta)}/x) \in \mathcal{L}([a, \infty)). \end{aligned}$$

Since $k = 0$ here, (6.18) gives after normalization of constants,

$$y_+ = [1 + o(1)]x^{-(\alpha + 1)}(\ln x)^{-(\gamma + \beta)/2}, \quad y_- = [1 + o(1)].$$

This gives $y_+ \in \mathcal{L}_w^2([a, \infty))$ and $y_- \notin \mathcal{L}_w^2([a, \infty))$. In (3.2) choose $y_4 = y_+$, $y_3 = y_-$. Then

$$\begin{aligned} \int_a^x |K(x, s)|^2 w(s) ds &= \int_a^x |y_4(x)y_3(s)|^2 w(s) ds \\ &= O\left(x^{-2(\alpha + 1)}(\ln x)^{-(\gamma + \beta)} \int_a^x s^{\alpha} ds\right) \\ &= O\left(x^{-\alpha - 1}(\ln x)^{-(\gamma + \beta)}\right). \end{aligned} \tag{6.25}$$

Since $\gamma + \beta > 1$, we have that

$$\int_a^{\infty} \int_a^x |K(x, s)|^2 w(s)w(x) ds dx = O\left(\int_a^{\infty} x^{-1}(\ln x)^{-(\gamma + \beta)} dx\right) < \infty,$$

and thus by (3.4) the resolvent is Hilbert-Schmidt.

Another approach to examples such as (6.24) when the range of parameters falls outside the scope of Theorem 6.1 is by a Kummer-Liouville transformation, see for example [1, 2]. For example in (6.24) with $\delta \geq \beta - 2$ and p real choose in (6.4)

$$y(x) = \mu(x)Y(t), \quad t = \int_a^x \gamma(s) ds, \quad \gamma := [w/p]^{1/2}, \quad \mu(x) := [pw]^{-1/2}.$$

Then

$$L[y] = zy, \Leftrightarrow \frac{1}{\tilde{w}} [-(\tilde{p}Y^*)' + \tilde{q}Y] = zY, \quad \cdot = d/dt, \quad (6.26)$$

where

$$\tilde{p} = p\mu^2\gamma = 1, \quad \tilde{w} = w\mu^2/\gamma = 1, \quad \tilde{q} = -\mu\gamma^{-1}(p\mu')' + \mu^2\gamma^{-1}q.$$

The first summand of \tilde{q} is independent of q . It involves two differentiations. In the coefficients of (6.24) this gives in the first summand as a linear combination of $p_0(\ln x)^{\beta-\delta}$, $p_0(\ln x)^{\beta-\delta-1}$, and $p_0(\ln x)^{\beta-\delta-2}$. The second gives $\mu^2\gamma^{-1}q = q/w = q_0(\ln x)^{\gamma-\delta}$. Thus \tilde{q} is a sum of four logarithmic terms. For example with $\delta = \beta$, it follows that $t = \ln x$, and \tilde{q} has the form $\tilde{q} = q_0t^{\gamma-\delta} + c_0 + c_1/t + c_2/t^2$. Equations of this type can be handled by the methods of [5, 7]. In particular we see that for $0 < \alpha < 1$ the operator is not Hilbert-Schmidt.

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