

EXPLORING MULTIPLE SOLUTIONS AND NUMERICAL APPROACHES FOR A SIXTH-ORDER BOUNDARY VALUE PROBLEM

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Abstract. We analyze the existence of multiple solutions for a sixth-order boundary value problem. Firstly, we introduce an operator that transforms the problem into a fixed-point problem and delineate its key properties. Subsequently, we investigate the existence of solutions in the functional space $C^1[0, 1]$, employing the fixed-point theorem of Avery-Peterson. We then provide non-trivial examples and establish a theorem based on the Banach-Piccard theorem, motivating the definition of a numerical method based on the compression principle for the problem. Additionally, we discuss the utilization of nonlinear optimization methods for the problem and compare them with the classical method based on the contraction principle.

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REFERENCES

- [1] M. ADJUSTOVS AND A. LEPINS, *Extremal solutions of a boundary value problem for a sixth-order equation*, Differential Equations, **50** (2), 141–146, 2014.
- [2] R. P. AGARWAL, B. KOVACS AND D. O'REGAN, *Existence of positive solution for a sixth-order differential system with variable parameters*, Journal of Applied Mathematics and Computing, **44**, 437–454, 2014.
- [3] R. P. AGARWAL, B. KOVACS AND D. O'REGAN, *Positive solutions for a sixth-order boundary value problem with four parameters*, Boundary Value Problems, **2013** (1), 1–22, 2013.
- [4] MAGNUS ALVELID, *Sixth order differential equation for sandwich beam deflection including transverse shear*, Composite Structures, **102**, 29–37, 2013.
- [5] ZOUAOUI BEKRI AND SLIMANE BENAICHA, *Positive solutions for boundary value problem of sixth-order elastic beam equation*, Open Journal of Mathematical Sciences, **4** (1), 9–17, 2020.
- [6] GABRIELE BONANNO, PASQUALE CANDITO AND DONAL O'REGAN, *Existence of Nontrivial Solutions for Sixth-Order Differential Equations*, Mathematics, **9** (16), 2021.
- [7] SUBRAHMANYAN CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, Courier Corporation, 2013.
- [8] J. V. CHAPAROVA, L. A. PELETIER, S. A. TERSIAN FAZHAN GENG AND YONGGANG YE, *Existence and nonexistence of nontrivial solutions of semilinear sixth-order ordinary differential equations*, Appl. Math. Lett., **17** (10), 1207–1212, 2004.
- [9] T. GARBUIZA, *On solutions of one 6-th order nonlinear boundary value problem*, Mathematical Modelling and Analysis, **13** (3), 349–355, 2008.
- [10] SUQUIN GE, WANYI WANG AND QIUXIA YANG, *Dependence of eigenvalues of sixth-order boundary value problems on the boundary*, Bull. Aust. Math. Soc., **90** (3), 457–468, 2014.
- [11] K. GHANBARI AND H. MIRZAEI, *On the isospectral sixth order Sturm-Liouville equation*, J. Lie Theory, **23** (4), 921–935, 2013.
- [12] GARY A. GLATZMAIER, *Numerical simulations of stellar convective dynamos III. At the base of the convection zone*, Geophysical & Astrophysical Fluid Dynamics, **31** (1–2), 137–150, 1985.

- [13] J. R. GRAEF AND B. YANG, *Boundary value problems for sixth order nonlinear ordinary differential equations*, Dynam. Systems Appl., **10** (4), 465–475, 2001.
- [14] T. GYULOV, *Trivial and nontrivial solutions of a boundary value problem for a sixth-order ordinary differential equation*, Comptes Rendus-Academie Bulgarie Des Science, **58** (9), 1013–1018, 2005.
- [15] FAOUZI HADDOUCHI AND NOURREDINE HOUARI, *Solvability of a sixth-order boundary value problem with multi-point and multi-term integral boundary conditions*, Mathematical Methods in the Applied Sciences, **48** (3), 3400–3413, 2025.
- [16] C. T. KELLEY, *Iterative methods for optimization*, SIAM, 1999.
- [17] AMMAR KHANFER, LAZHAR BOUGOFFA AND NAWAL ALHELALI, *On the Sixth-Order Beam Equation of Small Deflection with Variable Parameters*, Mathematics, **13** (5), 2025.
- [18] A. L. M. MARTINEZ, E. V. CASTELANI, C. A. P. MARTINEZ, G. M. BRESSAN AND R. M. SOUZA, *Multiple Solutions To A Third-Order Three-Point Nonhomogeneous Boundary Value Problem Aided By Nonlinear Programming Methods*, Differential Equations & Applications, **13** (1), 35–49, 2021.
- [19] A. L. M. MARTINEZ, M. R. A. FERREIRA AND E. V. CASTELANI, *Theoretical and Numerical Aspects of a Third-order Three-point Nonhomogeneous Boundary Value Problem*, Trends in Applied and Computational Mathematics, **20** (3), 417–427, 2019.
- [20] A. L. M. MARTINEZ, C. A. P. MARTINEZ, G. M. BRESSAN, R. MOLINA SOUZA AND E. W. STIEGELMEIER, *Multiple Solutions for a Sixth Order Boundary Value Problem*, Trends in Computational and Applied Mathematics, **22**, 2–12, 2021.
- [21] A. L. M. MARTINEZ, C. A. PENDEZA MARTINEZ, T. S. PINTO AND E. V. CASTELANI, *Analysis of Solutions for a Sixth-Order Boundary Value Problem and Numerical Search using Nonlinear Programming Techniques*, Engineering Letters, **32** (3), 488–492, 2024.
- [22] C. A. P. MARTINEZ, A. L. M. MARTINEZ, G. M. BRESSAN, E. V. CASTELANI AND R. M. SOUZA, *Multiple solutions for a fourth order equation with nonlinear boundary conditions: theoretical and numerical aspects*, Differential Equations & Applications, **11** (3), 335–348, 2019.
- [23] M. MÖLLER AND B. ZINSOU, *Sixth order differential operators with eigenvalue dependent boundary conditions*, Applicable Analysis and Discrete Mathematics, 378–389, 2013.
- [24] SAEID SHOKOOH, *Existence of solutions for a sixth-order nonlinear equation*, Rendiconti del Circolo Matematico di Palermo Series 2, **72** (8), 4251–4271, 2023.
- [25] JUKI TOOMRE, J.-P. ZAHN, JEAN LATOUR AND EDWARD A. SPIEGEL, *Stellar convection theory. II. Single-mode study of the second convection zone in an A-type star*, Astrophysical Journal, **207**, 545–563, 1976.
- [26] B. YANG, *Positive solutions to a nonlinear sixth order boundary value problem*, Differential Equations & Applications, **11** (2), 307–317, 2019.