

OPIAL TYPE INTEGRAL INEQUALITIES FOR FRACTIONAL DERIVATIVES II

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Abstract. A certain class of functions is used to apply results related to opial type inequalities. Also applications of Riemann-Liouville fractional integral and Caputo fractional derivative with respect to this class of functions are given.

1. Introduction and preliminaries

In [2] we gave applications of Riemann-Liouville fractional integral, Caputo fractional derivative and integral representation of Riemann-Liouville fractional derivative [6], on opial type inequalities considering a particular class of functions. Here we prove similar results by using another class $U(v, K)$ of functions $u : [a, b] \rightarrow \mathbb{R}$ which admits representation [5, p. 238],

$$u(x) = \int_x^b K(x, t)v(t)dt. \quad (1.1)$$

where v is a continuous function and K is an arbitrary non-negative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a, b]$.

We can observe that the following result holds for the class of functions $U(v, K)$.

THEOREM 1. *Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U(v, K)$ where*

$$\left(\int_x^b (K(x, t))^p dt \right)^{\frac{1}{p}} \leq M, \quad p^{-1} + q^{-1} = 1.$$

Then

$$\int_a^b |u(x)|^{1-q} \phi'(|u(x)| |v(x)|^q) dx \leq \frac{q}{M^q} \phi \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right). \quad (1.2)$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse of the inequality in (1.2) holds.

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Proof. Proof is on the same lines as the proof of such theorem in [7], (see also theorem 8.15 in [5, p. 237, 238]). \square

To prove exponential-convexity of a class of certain functions the following Definition and Proposition [3], help us.

DEFINITION 1. A function $h : (a, b) \rightarrow \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^n u_i u_j h(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$ and all choices $u_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ and $x_i \in (a, b)$, such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

PROPOSITION 1. Let $h : (a, b) \rightarrow \mathbb{R}$. The followings are equivalent.

(i) h is exponentially convex.

(ii) h is continuous and

$$\sum_{i,j=1}^n u_i u_j h\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for every $u_i \in \mathbb{R}$ and every $x_i, x_j \in (a, b)$, $1 \leq i, j \leq n$.

(iii) h is continuous and for every $x_i \in (a, b)$, $i = 1, 2, \dots, n$,

$$\det \left[h\left(\frac{x_i + x_j}{2}\right) \right]_{i,j=1}^k \geq 0, \quad k = 1, 2, \dots, n.$$

In [3] we also have the following corollary.

COROLLARY 1. If $h : (a, b) \rightarrow (0, \infty)$ is exponentially convex function then h is a log-convex function:

We present the paper in such a way that section 2 contains mean value theorems, exponential convexity, Cauchy's means for a class of linear functionals. In section 3 we give theorems for Riemann-Liouville fractional integral and Caputo fractional derivative.

2. Preparatory inequalities

DEFINITION. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. We define the linear functional $\beta_h(u, v)$ as:

$$\beta_h(u, v) = \frac{q}{M^q} h\left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}}\right) - \int_a^b |u(x)|^{1-q} h'(|u(x)|) |v(x)|^q dx. \quad (2.1)$$

We have proved the following Lemma in [2].

LEMMA 1. Let $h \in C^2(I)$ $I \subseteq (0, \infty)$, and $g(x) = x^q$, $q > 1$ with

$$m' \leq \frac{\xi h''(\xi) - (q-1)h'(\xi)}{q^2 \xi^{2q-1}} \leq M' \text{ for all } \xi \in I.$$

Then the functions ϕ_1, ϕ_2 defined as:

$$\phi_1(x) = \frac{M'x^{2q}}{2} - h(x), \quad \phi_2(x) = h(x) - \frac{m'x^{2q}}{2},$$

are convex functions with respect to $g(x) = x^q$, that is $\phi_i(x^{\frac{1}{q}})$, $i = 1, 2$, are convex.

THEOREM 2. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval, then there exists $\xi \in I$ such that the following equality holds

$$\beta_h(u, v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \right). \tag{2.2}$$

Proof. Suppose that $\min_{y \in I}(\psi(y)) = m_1$ and $\max_{y \in I}(\psi(y)) = M_1$ where

$$\psi(y) = \frac{yh''(y) - (q-1)h'(y)}{q^2 y^{2q-1}}.$$

Using ϕ_1 from Lemma 1 instead of ϕ in (1.2) we get

$$\begin{aligned} & \frac{q}{M^q} h \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right) - \int_a^b |u(x)|^{1-q} h'(|u(x)|) |v(x)|^q dx \\ & \leq \frac{qM_1}{2} \left(M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \right). \end{aligned} \tag{2.3}$$

Similarly, using ϕ_2 from Lemma 1 instead of ϕ in (1.2) we get

$$\begin{aligned} & \frac{q}{M^q} h \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right) - \int_a^b |u(x)|^{1-q} h'(|u(x)|) |v(x)|^q dx \\ & \geq \frac{qm_1}{2} \left(M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \right). \end{aligned} \tag{2.4}$$

By combining the above two inequalities and using the fact that

$$m \leq \frac{yh''(y) - (q-1)h'(y)}{q^2 y^{2q-1}} \leq M$$

there exist $\xi \in I$ such that we get (2.2). \square

THEOREM 3. Let $h_1, h_2 : [0, \infty) \rightarrow \mathbb{R}$ be the functions with assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval and

$$M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \neq 0.$$

Then there exists $\xi \in I$ such that

$$\frac{\beta_{h_1}(u, v)}{\beta_{h_2}(u, v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)},$$

provided the denominators are not equal to zero.

Proof. The proof is similar to the proof of such theorems for example see in [4]. \square

Throughout the paper we frequently use the following family of convex functions with respect to $g(x) = x^q$ ($q > 1$) on $(0, \infty)$.

$$\varphi_s(x) = \begin{cases} \frac{q^2}{s(s-q)}x^s, & s \neq 0, q; \\ -q \log x, & s = 0; \\ qx^q \log x, & s = q. \end{cases} \tag{2.5}$$

In the following we use $\Lambda_{\varphi_s}(u, v)$ in the place of $\beta_{\varphi_s}(u, v)$, when we put $h = \varphi_s$ in equation (2.1), that is

$$\Lambda_{\varphi_s}(u, v) = \begin{cases} \frac{q^2}{M^q s(s-q)} \left(qM^s \left(\int_a^b |v(x)|^q dx \right)^{\frac{s}{q}} - sM^q \int_a^b |u(x)|^{s-q} |v(x)|^q dx \right), & s \neq 0, q; \\ \frac{q}{M^q} \left(-q \log \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right) + M^q \int_a^b |u(x)|^{-q} |v(x)|^q dx \right), & s = 0; \\ \frac{q^2}{M^q} \left(M^q \int_a^b |v(x)|^q \log \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right) + \int_a^b (1+q \log |u(x)|) |v(x)|^q dx \right), & s = q. \end{cases} \tag{2.6}$$

THEOREM 4. For $\Lambda_{\varphi_s}(u, v)$ defined above we have

- a) for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}$ the matrix $A = \left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}(u, v) \right]_{i,j=1}^n$, is a positive semi-definite matrix.
- b) the function $s \mapsto \Lambda_{\varphi_s}(u, v)$, $s \in \mathbb{R}$ is exponentially convex.
- c) if $\Lambda_{\varphi_s}(u, v)$ is positive, then the function $s \mapsto \Lambda_{\varphi_s}(u, v)$, $s \in \mathbb{R}$ is log-convex.

Proof. a) Define the function $f(x) = \sum_{i,j=1}^n u_i u_j \varphi_{p_{ij}}(x)$, where $p_{ij} = \frac{p_i+p_j}{2}$.

Set

$$F(x) = f(x^{\frac{1}{q}}) = \sum_{i,j=1}^n u_i u_j \varphi_{p_{ij}}(x^{\frac{1}{q}}).$$

Then

$$F''(x) = \left(\sum_{i=1}^n u_i x^{\frac{p_i-2q}{2q}} \right)^2 \geq 0.$$

This implies that f is convex with respect to $g(x) = x^q$, and also $f(0) = 0$. So using this f in the place of h in (2.1) we have

$$\sum_{i,j=1}^n u_i u_j \Lambda_{\varphi_{p_{ij}}}(u, v) \geq 0. \tag{2.7}$$

Hence the matrix, $A = \left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}(u, v) \right]_{n \times n}$ is positive semi-definite.

b) Since after some computation we have $\lim_{s \rightarrow 0} \Lambda_{\varphi_s}(u, v) = \Lambda_{\varphi_0}(u, v)$ and also $\lim_{s \rightarrow q} \Lambda_{\varphi_s}(u, v) = \Lambda_{\varphi_q}(u, v)$, so $\Lambda_{\varphi_s}(u, v)$ is continuous, then by (2.7) and Proposition 1 we have $s \mapsto \Lambda_{\varphi_s}(u, v)$ is exponentially convex.

c) As $\Lambda_{\varphi_s}(u, v)$ is positive and exponentially convex so by Corollary 1, $\Lambda_{\varphi_s}(u, v)$ is log-convex. \square

If we put $h_1 = \varphi(s)$; $h_2 = \varphi_r$ in Theorem 3, then we have a mean defined as:

$$N_{s,r}^{[q]}(u, v) = \left(\frac{\beta_{\varphi_s}(u, v)}{\beta_{\varphi_r}(u, v)} \right)^{\frac{1}{s-r}}, \quad s \neq r \tag{2.8}$$

that is

$$N_{s,r}^{[q]}(u, v) = \left(\frac{r(r-q) q M^s (\int_a^b |v(x)|^q dx)^{\frac{s}{q}} - s M^q \int_a^b |u(x)|^{s-q} |v(x)|^q dx}{s(s-q) q M^r (\int_a^b |v(x)|^q dx)^{\frac{r}{q}} - r M^q \int_a^b |u(x)|^{r-q} |v(x)|^q dx} \right)^{\frac{1}{s-r}}, \tag{2.9}$$

$s, r \neq q, \quad s \neq r.$

In limiting cases we have:

When s goes to r

$$N_{r,r}^{[q]}(u, v) = \exp \left(\frac{A}{B} - \frac{2r-q}{r(r-q)} \right), \quad r \neq q, \tag{2.10}$$

where

$$A = q M^r \left(\int_a^b |v(x)|^q dx \right)^{\frac{r}{q}} \log \left(M \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \right) - M^q \left(\int_a^b |u(x)|^{r-q} |v(x)|^q dx + r \int_a^b |u(x)|^{r-q} \log(|u(x)|) |v(x)|^q dx \right),$$

and

$$B = q M^r \left(\int_a^b |v(x)|^q dx \right)^{\frac{r}{q}} - r M^q \int_a^b |u(x)|^{r-q} |v(x)|^q dx.$$

In (2.10) when r goes to q we get

$$N_{s,q}^{[q]}(u, v) = N_{q,s}^{[q]}(u, v) = \left(\frac{q(qM^s(\int_a^b |v(x)|^q dx)^{\frac{s}{q}} - sM^q \int_a^b |u(x)|^{s-q} |v(x)|^q dx)}{s(s-q)M^q(q \int_a^b |v(x)|^q dx \log(M(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) - \lambda)} \right)^{\frac{1}{s-q}}$$

(2.11)

where $\lambda = \int_a^b |v(x)|^q dx + q \int_a^b \log(|u(x)|) |v(x)|^q dx$.

When s goes to q we have

$$N_{q,q}^{[q]}(u, v) = \exp \left(\frac{1}{2} \left(\frac{P}{Q} - \frac{2}{q} \right) \right), \tag{2.12}$$

where

$$P = qM^q(q \int_a^b |v(x)|^q dx (\log(M(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}))^2 - (2 \int_a^b \log |u(x)| |v(x)|^q dx + q \int_a^b (\log |u(x)|)^2 |v(x)|^q dx),$$

and

$$Q = qM^q(q \int_a^b |v(x)|^q dx \log(M(\int_a^b |v(x)|^q dx)^{\frac{1}{q}}) - (\int_a^b |v(x)|^q dx + q \int_a^b \log |u(x)| |v(x)|^q dx)).$$

Now we prove monotonicity of means $N_{s,r}^{[q]}(u, v)$.

THEOREM 5. *Let $t, s, l, m \in \mathbb{R}_+$ such that $t \leq l, s \leq m$. Then*

$$N_{t,s}^{[q]}(u, v) \leq N_{l,m}^{[q]}(u, v).$$

Proof. The following inequality holds for convex function φ see in [5, p. 4]

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{2.13}$$

where $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$.

Since by Theorem 4, $\Lambda_{\varphi_s}(u, v)$ is log-convex, we can put in (2.14):

$$\varphi = \log \Lambda_{\varphi_s}(u, v), \quad x_1 = s, \quad x_2 = t, \quad y_1 = l, \quad y_2 = m.$$

We get

$$\frac{\log \Lambda_{\varphi_t}(u, v) - \log \Lambda_{\varphi_s}(u, v)}{t - s} \leq \frac{\log \Lambda_{\varphi_m}(u, v) - \log \Lambda_{\varphi_l}(u, v)}{m - l} \quad s \neq t, \quad l \neq m,$$

therefore we get

$$\left(\frac{\Lambda_{\varphi_l}(u, v)}{\Lambda_{\varphi_s}(u, v)}\right)^{\frac{1}{l-s}} \leq \left(\frac{\Lambda_{\varphi_m}(u, v)}{\Lambda_{\varphi_l}(u, v)}\right)^{\frac{1}{m-l}}, \quad s \neq t, l \neq m. \tag{2.14}$$

From (2.15) we get our result for $t \neq s, l \neq m$ and for $t = s, l = m; t \neq s, l = m; t = s, l \neq m$ we can consider limiting cases. \square

3. Inequalities for fractional integrals and derivatives

As we are interested for the class $U(v, K)$ of functions $u : [a, b] \rightarrow \mathbb{R}$ having representation (1.1) so we use the following definition of Riemann-Liouville fractional integral.

DEFINITION 2. Let $\alpha > 0$. For any $f \in L(a, b)$ the Riemann-Liouville fractional integral of f of order α is defined by

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t) dt, \quad x \in [a, b]. \tag{3.1}$$

THEOREM 6. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval and let $v \in C[a, b]$, has Riemann-Liouville fractional integral of order $\alpha, \alpha > \frac{1}{q}$.

Then then there exists $\xi \in I$ such that we have

$$\beta_h(I_b^\alpha v, v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(\frac{(b-a)^{q\alpha-1}}{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}} \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |I_b^\alpha v(x)|^q |v(x)|^q dx \right). \tag{3.2}$$

Proof. From Theorem 2, we have

$$\beta_h(u, v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \right), \tag{3.3}$$

for some $\xi \in I$ and v has Riemann-Liouville fractional integral of order α , so

$$u(x) = I_b^\alpha v(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} v(t) dt, \quad x \in [a, b].$$

Here

$$K(x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}, & x \leq t \leq b; \\ 0, & a < t \leq x. \end{cases}$$

Let

$$U(x) = \left(\int_x^b (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(b-x)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}$$

$$U'(x) = -\frac{(\alpha-\frac{1}{q})(b-x)^{(\alpha-\frac{1}{q}-1)}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}} \leq 0, \quad \text{for } \alpha > \frac{1}{q}, \quad x \in [a,b].$$

$U(x)$ is decreasing in $[a,b]$, therefore

$$\max_{x \in [a,b]} (U(x)) = \frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}, \quad \text{for } \alpha > \frac{1}{q}.$$

That is

$$\left(\int_x^b K(x,t)^p \right)^{\frac{1}{p}} \leq \frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}},$$

so here $M = \frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}$, and by putting the values of $u(x)$ and M in (3.3) we get $\beta_h(I_b^\alpha v, v)$ as required in (3.2). \square

THEOREM 7. *Let $h_1, h_2 : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval and let $v \in C[a, b]$, has Riemann-Liouville fractional integral of order $\alpha, \alpha > \frac{1}{q}$.*

Then then there exists $\xi \in I$ such that we have

$$\frac{\beta_{h_1}(I_b^\alpha v, v)}{\beta_{h_2}(I_b^\alpha v, v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)}, \tag{3.4}$$

provided the denominators are not equal to zero.

Proof. By Theorem 3 we have

$$\frac{\beta_{h_1}(u, v)}{\beta_{h_2}(u, v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)},$$

for some $\xi \in I$ and from the proof of Theorem 6, we can easily get (3.4) with required conditions. \square

If v has Riemann-Liouville fractional integral of order α , $\alpha > \frac{1}{q}$. Then (2.7) becomes

$$\Lambda_{\varphi_s}(I_b^\alpha v, v) = \begin{cases} \left[\frac{q^2}{s(s-q)} \left(q \frac{(b-a)^{(s-q)(\alpha-\frac{1}{q})} D^{\frac{s}{q}}}{\Gamma^{s-q}(\alpha)(p\alpha-p+1)^{\frac{s-q}{p}}} - s \int_a^b |I_b^\alpha v(x)|^{s-q} |v(x)|^q dx \right), & s \neq 0, q; \right. \\ \left. q \left(-q \frac{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}} \log\left(\frac{(b-a)^{\alpha-\frac{1}{q}} D^{\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}\right) + \int_a^b |I_b^\alpha v(x)|^{-q} |v(x)|^q dx \right), & s = 0; \right. \\ \left. q^2 \left(D \log\left(\frac{(b-a)^{\alpha-\frac{1}{q}} D^{\frac{1}{q}}}{\Gamma(\alpha)(p\alpha-p+1)^{\frac{1}{p}}}\right) + \frac{\Gamma^q(\alpha)(p\alpha-p+1)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}} \int_a^b (1+q \log |I_b^\alpha v(x)|) |v(x)|^q dx \right), & s = q, \right. \end{cases} \tag{3.5}$$

where $D = \int_a^b |v(x)|^q dx$.

THEOREM 8. For $\Lambda_{\varphi_s}(I_b^\alpha v, v)$ defined above we have

a) for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}$ the matrix $A = \left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}(I_b^\alpha v, v) \right]_{i,j=1}^n$, is a positive semi-definite matrix.

b) the function $s \mapsto \Lambda_{\varphi_s}(I_b^\alpha v, v)$, $s \in \mathbb{R}$ is exponentially convex.

c) if $\Lambda_{\varphi_s}(I_b^\alpha v, v)$ is positive, then the function $s \mapsto \Lambda_{\varphi_s}(I_b^\alpha v, v)$, $s \in \mathbb{R}$ is log-convex.

Proof. The proof is similar to the proof of Theorem 4. \square

If we put $h_1 = \varphi(s)$; $h_2 = \varphi(r)$ in Theorem 7, then we have a mean defined as:

$$\Pi_{s,r}^{[q]}(I_b^\alpha v, v) = \left(\frac{\beta_{\varphi_s}(I_b^\alpha v, v)}{\beta_{\varphi_r}(I_b^\alpha v, v)} \right)^{\frac{1}{s-r}}, \quad s \neq r \tag{3.6}$$

that is

$$\Pi_{s,r}^{[q]}(I_b^\alpha v, v) = \left(\frac{r(r-q) q \Gamma^{q-s}(\alpha)(p\alpha-p+1)^{\frac{q-s}{p}} (b-a)^{(s-q)(\alpha-\frac{1}{q})} D^{\frac{s}{q}} - s \int_a^b |I_b^\alpha v(x)|^{s-q} |v(x)|^q dx}{s(s-q) q \Gamma^{q-r}(\alpha)(p\alpha-p+1)^{\frac{q-r}{p}} (b-a)^{(r-q)(\alpha-\frac{1}{q})} D^{\frac{r}{q}} - r \int_a^b |I_b^\alpha v(x)|^{r-q} |v(x)|^q dx} \right)^{\frac{1}{s-r}}, \quad s, r \neq q, s \neq r. \tag{3.7}$$

In limiting cases we have:

$$\Pi_{r,r}^{[q]}(I_b^\alpha v, v) = \exp\left(\frac{A_1}{B_1} - \frac{2r-q}{r(r-q)}\right), \quad r \neq q, \tag{3.8}$$

$$\Pi_{s,q}^{[q]}(I_b^\alpha v, v) = \Pi_{q,s}^{[q]}(I_b^\alpha v, v) =$$

$$\left(\frac{q \left(\Gamma^{q-s}(\alpha)(p\alpha - p + 1)^{\frac{q-s}{p}} (b-a)^{(s-q)(\alpha-\frac{1}{q})} D^{\frac{s}{q}-s} \int_a^b |I_b^\alpha v(x)|^{s-q} |v(x)|^q dx \right)}{s(s-q) \left((\log D - 1)D - q \log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}} (b-a)^{-(\alpha-\frac{1}{q})}) - q \int_a^b \log |I_b^\alpha v(x)| |v(x)|^q dx \right)} \right)^{\frac{1}{s-q}},$$

$s \neq q. \quad (3.9)$

$$\prod_{q,q}^{[q]}(I_b^\alpha v, v) = \exp \left(\frac{1}{2} \left(\frac{P_1}{Q_1} - \frac{2}{q} \right) \right), \quad (3.10)$$

where A_1, B_1, P_1, Q_1 are as follows:

$$\begin{aligned} A_1 &= \Gamma^{q-r}(\alpha)(p\alpha - p + 1)^{\frac{q-r}{p}} (b-a)^{(r-q)(\alpha-\frac{1}{q})} D^{\frac{r}{q}} \log D \\ &\quad - q \log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}} (b-a)^{-(\alpha-\frac{1}{q})}) \Gamma^{q-r}(\alpha)(p\alpha - p + 1)^{\frac{q-r}{p}} \\ &\quad \times (b-a)^{(r-q)(\alpha-\frac{1}{q})} D^{\frac{r}{q}} - \int_a^b |I_b^\alpha v(x)|^{r-q} |v(x)|^q dx \\ &\quad - r \int_a^b |I_b^\alpha v(x)|^{r-q} \log |I_b^\alpha v(x)| |v(x)|^q dx, \\ B_1 &= q \Gamma^{q-r}(\alpha)(p\alpha - p + 1)^{\frac{q-r}{p}} (b-a)^{(r-q)(\alpha-\frac{1}{q})} D^{\frac{r}{q}} - r \int_a^b |I_b^\alpha v(x)|^{r-q} |v(x)|^q dx, \\ P_1 &= \frac{D(\log D)^2}{q} - D \log D \log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}} (b-a)^{-(\alpha-\frac{1}{q})}) \\ &\quad + q(\log(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}} (b-a)^{-(\alpha-\frac{1}{q})}))^2 - 2 \int_a^b \log |I_b^\alpha v(x)| |v(x)|^q dx \\ &\quad - q \int_a^b (\log |I_b^\alpha v(x)|)^2 |v(x)|^q dx, \\ Q_1 &= (\log D - 1)D - q \log \left(\Gamma(\alpha)(p\alpha - p + 1)^{\frac{1}{p}} (b-a)^{-(\alpha-\frac{1}{q})} \right) \\ &\quad - q \int_a^b \log |I_b^\alpha v(x)| |v(x)|^q dx. \end{aligned}$$

Further we prove monotonicity of above means.

THEOREM 9. *Let $t, s, l, m \in \mathbb{R}_+$ such that $t \leq l, s \leq m$.*

Then

$$\prod_{t,s}^{[q]}(I_b^\alpha v, v) \leq \prod_{l,m}^{[q]}(I_b^\alpha v, v).$$

Proof. The following inequality holds for convex function φ see in [5, p. 4],

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \quad (3.11)$$

where $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$.

Since by Theorem 8, $\Lambda_s(I_b^\alpha v, v)$ is log-convex, we can put in (3.11):

$$\varphi = \log \Lambda_\varphi(I_b^\alpha v, v), \quad x_1 = s, \quad x_2 = t, \quad y_1 = l, \quad y_2 = m,$$

we get for $s \neq t, l \neq m$

$$\frac{\log \Lambda_{\varphi_t}(I_b^\alpha v, v) - \log \Lambda_{\varphi_s}(I_b^\alpha v, v)}{t - s} \leq \frac{\log \Lambda_{\varphi_m}(I_b^\alpha v, v) - \log \Lambda_{\varphi_l}(I_b^\alpha v, v)}{m - l},$$

therefore we have

$$\left(\frac{\Lambda_{\varphi_t}(I_b^\alpha v, v)}{\Lambda_{\varphi_s}(I_b^\alpha v, v)} \right)^{\frac{1}{t-s}} \leq \left(\frac{\Lambda_{\varphi_m}(I_b^\alpha v, v)}{\Lambda_{\varphi_l}(I_b^\alpha v, v)} \right)^{\frac{1}{m-l}}. \tag{3.12}$$

From (3.12), we get our result for $t \neq s, l \neq m$ and for $t = s, l = m; t \neq s, l = m; t = s, l \neq m$ we can consider limiting cases. \square

To give more results we apply the following definition of Caputo fractional derivative (see [6, p. 92]).

DEFINITION 3. Let $\alpha > 0$ and $\alpha \notin \{1, 2, \dots\}$. The Caputo fractional derivative of order α for a function $f : [a, b] \rightarrow \mathbb{R}$ belonging to the space $AC^n[a, b]$ of absolutely continuous functions is defined by

$$D_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_x^b \frac{f^{(n)}(s)}{(x - s)^{\alpha - n + 1}} ds, \tag{3.13}$$

where $n = [\alpha] + 1$, and $[\alpha]$ stands for the largest integer not greater than α .

THEOREM 10. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h \in C^2(I)$ with $h(0) = 0$, where $I \subseteq (0, \infty)$ is compact interval and let $v \in AC^n[a, b]$ for even n , has Caputo fractional derivative of order $\alpha, \alpha \notin \{1, 2, 3, \dots\}$ and, $0 < \alpha - [\alpha] < \frac{1}{p}$, then there exists $\xi \in I$ such that the following equality holds

$$\beta_h(D_b^\alpha v, v^{(n)}) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(\frac{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}}{\Gamma^q([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{q}{p}}} \right. \\ \left. \times \left(\int_a^b |v^{(n)}(x)|^q dx \right)^2 - 2 \int_a^b |D_b^\alpha v(x)|^q |v^{(n)}(x)|^q dx \right). \tag{3.14}$$

Proof. From Theorem 2 we have

$$\beta_h(u, v) = \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(M^q \left(\int_a^b |v(x)|^q dx \right)^2 - 2 \int_a^b |u(x)|^q |v(x)|^q dx \right), \tag{3.15}$$

and $v \in AC^n[a, b]$, has Caputo fractional derivative of order α , and n is even so

$$u(x) = D_b^\alpha v(x) = \frac{1}{\Gamma(n - \alpha)} \int_x^b (t - x)^{n - \alpha - 1} v^{(n)}(t) dt.$$

Here

$$K(x, t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(t-x)^{n-\alpha-1}, & x \leq t \leq b; \\ 0, & a \leq t < x. \end{cases}$$

Let

$$V(x) = \left(\int_x^b (K(x, t))^p dt \right)^{\frac{1}{p}} = \frac{(b-x)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)(p(n-\alpha)-p+1)^{\frac{1}{p}}}, \quad n = [\alpha] + 1$$

$$V'(x) = -\frac{(p([\alpha]-\alpha)+1)^{\frac{1}{q}}(b-x)^{(n-\alpha-\frac{1}{q}-1)}}{p\Gamma([\alpha]-\alpha+1)} \leq 0,$$

for $0 < \alpha - [\alpha] < \frac{1}{p}$, $x \in [a, b]$. $V(x)$ is decreasing in $[a, b]$.

Therefore, $\max_{x \in [a, b]} (V(x)) = \frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}$, for $0 < \alpha - [\alpha] < \frac{1}{p}$.

That is

$$\left(\int_x^b K(x, t)^p \right)^{\frac{1}{p}} \leq \frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}, \quad 0 < \alpha - [\alpha] < \frac{1}{p},$$

so here $M = \frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}}$, for $0 < \alpha - [\alpha] < \frac{1}{p}$.

Therefore by putting $v = v^n$ and the values of $u(x)$ and M in (3.15) we get $\beta_h(D_b^\alpha v, v^{(n)})$ as required in (3.14). \square

THEOREM 11. *Let $h_1, h_2 : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$ such that $h_1(0) = h_2(0) = 0$, where $I \subseteq (0, \infty)$ is compact interval and let $v \in AC^n[a, b]$ for even n , has Caputo fractional derivative of order α , $\alpha \notin \{1, 2, 3, \dots\}$ and, $0 < \alpha - [\alpha] < \frac{1}{p}$, then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\beta_{h_1}(D_b^\alpha v, v^{(n)})}{\beta_{h_2}(D_b^\alpha v, v^{(n)})} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)}, \tag{3.16}$$

Proof. By Theorem 3 we have

$$\frac{\beta_{h_1}(u, v)}{\beta_{h_2}(u, v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)},$$

and from the proof of Theorem 10, we get (3.16) with required condition. \square

If $v \in AC^n[a, b]$ for even n , has Caputo fractional derivative of order α , $\alpha \notin \{1, 2, 3, \dots\}$ and, $0 < \alpha - [\alpha] < \frac{1}{p}$. Then (2.7) becomes

$$\Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)}) = \left\{ \begin{array}{l} \frac{q^2}{s(s-q)} \left(\frac{q(b-a)^{(s-q)([\alpha]-\alpha+\frac{1}{p})} E^{\frac{s}{q}}}{\Gamma^{s-q}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{s-q}{p}}} - s \int_a^b |D_b^\alpha v(x)|^{s-q} |v^{(n)}(x)|^q dx \right), \quad s \neq 0, q; \\ q \left(\frac{-q\Gamma^q([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q}{p}}}{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}} \log \left(\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}} E^{\frac{1}{q}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}} \right) \right. \\ \quad \left. + \int_a^b |D_b^\alpha v(x)|^{s-q} |v^{(n)}(x)|^q dx \right), \quad s = 0; \\ q^2 \left(E \log \left(\frac{(b-a)^{[\alpha]-\alpha+\frac{1}{p}} E^{\frac{1}{q}}}{\Gamma([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{1}{p}}} \right) + \frac{(b-a)^{q([\alpha]-\alpha+\frac{1}{p})}}{\Gamma^q([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q}{p}}} \right. \\ \quad \left. \times \int_a^b (1 + q \log |D_b^\alpha v(x)|) |v^{(n)}(x)|^q dx \right), \quad s = q, \end{array} \right. \tag{3.17}$$

where $E = \int_a^b |v^{(n)}(x)|^q dx$.

THEOREM 12. For $\Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})$ defined above we have:

- a) The matrix $A = \left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}(D_b^\alpha v, v^{(n)}) \right]_{i,j=1}^n$, is a positive-semidefinite matrix.
- b) The function $s \mapsto \Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})$ is exponentially convex.
- c) $\Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})$ is log-convex.

Proof. For proof see the proof of Theorem 4. \square

If we put $h_1 = \varphi(s)$, $h_2 = \varphi(r)$ in Theorem 11, then we have a mean defined as:

$$\Pi_{s,r}^{[q]}(D_b^\alpha v, v^{(n)}) = \left(\frac{\beta_{\varphi_s}(D_b^\alpha v, v^{(n)})}{\beta_{\varphi_r}(D_b^\alpha v, v^{(n)})} \right)^{\frac{1}{s-r}}, \quad s \neq r \tag{3.18}$$

that is

$$\Pi_{s,r}^{[q]}(D_b^\alpha v, v^{(n)}) = \left(\frac{r(r-q) q\Gamma^{q-s}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-s}{p}} (b-a)^{(s-q)([\alpha]-\alpha+\frac{1}{p})} E^{\frac{s}{q}} - L_1}{s(s-q) q\Gamma^{q-r}([\alpha]-\alpha+1)(p([\alpha]-\alpha)+1)^{\frac{q-r}{p}} (b-a)^{(r-q)([\alpha]-\alpha+\frac{1}{p})} E^{\frac{r}{q}} - M_1} \right)^{\frac{1}{s-r}}, \tag{3.19}$$

$s, r \neq q, s \neq r.$

where

$$L_1 = s \int_a^b |D_b^\alpha v(x)|^{s-q} |v^{(n)}(x)|^q dx,$$

$$M_1 = r \int_a^b |D_b^\alpha v(x)|^{r-q} |v^{(n)}(x)|^q dx.$$

In limiting cases we have:

$$\Pi_{r,r}^{[q]}(D_b^\alpha v, v^{(n)}) = \exp\left(\frac{A_2}{B_2} - \frac{2r-q}{r(r-q)}\right), \quad r \neq q, \tag{3.20}$$

$$\begin{aligned} \Pi_{s,q}^{[q]}(D_b^\alpha v, v^{(n)}) &= \Upsilon_{q,s}^{[q]}(D_b^\alpha v, v^{(n)}) = \\ &\left(\frac{q \left(\Gamma^{q-s}([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{q-s}{p}} (b-a)^{(s-q)([\alpha] - \alpha + \frac{1}{p})} E^{\frac{s}{q}} - L_1 \right)}{s(s-q) \left((\log E - 1)E - q \log(\Gamma([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{1}{p}} (b-a)^{-([\alpha] - \alpha + \frac{1}{p})} - M_2 \right)} \right)^{\frac{1}{s-q}}; \end{aligned}$$

$s \neq q, \tag{3.21}$

$$\Pi_{q,q}^{[q]}(D_b^\alpha v, v^{(n)}) = \exp\left(\frac{1}{2} \left(\frac{P_2}{Q_2} - \frac{2}{q} \right)\right), \tag{3.22}$$

where A_2, B_2, M_2, P_2, Q_2 are as follows:

$$\begin{aligned} A_2 &= \Gamma^{q-r}([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{q-r}{p}} (b-a)^{(r-q)([\alpha] - \alpha + \frac{1}{p})} E^{\frac{r}{q}} \log E \\ &\quad - q \log(\Gamma([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{1}{p}} (b-a)^{-([\alpha] - \alpha + \frac{1}{p})} \Gamma^{q-r}([\alpha] - \alpha + 1) \\ &\quad \times (p([\alpha] - \alpha) + 1)^{\frac{q-r}{p}} (b-a)^{(r-q)([\alpha] - \alpha + \frac{1}{p})} E^{\frac{r}{q}} - \int_a^b |D_b^\alpha v(x)|^{r-q} |v^{(n)}(x)|^q dx \\ &\quad - r \int_a^b |D_b^\alpha v(x)|^{r-q} \log |D_b^\alpha v(x)| |v^{(n)}(x)|^q dx, \end{aligned}$$

$$\begin{aligned} B_2 &= q \Gamma^{q-r}([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{q-r}{p}} (b-a)^{(r-q)([\alpha] - \alpha + \frac{1}{p})} E^{\frac{r}{q}} \\ &\quad - r \int_a^b |D_b^\alpha v(x)|^{r-q} |v^{(n)}(x)|^q dx, \end{aligned}$$

$$M_2 = q \int_a^b \log |D_b^\alpha v(x)| |v^{(n)}(x)|^q dx,$$

$$\begin{aligned} P_2 &= \frac{E(\log E)^2}{q} - E \log E \log(\Gamma([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{1}{p}} (b-a)^{-([\alpha] - \alpha + \frac{1}{p})} \\ &\quad + q(\log(\Gamma([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{1}{p}} (b-a)^{-([\alpha] - \alpha + \frac{1}{p})}))^2 \\ &\quad - 2 \int_a^b \log |D_b^\alpha v(x)| |v(x)|^q dx - q \int_a^b (\log |D_b^\alpha v(x)|)^2 |v(x)|^q dx, \end{aligned}$$

$$\begin{aligned} Q_2 &= (\log E - 1)E - q \log(\Gamma([\alpha] - \alpha + 1)(p([\alpha] - \alpha) + 1)^{\frac{1}{p}} (b-a)^{-([\alpha] - \alpha + \frac{1}{p})} \\ &\quad - q \int_a^b \log |D_b^\alpha v(x)| |v^{(n)}(x)|^q dx. \end{aligned}$$

Now we prove monotonicity.

THEOREM 13. *Let $t, s, l, m \in \mathbb{R}_+$ such that $t \leq l, s \leq m$.*

Then

$$\Pi_{t,s}^{[q]}(D_b^\alpha v, v^{(n)}) \leq \Pi_{l,m}^{[q]}(D_b^\alpha v, v^{(n)}).$$

Proof. The following inequality holds for convex function φ see in [8, p. 4],

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \tag{3.23}$$

where $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$.

Since by Theorem 12, $\Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})$ is log-convex, we can put in (3.23): $\varphi = \log \Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})$, $x_1 = s, x_2 = t, y_1 = l, y_2 = m$. We get for $s \neq t, l \neq m$

$$\frac{\log \Lambda_{\varphi_l}(D_b^\alpha v, v^{(n)}) - \log \Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})}{t - s} \leq \frac{\log \Lambda_{\varphi_m}(D_b^\alpha v, v^{(n)}) - \log \Lambda_{\varphi_l}(D_b^\alpha v, v^{(n)})}{m - l}, \tag{3.24}$$

that is

$$\left(\frac{\Lambda_{\varphi_l}(D_b^\alpha v, v^{(n)})}{\Lambda_{\varphi_s}(D_b^\alpha v, v^{(n)})} \right)^{\frac{1}{t-s}} \leq \left(\frac{\Lambda_{\varphi_m}(D_b^\alpha v, v^{(n)})}{\Lambda_{\varphi_l}(D_b^\alpha v, v^{(n)})} \right)^{\frac{1}{m-l}}, \quad s \neq t, \quad l \neq m. \tag{3.25}$$

From (3.24) we get our result for $t \neq s, l \neq m$ and for $t = s, l = m; t \neq s, l = m; t = s, l \neq m$ we can consider limiting cases. \square

In the following result [1] one can see composition identity for Caputo fractional derivatives.

LEMMA 2. *Let $v > \gamma \geq 0, n = [v] + 1, m = [\gamma] + 1$ and $f \in AC^n([a, b])$. Suppose that one of the following conditions hold:*

- (a) $v, \gamma \notin \mathbb{N}_0$ and $f^i(b) = 0$ for $i = m, m + 1, \dots, n - 1$.
- (b) $v \in \mathbb{N}, \gamma \notin \mathbb{N}_0$ and $f^i(b) = 0$ for $i = m, m + 1, \dots, n - 2$.
- (c) $v \notin \mathbb{N}, \gamma \in \mathbb{N}_0$ and $f^i(b) = 0$ for $i = m - 1, \dots, n - 1$.
- (d) $v \in \mathbb{N}, \gamma \in \mathbb{N}_0$ and $f^i(b) = 0$ for $i = m - 1, \dots, n - 2$.

Then

$$D_b^\gamma f(t) = \frac{1}{\Gamma(v - \gamma)} \int_t^b (s - t)^{v - \gamma - 1} D_b^v f(s) ds. \tag{3.26}$$

By using Lemma 2 previous results can be proved, stated as follows:

THEOREM 14. *Let ϕ , q and p be defined as in Theorem 1, and $0 < \gamma < v - \frac{1}{q}$. If one of the conditions in Lemma 2 is satisfied, then*

$$\begin{aligned} & \int_a^b |D_b^\gamma u(x)|^{1-q} \phi'(|D_b^\gamma u(x)|) |D_b^\nu v(x)|^q dx \\ & \leq \frac{q\Gamma^q(v-\gamma)(p(v-\gamma)-p+1)^{\frac{q}{p}}}{(b-a)^{q(v-\gamma-\frac{1}{q})}} \\ & \times \phi\left(\frac{(b-a)^{v-\gamma-\frac{1}{q}}}{\Gamma(v-\gamma)(p(v-\gamma)-p+1)^{\frac{1}{p}}}\left(\int_a^b |D_b^\nu v(x)|^q dx\right)^{\frac{1}{q}}\right), \end{aligned} \tag{3.27}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse of the inequality (3.27) holds.

THEOREM 15. *Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the function with assumptions of Theorem 1. If $h \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval, also let $0 < \gamma < v - \frac{1}{q}$ and one of the conditions in Lemma 2 is satisfied, then there exists $\xi \in I$ such that*

$$\begin{aligned} \alpha_h(D_b^\gamma v, D_b^\nu v) &= \frac{\xi h''(\xi) - (q-1)h'(\xi)}{2q^2 \xi^{2q-1}} \left(\frac{(b-a)^{q(v-\gamma-\frac{1}{q})}}{\Gamma^q(v-\gamma)(p(v-\gamma)-p+1)^{\frac{q}{p}}} \right. \\ & \times \left. \left(\int_a^b |D_b^\nu v(x)|^q dx \right)^2 - 2 \int_a^b |D_b^\gamma v(x)|^q |D_b^\nu v(x)|^q dx \right). \end{aligned}$$

THEOREM 16. *Let $h_1, h_2 : [0, \infty) \rightarrow \mathbb{R}$ be the functions with assumptions of Theorem 1. If $h_1, h_2 \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval, let $0 < \gamma < v - \frac{1}{q}$ and one of the conditions in Lemma 2 is satisfied, then there exists $\xi \in I$ such that*

$$\frac{\alpha_{h_1}(D_b^\gamma v, D_b^\nu v)}{\alpha_{h_2}(D_b^\gamma v, D_b^\nu v)} = \frac{\xi h_1''(\xi) - (q-1)h_1'(\xi)}{\xi h_2''(\xi) - (q-1)h_2'(\xi)},$$

provided that denominators are not equal to zero.

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