

TIME-FRACTIONAL DIFFUSION EQUATION WITH DYNAMICAL BOUNDARY CONDITION

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Abstract. We establish the unique solvability in Hölder spaces for an initial-boundary problem for fractional diffusion equation with fractional dynamic boundary condition.

1. Introduction

Let $D_{*,t}^\alpha$ ($0 < \alpha < 1$) be the Caputo derivative of order α

$$D_{*,t}^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} (u(x,\tau) - u(x,0)) d\tau,$$

here Γ is Euler's Gamma function (see [5]).

Let Ω be a bounded domain in \mathbb{R}^n with $C^{2+\theta}$ -boundary Σ ($\theta \in (0, 1)$). Denote $\Omega_T = \Omega \times (0, T]$, $\Sigma_T = \Sigma \times (0, T]$, $T > 0$.

We need to find the function $u(x,t)$, satisfying the equation

$$D_{*,t}^\alpha u(x,t) - \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) u(x,t) = f(x,t), \quad (x,t) \in \Omega_T, \quad (1)$$

with initial

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad (2)$$

and dynamical boundary conditions

$$D_{*,t}^\alpha u(x,t) + \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) u(x,t) = \psi(x,t), \quad (x,t) \in \Sigma_T, \quad (3)$$

where

$$\begin{aligned} \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) &= \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n a_i(x,t) \frac{\partial}{\partial x_i} - a_0(x,t), \\ \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) &= \sum_{i=1}^n b_i(x,t) \frac{\partial}{\partial x_i} + b_0(x,t). \end{aligned}$$

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We assume that the inequalities

$$v\xi^2 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq \mu\xi^2, \quad \text{for all } (x,t) \in \Omega_T, \quad (4)$$

$$\sum_{i=1}^n b_i(x,t)n_i(x) \leq -\delta < 0, \quad \text{for all } (x,t) \in \Sigma_T, \quad (5)$$

are valid (here $n(x)$ is the unit outward normal to Σ at the point x) and compatibility condition

$$\psi(x,0) - \mathcal{B}\left(x,0,\frac{\partial}{\partial x}\right)u_0(x) = \mathcal{A}\left(x,0,\frac{\partial}{\partial x}\right)u_0(x) + f(x,0), \quad x \in \Sigma \quad (6)$$

is fulfilled.

This problem is obtained from the standard one (see, for exaple, [1, 3, 7, 9, 27]) for a second-order parabolic equation with dynamical boundary condition by replacing the first-order time derivatives with the Caputo derivatives of order α . We refer the reader also to [8] for an extensive derivation and physical interpretation of dynamic boundary condition for the heat equation.

Fractional partial differential equations received much attention in the literature because of numerous applications in physics, chemistry, hydrology and engineering ([10, 20, 21, 23, 30]).

As for equation (1), S. D. Eidelman and A. N. Kochubei constructed and investigated in [6] the fundamental solution in \mathbb{R}^n . Ph. Clément, S.-O. Londen and G. Simonett obtained in [4] existence, uniqueness and continuation on abstract quasilinear parabolic equation with time-fractional derivative. In [25] A. V. Pskhu construct a fundamental solution of a diffusion-wave equation with Dzhrbashyan-Nersesyan fractional differentiation operator with respect to the time variable. He give a solution of the Cauchy problem and prove the uniqueness theorem in the class of functions satisfying an analogue of Tychonoff's condition.

Other results regarding to solvability of initial-boundary problems to fractional diffusion equation can be found in [11, 13, 14, 18, 22, 24, 26, 28] and literature therein.

As for mathematical treatment of condition (3), M. Kirane, N. Tatar in [12] have analyzed the issue of local and global solutions for elliptic systems with nonlinear fractional dynamic boundary condition. N. Vasylyeva obtained in [31, 32, 33] coercive estimates of the solution to the Poisson equation with a boundary condition comprising the fractional derivative in time and prove the existence and uniqueness of the classical solution for corresponding moving boundary problems locally in time.

In this paper we prove the well-posedness and regularity of the solution to problem (1)–(3). Here we extend the results of [16], where this problem (1)–(3) was considered in one-dimensional case. In our analysis we follow very closely the approach of G. I. Bizhanova and V. A. Solonnikov (see [3]).

Below constants (always independent on x and t) will be denoted by the same letter C , even if they may vary from line to line. Sometimes we write, e.g., $C(p, q)$, when we want to emphasize the dependence of C on particular parameters p, q .

2. Functional spaces and main result

Let $\theta \in (0, 1)$, $x' = (x_1, \dots, x_{n-1})$ throughout this paper. Let Q be a domain in \mathbb{R}^n , $Q_T = Q \times (0, T]$.

We shall use the following notations $D_x^l = \frac{\partial^{|l|}}{\partial x_1^{l_1} \dots \partial x_n^{l_n}}$, $l = (l_1, \dots, l_n)$, $l_i \geq 0$, $i = 1, \dots, n$, $|l| = l_1 + \dots + l_n$.

We define $C^{k+\theta}(Q)$, ($k \in \mathbb{N} \cup \{0\}$), as the space of functions $f(x)$, $x \in Q$, with the norm

$$|f|_Q^{(k+\theta)} = |f|_Q^{(k)} + [f]_Q^{(k+\theta)},$$

here

$$|f|_Q^{(k)} = \sum_{|l| \leq k} |D_x^l f|_Q, \quad |f|_Q = \sup_{x \in Q} |f(x)|,$$

$$[w]_Q^{(k+\theta)} = \sum_{|l|=k} \sup_{x, y \in Q} |D_x^l f(x) - D_x^l f(y)| |x - y|^{-\theta}.$$

By $C_\alpha^\theta(Q_T)$ we denote the set of functions $f(x, t)$ having a finite norm

$$|f|_{\alpha, Q_T}^{(\theta)} = |f|_{Q_T} + [f]_{\alpha, Q_T}^{(\theta)},$$

where

$$|f|_{Q_T} = \sup_{t \in (0, T)} \sup_{x \in Q} |f(x, t)|,$$

$$[f]_{\alpha, Q_T}^{(\theta)} = \langle f \rangle_{x, Q_T}^{(\theta)} + \langle f \rangle_{t, Q_T}^{(\theta, \frac{\alpha}{2})}$$

$$\langle f \rangle_{x, Q_T}^{(\theta)} = \sup_{t \in (0, T)} \sup_{x, y \in Q} |f(x, t) - f(y, t)| |x - y|^{-\theta},$$

$$\langle f \rangle_{t, Q_T}^{(\theta)} = \sup_{t, \tau \in (0, T)} \sup_{x \in Q} |f(x, t) - f(x, \tau)| |t - \tau|^{-\theta}.$$

By definition, the space $C_\alpha^{k+\theta}(Q_T)$, $k \in \mathbb{N}$ consists of functions $f(x, t)$ with a finite norm

$$|f|_{\alpha, Q_T}^{(k+\theta)} = \sum_{|l|+2m \leq k} |(D_{*,t}^\alpha)^m D_x^l f|_{Q_T} + [f]_{\alpha, Q_T}^{(k+\theta)},$$

$$[f]_{\alpha, Q_T}^{(k+\theta)} = \langle f \rangle_{\alpha, Q_T}^{(k+\theta)} + \langle f \rangle_{t, Q_T}^{((k+\theta)\frac{\alpha}{2})}$$

$$\langle f \rangle_{\alpha, Q_T}^{(k+\theta)} = \sum_{|l|+2m=k} \langle (D_{*,t}^\alpha)^m D_x^l f \rangle_{\alpha, Q_T}^{(\theta)},$$

$$\langle f \rangle_{t, Q_T}^{((k+\theta)\frac{\alpha}{2})} = \sum_{|l|+2m=k-1} \langle (D_{*,t}^\alpha)^m D_x^l f \rangle_{t, Q_T}^{((1+\theta)\frac{\alpha}{2})}.$$

The symbol $C_{\alpha,0}^{k+\theta}(Q_T)$ denotes the subspace of $C_\alpha^{k+\theta}(Q_T)$, whose elements $f(x, t)$ have the property $(D_{*,t}^\alpha)^m f|_{t=0} = 0$, where $m = 0, \dots, [\frac{k+\theta}{2}]$. With the help of local coordinates and partition of unity, all these spaces can be introduced on manifold Σ_T .

REMARK 1. We emphasize that in definition of $C_{\alpha}^{k+\theta}(Q_T)$ we use “repeated” derivative $(D_{*t}^{\alpha})^m$ rather than “multiple” one $(D_{*t}^{\alpha m})$. Our choice is explained by the following example. As is well-known (see for instance [28]) Fourier method for fractional diffusion equation is based on the application of the Mittag-Leffler functions $E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}$. By the definition of the Caputo derivative we have

$$D_{*t}^{\alpha}(D_{*t}^{\alpha}E_{\alpha,1}(\lambda t^{\alpha})) = \lambda^2 E_{\alpha,1}(\lambda t^{\alpha})$$

and (for example $\alpha < 1/2$)

$$D_{*t}^{2\alpha}E_{\alpha,1}(\lambda t^{\alpha}) = \frac{\lambda t^{-\alpha}}{\Gamma(1-2\alpha)} + \lambda^2 E_{\alpha,1}(\lambda t^{\alpha}),$$

i.e. $(D_{*t}^{\alpha})^2 E_{\alpha,1}(\lambda t^{\alpha})$ is smooth, but in the contrary $D_{*t}^{2\alpha} E_{\alpha,1}(\lambda t^{\alpha})$ is singular at the point $t = 0$.

An important role in the investigation of problem (1)–(3) is played by the estimates of the following model problem in $\mathbb{R}_+^n = \{x_n > 0\}$:

$$D_{*t}^{\alpha} w(x, t) - \Delta w(x, t) = 0, \quad (x, t) \in \mathbb{R}_{+,T}^n, \tag{7}$$

$$w(x, 0) = 0, \quad x \in \mathbb{R}_+^n, \tag{8}$$

$$w(x, t) \rightarrow 0, \quad |x| \rightarrow \infty, \tag{9}$$

$$D_{*t}^{\alpha} w(x, t) + \sum_{i=1}^n h_i \frac{\partial w}{\partial x_i}(x, t) = \varphi(x', t), \quad x_n = 0. \tag{10}$$

We assume that constants $h_i, (i = 1, \dots, n)$ and the function φ are such that

$$\varphi \in C_{\alpha,0}^{k+\theta}(\mathbb{R}_T^{n-1}), \quad k \in \mathbb{N}, \tag{11}$$

$$\varphi(x, t) = 0, \quad |x'| > R_0, \quad h_n \leq -\delta_0, \quad |h'| \leq M_0 \tag{12}$$

for some fixed positive parameters R_0, δ_0, M_0 .

The main results of this paper are the following.

THEOREM 1. *Suppose that assumptions (11)–(12) hold. Then model problem (7)–(10) has a unique solution $w \in C_{\alpha,0}^{k+1+\theta}(\mathbb{R}_{+,T}^n)$, $D_{*t}^{\alpha} w|_{x_n=0} \in C_{\alpha,0}^{k+\theta}(\mathbb{R}_T^{n-1})$, satisfying estimate*

$$|w|_{\alpha, \mathbb{R}_{+,T}^n}^{(k+1+\theta)} + |D_{*t}^{\alpha} w|_{x_n=0}|_{\alpha, \mathbb{R}_T^{n-1}}^{(k+\theta)} \leq C(T) |\varphi|_{\alpha, \mathbb{R}_T^{n-1}}^{(k+\theta)}. \tag{13}$$

THEOREM 2. *Suppose that*

$$\Sigma \in C^{2+\theta}, \quad a_{ij}, a_i, a_0 \in C_{\alpha}^{\theta}(\Omega_T), \quad b_i, b_0 \in C_{\alpha}^{1+\theta}(\Sigma_T), \quad i, j = 1, \dots, n \tag{14}$$

and assumptions (4), (5), (6) hold. Then for every functions $u_0 \in C^{2+\theta}(\Omega)$, $f \in C_{\alpha}^{\theta}(\Omega_T)$, $\psi \in C_{\alpha}^{1+\theta}(\Sigma_T)$, problem (1)–(3) has a unique solution $u \in C_{\alpha}^{2+\theta}(\Omega_T)$, satisfying the estimate

$$|u|_{\alpha, \Omega_T}^{(2+\theta)} + |D_{*t}^{\alpha} u|_{\alpha, \Sigma_T}^{(1+\theta)} \leq C(T) \left(|u_0|_{\Omega}^{(2+\theta)} + |f|_{\alpha, \Omega_T}^{(\theta)} + |\psi|_{\alpha, \Sigma_T}^{(1+\theta)} \right). \tag{15}$$

REMARK 2. Assumptions of this theorem implies (see (6))

$$\mathcal{A} \left(x, 0, \frac{\partial}{\partial x} \right) u_0(x) + f(x, 0) \in C^{1+\theta}(\Sigma). \tag{16}$$

3. Model problem

3.1. Preliminaries

First of all we present some definitions and results concerning with fractional calculus that will be used in the sequel. For more information see [5, 26, 29].

Let $\nu \in \mathbb{R}_+$. The Riemann-Liouville fractional integral is defined by

$$J_t^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \tau)^{\nu-1} f(\tau) d\tau$$

The Riemann-Liouville fractional derivative is defined by

$$D_t^\nu f(t) = D_t^m J_t^{m-\nu} f(t), \quad m = [\nu] + 1.$$

For $\nu = 0$ we set $J_t^0 f(t) = f(t)$, $D_t^0 f(t) = f(t)$. It easy to see that

$$D_t^\nu f(t) = D_{*,t}^\nu f(t), \quad \text{if } f(0) = 0, \quad \nu \in (0, 1). \tag{17}$$

Below we use the Wright functions (see [25, 26])

$$\phi(-\alpha, \delta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\delta - \alpha k)},$$

with properties (see [25, 26, 15]) ($z > 0$)

$$|\phi(-\alpha, -\gamma, -z)| \leq c \exp(-\sigma z^{\frac{1}{1-\alpha}}) \begin{cases} 1, & \text{if } \gamma \notin \mathbb{N} \cup 0, \\ z, & \text{if } \gamma \in \mathbb{N} \cup 0. \end{cases} \tag{18}$$

$$\begin{aligned} D_y^\nu y^{\delta-1} \phi(-\alpha, \delta, -cy^{-\alpha}) &= y^{\delta-\nu-1} \phi(-\alpha, \delta - \nu, -cy^{-\alpha}), \\ J_y^\nu y^{\delta-1} \phi(-\alpha, \delta, -cy^{-\alpha}) &= y^{\delta+\nu-1} \phi(-\alpha, \delta + \nu, -cy^{-\alpha}), \\ \frac{d}{dz} \phi(-\alpha, \delta, -z) &= -\phi(-\alpha, \delta - \alpha, -z) \end{aligned} \tag{19}$$

and

$$\phi(-\alpha, 0, -z) = \alpha z \phi(-\alpha, 1 - \alpha, -z) \quad \int_0^\infty \phi(-\alpha, 1 - \alpha, -z) dz = 1. \tag{20}$$

We remind that the fundamental solution Γ_α to the Cauchy problem for the equation $D_{*,t}^\alpha u(x,t) - \Delta u(x,t) = f(x,t)$ in $\mathbb{R}^n \times (0, T]$ can be represented in the form (see [15])

$$\begin{aligned} \Gamma_\alpha(x,t) &= (4\pi)^{-n/2} \int_0^\infty \lambda^{-n/2} \exp\left(-\frac{|x^2|}{4\lambda}\right) t^{-1} \phi(-\alpha, 0, -\lambda t^{-\alpha}) d\lambda \\ &= \int_0^\infty \Gamma_1(x, \lambda) t^{-1} \phi(-\alpha, 0, -\lambda t^{-\alpha}) d\lambda, \end{aligned} \tag{21}$$

where $\Gamma_1(x, t) = (4\pi)^{-n/2} t^{-n/2} \exp(-\frac{|x|^2}{4t})$ is the fundamental solution for the heat equation. It is easy to check

$$D_{*t}^\alpha \Gamma_\alpha(x, t) - \Delta \Gamma_\alpha(x, t) = 0, \quad x \neq 0, \quad t \neq 0. \tag{22}$$

Besides we use the inequality (see (3.27) in [15])

$$\sum_{i=1}^n (x_i - h_i \lambda)^2 \geq C \left(\sum_{i=1}^{n-1} x_i^2 + \lambda^2 \right) + x_n^2, \tag{23}$$

here $x_n, \lambda \geq 0$ and C depends on δ_0, M_0 from (12).

3.2. Representation of Green function G and solution w

We take the Fourier transform on the tangent spatial variables x' and the Laplace transform on t

$$F[v] = \int_{\mathbb{R}^{n-1}} v(x', x_n, t) \exp(-ix' \cdot \xi) dx, \quad \xi = (\xi_1, \dots, \xi_{n-1}),$$

$$L[v] = \int_0^\infty v(x, t) \exp(-pt) dt.$$

Problem (7)–(10) reduces then to the ordinary differential equation ($\tilde{w} = F[L[w]]$)

$$p^\alpha \tilde{w}(\xi, x_n, p) + |\xi|^2 \tilde{w}(\xi, x_n, p) - \tilde{w}_{x_n x_n}(\xi, x_n, p) = 0, \quad x_n > 0,$$

with the boundary conditions

$$(p^\alpha \tilde{w} + h_n \tilde{w}_{x_n} + ih' \cdot \xi \tilde{w})|_{x_n=0} = \tilde{\varphi}(\xi, p), \quad \tilde{w} \rightarrow 0, \quad x_n \rightarrow \infty.$$

Assumption (12) allows us to write

$$\begin{aligned} \tilde{w}(\xi, x_n, p) &= \frac{\exp(-\sqrt{p^\alpha + |\xi|^2} x_n)}{p^\alpha - h_n \sqrt{p^\alpha + |\xi|^2} + ih' \cdot \xi} \tilde{\varphi}(\xi, p) \\ &= \int_0^\infty \exp(-\sqrt{p^\alpha + |\xi|^2} (x_n - h_n \lambda) - ih' \cdot \xi \lambda - p^\alpha \lambda) d\lambda \tilde{\varphi}(\xi, p) \\ &\equiv \tilde{G}(\xi, x_n, p) \tilde{\varphi}(\xi, p). \end{aligned}$$

Since (see formula (3.2.7) in [26]) $L[t^{-1} \phi(-\alpha, 0, -\lambda t^{-\alpha})] = \exp(-p^\alpha)$ and (see [15]) $L^{-1} F'^{-1}[\exp(-\sqrt{p^\alpha + |\xi|^2} x_n) = -2 \frac{\partial \Gamma_\alpha(x, t)}{\partial x_n}$, we get similar to [3]

$$\begin{aligned} G(x, t) &= -2 \left(\frac{\partial \Gamma_\alpha}{\partial x_n} * t^{-1} \phi(-\alpha, 0, \cdot) \right)_1(x, t) \\ &\equiv -2 \int_0^t d\tau \int_0^\infty \frac{\partial \Gamma_\alpha(x - h\lambda, t - \tau)}{\partial x_n} \tau^{-1} \phi(-\alpha, 0, -\lambda \tau^{-\alpha}) d\lambda, \end{aligned} \tag{24}$$

and

$$w(x, t) = (G * \varphi)(x, t) \equiv \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G(x' - y', x_n, t - \tau) \varphi(y', \tau) dy'. \tag{25}$$

Besides we remark that the change of variables $\lambda \rightarrow \sigma = \lambda \tau^{-\alpha}$ leads to

$$G(x, t) = -2 \int_0^t d\tau \int_0^\infty \frac{\partial \Gamma_\alpha(x - h\sigma\tau^\alpha, t - \tau)}{\partial x_n} \tau^{\alpha-1} \phi(-\alpha, 0, -\sigma) d\sigma. \tag{26}$$

3.3. Properties of Green function G

LEMMA 1. *The following identities are valid*

$$D_t^\alpha G(x, t) + \sum_{i=1}^n h_i \frac{\partial G}{\partial x_i}(x, t) = -2 \frac{\partial \Gamma_\alpha}{\partial x_n}(x, t), \tag{27}$$

$$\begin{aligned} \frac{\partial G}{\partial x_n}(x, t) &= -2\Gamma_\alpha(x, t) \\ &+ 2 \sum_{i=1}^n \int_0^t d\tau \int_0^\infty h_i \frac{\partial \Gamma_\alpha(x - h\eta, \tau)}{\partial x_i} (t - \tau)^{-1} \phi(-\alpha, 0, -\eta(t - \tau)^{-\alpha}) d\eta \\ &+ 2\Delta' \int_0^t d\tau \int_0^\infty \Gamma_\alpha(x - h\eta, \tau) (t - \tau)^{-1} \phi(-\alpha, 0, -\eta(t - \tau)^{-\alpha}) d\eta, \end{aligned} \tag{28}$$

here $\Delta' = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Proof. From (24) and change the order of integration we obtain (see (24))

$$J_t^{1-\alpha} G = -2 \left(\frac{\partial \Gamma_\alpha}{\partial x_n} * J_t^{1-\alpha} (t^{-1} \phi(-\alpha, 0, \cdot)) \right)_1.$$

Then we use (19) and change the corresponding variables to set

$$J_t^{1-\alpha} G(x, t) = -2 \int_0^t ds \int_0^\infty \frac{\partial \Gamma_\alpha}{\partial x_n}(x - h(t-s)^\alpha \sigma, s) \phi(-\alpha, 1 - \alpha, -\sigma) d\sigma.$$

(compare with (26)).

By definition $D_t^\alpha G(x, t) = D_t(J_t^{1-\alpha} G(x, t))$. Relations (20) lead to

$$\begin{aligned} D_t^\alpha G(x, t) &= -2 \frac{\partial \Gamma_\alpha}{\partial x_n}(x, t) \\ &+ \sum_{i=1}^n 2 \int_0^t ds \int_0^\infty \frac{\partial^2 \Gamma_\alpha}{\partial x_n \partial x_i}(x - h(t-s)^\alpha \sigma, s) (t-s)^{\alpha-1} \phi(-\alpha, 0, -\sigma) d\sigma. \end{aligned} \tag{29}$$

Clearly from (26) we have

$$\sum_{i=1}^n h_i \frac{\partial G}{\partial x_i}(x, t) = -2 \sum_{i=1}^n \int_0^t d\tau \int_0^\infty h_i \frac{\partial^2 \Gamma_\alpha}{\partial x_n \partial x_i}(x - h\tau^\alpha \sigma, t - \tau) \tau^{\alpha-1} \phi(-\alpha, 0, -\sigma) d\sigma. \quad (30)$$

Comparing (29) and (30), we get (27).

By (24), (22), we obtain

$$\begin{aligned} G_{x_n}(x, t) &= -2 \int_0^t d\tau \int_0^\infty \frac{\partial^2 \Gamma_\alpha}{\partial x_n^2}(x - h\lambda, \tau)(t - \tau)^{-1} \phi(-\alpha, 0, -\lambda(t - \tau)^{-\alpha}) d\lambda \\ &= 2 \int_0^t d\tau \int_0^\infty \Delta' \Gamma_\alpha(x - h\lambda, \tau)(t - \tau)^{-1} \phi(-\alpha, 0, -\lambda(t - \tau)^{-\alpha}) d\lambda \\ &\quad - 2 \int_0^t d\tau \int_0^\infty D_\tau^\alpha \Gamma_\alpha(x - h\lambda, \tau)(t - \tau)^{-1} \phi(-\alpha, 0, -\lambda(t - \tau)^{-\alpha}) d\lambda \\ &\equiv G_1(x, t) + G_2(x, t). \end{aligned}$$

In view of (21), (19) we see $D_t^\alpha \Gamma_\alpha = D_t(J_t^{1-\alpha} \Gamma_\alpha) = J_t^{1-\alpha}(D_t \Gamma_\alpha)$. The rule of fractional integration by parts (see [26]) and (19) yield

$$G_2(x, t) = -2 \int_0^t d\tau \int_0^\infty \Gamma_{\alpha, \tau}(x - h\lambda, \tau)(t - \tau)^{-\alpha} \phi(-\alpha, 1 - \alpha, -\lambda(t - \tau)^{-\alpha}) d\lambda.$$

First we integrate by parts with respect with respect to τ . The estimates of Γ_α , obtained in [15], and second identity in (20) lead to

$$G_2(x, t) = -2\Gamma_\alpha(x, t) + 2 \int_0^t d\tau \int_0^\infty \Gamma_\alpha(x - h\lambda, \tau) D_s(\tau^{-\alpha} \phi(-\alpha, 1 - \alpha, -\lambda s^{-\alpha}))|_{s=t-\tau} d\lambda. \quad (31)$$

Then we apply

$$\frac{d}{d\lambda}(\tau^{-1} \phi(-\alpha, 0, -\lambda \tau^{-\alpha})) = D_\tau(\tau^{-\alpha} \phi(-\alpha, 1 - \alpha, -\lambda \tau^{-\alpha}))$$

(see (19)) and integrate by parts with respect to λ in (31). In this way we established (28) from (29), (31). \square

LEMMA 2. *The function G satisfies the following estimates*

$$\int_0^\infty |D_x^l G(x, t)| dt \leq C|x|^{2-(n+|l|)}, \quad n + |l| > 2, \quad (32)$$

$$\int_{\mathbb{R}^{n-1}} |D_t^k G(x, t)| dx' \leq Ct^{\alpha-1-k}, \quad k = 0, 1, \quad (33)$$

$$\int_{\mathbb{R}^{n-1}} |D_t^k J_t^{1-\alpha} G(x,t)| dx' \leq C t^{-k}, \quad k = 0, 1, 2, \tag{34}$$

$$\int_{\mathbb{R}^{n-1}} |D_t^k G_{z_i}(z', x_n, t)| |z'|^\theta dz' \leq C t^{\frac{\alpha\theta}{2}-1-k}, \quad k = 0, 1, \quad i = 1, \dots, n-1, \tag{35}$$

Proof. In accordance with definition of G we have

$$\begin{aligned} I_t &= \int_0^\infty |D_x^l G(x,t)| dt \\ &\leq C \int_0^\infty dt \int_0^t d\tau \int_0^\infty d\mu \int_0^\infty |D_x^l \Gamma_{1,x_n}(x-h\eta, \mu)| \\ &\quad \times \tau^{-1} |\phi(-\alpha, 0, -\mu\tau^{-\alpha})| (t-\tau)^{-1} |\phi(-\alpha, 0, -\eta(t-\tau)^{-\alpha})| d\eta, \end{aligned}$$

Formula (20), changes of the variable $t \rightarrow \zeta = \frac{\eta}{(t-\tau)^\alpha}$ and the order of integration lead to the inequalities

$$\begin{aligned} I_t &\leq C \int_0^\infty d\tau \int_\tau^\infty dt \int_0^\infty d\mu \int_0^\infty |D_x^l \Gamma_{1,x_n}(x-h\eta, \mu)| \\ &\quad \times \tau^{-1} |\phi(-\alpha, 0, -\mu\tau^{-\alpha})| \frac{\eta}{(t-\tau)^{1+\alpha}} |\phi(-\alpha, 1-\alpha, -\eta(t-\tau)^{-\alpha})| d\eta \tag{36} \\ &\leq C \int_0^\infty d\tau \int_0^\infty d\mu \int_0^\infty |D_x^l \Gamma_{1,x_n}(x-h\eta, \mu)| \tau^{-1} |\phi(-\alpha, 0, -\mu\tau^{-\alpha})| d\eta. \end{aligned}$$

The classical estimate

$$|D_t^k D_x^l \Gamma_1(x,t)| \leq C t^{-\frac{2k+|l+n}{2}} \exp\left(-C \frac{|x|^2}{t}\right) \tag{37}$$

and (23) with the change of variables $\eta \rightarrow z = \frac{\eta}{\mu^{1/2}}$ gives

$$\begin{aligned} I_t &\leq C \int_0^\infty d\tau \int_0^\infty d\mu \int_0^\infty \mu^{-\frac{|l+n+1}{2}} \exp\left(-C \frac{|x|^2 + \eta^2}{\mu}\right) \tau^{-1} |\phi(-\alpha, 0, -\mu\tau^{-\alpha})| d\eta \\ &\leq C \int_0^\infty d\tau \int_0^\infty \mu^{-\frac{|l+n}{2}} \exp\left(-C \frac{|x|^2}{\mu}\right) \tau^{-1} |\phi(-\alpha, 0, -\mu\tau^{-\alpha})| d\mu. \end{aligned}$$

We use (18), (20), change of variables $\tau \rightarrow \rho = \frac{\mu}{\tau^\alpha}$ and then $\mu \rightarrow \zeta = \frac{|x|^2}{\mu}$ to deduce

$$I_t \leq C |x|^{2-(n+|l|)} \int_0^\infty \zeta^{n+|l|-3} \exp(-C\zeta^2) d\zeta \leq C |x|^{2-(n+|l|)},$$

when $n + j - 3 > 1$. This proves (32).

Since

$$\int_{\mathbb{R}^{n-1}} |\Gamma_{1,x_n}(x - b\mu, t)| dx' \leq C \frac{x_n - b\mu}{\mu^{3/2}} \exp\left(-C \frac{|x|^2}{\mu}\right),$$

we have

$$\begin{aligned} I &= \int_{\mathbb{R}^{n-1}} |G(x, t)| dx' \\ &\leq C \int_{\mathbb{R}^{n-1}} dx' \int_0^t d\tau \int_0^\infty d\mu \int_0^\infty \frac{x_n - b\mu}{\mu^{3/2}} \exp\left(-C \frac{|x - b\mu|^2}{\mu}\right) (t - \tau)^{-1} \\ &\quad \times \left| \phi\left(-\alpha, 0, -\frac{\mu}{(t - \tau)\alpha}\right) \right| \tau^{-1} \left| \phi\left(-\alpha, 0, -\frac{\eta}{\tau\alpha}\right) \right| d\eta. \end{aligned}$$

We divide an integral over $(0, T)$ into the ones on the intervals $(0, t/2)$ and $(t/2, t)$. We denote corresponding integrals by I' and I'' .

We have

$$\begin{aligned} I' &\leq \frac{C}{t} \int_0^{t/2} \tau^{\alpha-1} d\tau \int_0^\infty \left| \phi\left(-\alpha, 0, -\frac{\eta}{\tau\alpha}\right) \right| \frac{d\eta}{\tau^\alpha} \int_0^\infty \frac{x_n - b\mu}{\mu^{3/2}} \exp\left(-C \frac{|x - b\mu|^2}{\mu}\right) d\mu \\ &\leq \frac{C}{t} \int_0^{t/2} \tau^{\alpha-1} d\tau \leq Ct^{\alpha-1}, \end{aligned} \tag{38}$$

here we change the variables $\eta \rightarrow \zeta = \frac{\eta}{\tau\alpha}$, $\mu \rightarrow \xi = \frac{x - b\mu}{\mu^{1/2}}$ and use estimate (18). In I'' we apply again (18) to estimate Wright functions

$$\begin{aligned} I'' &\leq \frac{C}{t} \int_{t/2}^t d\tau \int_0^\infty d\mu \int_0^\infty \frac{x_n - b\mu}{\mu^{3/2}} \exp\left(-C \frac{|x - b\mu|^2}{\mu}\right) \\ &\quad \times (t - \tau)^{-1} \left| \phi\left(-\alpha, 0, -\frac{\mu}{(t - \tau)\alpha}\right) \right| \exp\left(-C \left(\frac{\eta}{t\alpha}\right)^{\frac{1}{1-\alpha}}\right) d\eta \\ &\leq Ct^{\alpha-1} \int_0^\infty \exp\left(-C \left(\frac{\eta}{t\alpha}\right)^{\frac{1}{1-\alpha}}\right) \frac{\eta}{t^\alpha} \int_0^\infty \frac{x_n - b\mu}{\mu^{3/2}} \exp\left(-C \frac{|x - b\mu|^2}{\mu}\right) d\mu \\ &\quad \times \int_0^\infty \left| \phi\left(-\alpha, 1 - \alpha, -\frac{\mu}{\sigma\alpha}\right) \right| \frac{\mu}{\sigma^{1+\alpha}} d\sigma \leq Ct^{\alpha-1} \end{aligned} \tag{39}$$

Summing up (38), (39) we obtain (33) for $k = 0$.

Next we calculate $G_t, J_t^{1-\alpha}G, D_t^\alpha G, D_t^\alpha G$. Similarly [2] we have

$$\begin{aligned}
 G_t(x,t) &= -2 \frac{\partial}{\partial t} \left(\int_0^{t/2} d\tau \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, t-\tau) \tau^{-1} \phi(-\alpha, 0, -\eta \tau^{-\alpha}) d\eta + \right. \\
 &\quad \left. + \int_0^{t/2} d\tau \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, \tau) (t-\tau)^{-1} \phi(-\alpha, 0, -\eta (t-\tau)^{-\alpha}) d\eta \right) \\
 &= -4 \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, t/2) (t/2)^{-1} \phi(-\alpha, 0, -\eta (t/2)^{-\alpha}) d\eta \\
 &\quad - 2 \int_0^{t/2} d\tau \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, t-\tau) \tau^{-1} \phi(-\alpha, 0, -\eta \tau^{-\alpha}) d\eta \\
 &\quad - 2 \int_0^{t/2} d\tau \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, \tau) (t-\tau)^{-2} \phi(-\alpha, -1, -\eta \tau^{-\alpha}) d\eta
 \end{aligned} \tag{40}$$

By (19) we obtain

$$J_t^\alpha G(x,t) = -2 \int_0^t d\tau \int_0^\infty \Gamma_{\alpha,x_n}(x-h\eta, t-\tau) \tau^{-\alpha} \phi(-\alpha, 1-\alpha, -\eta \tau^{-\sigma}) d\eta. \tag{41}$$

We rewrite (27) in the form

$$D_t^\alpha G(x,t) = - \sum_{i=1}^n h_i \frac{\partial G}{\partial x_i}(x,t) - 2 \frac{\partial \Gamma_\alpha}{\partial x_n}(x,t), \tag{42}$$

It follows from (42) that

$$D_t^{\alpha+1} G(x,t) = - \sum_{i=1}^n h_i \frac{\partial^2 G}{\partial x_i \partial t}(x,t) - 2 \frac{\partial^2 \Gamma_\alpha}{\partial x_n \partial t}(x,t). \tag{43}$$

Arguing as above we established the rest of estimates in (33), (34) from representations (40), (41), (42), (43).

By the same reason we restrict our attention in (35) to the case $k = 0$.

Preliminary we have from (23), (37)

$$\int_{\mathbb{R}^{n-1}} \left| \frac{\partial^2 \Gamma_1}{\partial x_n \partial x_i}(x' - y' - h'\eta, x_n - h_n \eta, \mu) \right| |x' - y'|^\theta dy' \leq \mu^{\theta/2-3/2} \exp(-C\eta^2/\mu).$$

Then this estimate yields

$$\begin{aligned}
 \int_{\mathbb{R}^{n-1}} |G_{z_i}(z', x_n, t)| |z'|^\theta dz' &\leq C \int_0^t d\tau \int_0^\infty \mu^{\theta/2-3/2} \exp(-C\eta^2/\mu) \tau^{-1} \left| \phi\left(-\alpha, 0, -\frac{\mu}{\tau^\alpha}\right) \right| \\
 &\quad \times (t-\tau)^{-1} \left| \phi\left(-\alpha, 0, -\frac{\eta}{(t-\tau)^\alpha}\right) \right| d\mu.
 \end{aligned}$$

We divide this integral into two ones on the intervals $(0, t/2)$, $(t/2, t)$ and denote its by L' and L'' . Next we use (18), (20) to obtain

$$\begin{aligned} L' &\leq C \int_0^{t/2} d\tau \int_0^\infty d\mu \int_0^\infty \tau^{\alpha\theta/2} \left(\frac{\mu}{\tau^\alpha}\right)^{\theta/2} \frac{\eta}{\mu^{3/2}} \exp\left(-C\frac{\eta^2}{\mu}\right) \\ &\quad \times \tau^{-1} \exp\left(-C\left(\frac{\mu}{\tau^\alpha}\right)^{\frac{1}{1-\alpha}}\right) (t-\tau)^{-1-\alpha} \exp\left(-C\left(\frac{\eta}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}\right) d\eta \\ &\leq Ct^{-1} \int_0^{t/2} \tau^{\frac{\alpha\theta}{2}-1} d\tau \int_0^\infty \exp\left(-C\left(\frac{\eta}{t^\alpha}\right)^{\frac{1}{1-\alpha}}\right) \frac{d\eta}{t^\alpha} \int_0^\infty \exp\left(-C\frac{\eta^2}{\mu}\right) \frac{\eta d\mu}{\mu^{3/2}} \leq Ct^{\frac{\alpha\theta}{2}-1}, \end{aligned}$$

and

$$\begin{aligned} L'' &\leq C \int_0^{t/2} d\sigma \int_0^\infty d\mu \int_0^\infty t^{\alpha\theta/2} \left(\frac{\mu}{t^\alpha}\right)^{\theta/2} \mu^{-3/2} \exp\left(-C\frac{\eta^2}{\mu}\right) \\ &\quad \times \mu t^{-1-\alpha} \exp\left(-C\left(\frac{\mu}{t^\alpha}\right)^{\frac{1}{1-\alpha}}\right) \exp\left(-C\left(\frac{\eta}{\sigma^\alpha}\right)^{\frac{1}{1-\alpha}}\right) \frac{\eta}{\sigma^{1+\alpha}} d\eta \\ &\leq Ct^{\alpha\theta/2-1} \int_0^\infty \exp\left(-C\left(\frac{\mu}{t^\alpha}\right)^{\frac{1}{1-\alpha}}\right) \frac{d\mu}{t^\alpha} \int_0^\infty \exp\left(-C\frac{\eta^2}{\mu}\right) \frac{d\eta}{\mu^{1/2}} \\ &\quad \times \int_0^\infty \exp\left(-C\left(\frac{\eta}{\sigma^\alpha}\right)^{\frac{1}{1-\alpha}}\right) \frac{\eta}{\sigma^{1+\alpha}} d\sigma \leq Ct^{\frac{\alpha\theta}{2}-1}. \end{aligned}$$

The proof of Lemma is finished. \square

Next we study properties of the function

$$K(x, t) = \int_{\mathbb{R}^{n-1}} G_{x_n}(x, t) dx'.$$

LEMMA 3. *The following inequalities hold*

$$\int_0^T |K(x_n, t)| dt \leq C(T), \tag{44}$$

$$\int_0^T |K(x_n, t) - K(z_n, t)| dt \leq C(T, \theta) |x_n - z_n|^\theta. \tag{45}$$

Proof. By (28), (24) we have

$$K(x_n, t) = -2 \int_{\mathbb{R}^{n-1}} \Gamma_\alpha(x, t) dx' - h_n \int_{\mathbb{R}^{n-1}} G(x, t) dx'. \tag{46}$$

The first integral is estimated in [15] (see Lemma 3.5)

$$\int_{\mathbb{R}^{n-1}} |\Gamma_\alpha(x, t)| dx' \leq Ct^{\alpha/2-1},$$

the second one is estimated in (33). Thus we have

$$\int_0^T |K(x_n, t)| dt \leq C \left(\int_0^T t^{\alpha/2-1} dt + \int_0^T t^{\alpha-1} dt \right) \leq C(T). \tag{47}$$

We use inequalities ($h_n < 0$ by (12))

$$\begin{aligned} \exp(-C(x_n - h_n \eta)^2 / \mu) &\leq \exp(-Cx_n^2 / \mu - (\delta_0 \eta)^2 / \mu), \\ \exp(-Cx_n^2 / \mu) &\leq x_n^{\theta-1} \mu^{\frac{1-\theta}{2}} \frac{x_n^{1-\theta}}{\mu^{\frac{1-\theta}{2}}} \exp(-Cx_n^2 / \mu) \leq x_n^{\theta-1} \mu^{\frac{1-\theta}{2}} \end{aligned}$$

and representation (46) to get after routine calculations

$$\begin{aligned} \int_0^T |K(x_n, t) - K(z_n, t)| dt &\leq C \left(\int_{z_n}^{x_n} d\xi \int_0^T |\Gamma_{\alpha, x_n}(x', \xi, t)| dt + \int_{z_n}^{x_n} d\xi \int_0^T |G_{x_n}(x', \xi, t)| dt \right) \\ &\leq C \int_{z_n}^{x_n} d\xi \int_0^T \xi^{\theta-1} t^{\frac{1-\theta}{2}-1} dt \leq CT^{\frac{1-\theta}{2}} |x_n - z_n|^\theta. \end{aligned}$$

The proof of (45) is finished. \square

To this end we need the following result.

LEMMA 4. *Let $j \neq n$, $a > 0$, then the estimate is valid*

$$I_a = \int_0^\infty \left| \int_{|x'-y'| \leq a} G_{x_j}(x' - y', x_n, t) dy' \right| dt \leq C, \tag{48}$$

here the constant C doesn't depend on a .

Proof. We shall closely follow the line of the paper [3] (see estimate of L_3).

If $n \geq 3$, then we get from (36), ($|l| = 0$)

$$\begin{aligned} I_a &= \int_0^\infty \left| \int_{|x'-y'|=a} G(x' - y', x_n, t) dS_{y'} \right| dt \leq C \int_{|x'-y'|=a} \left(\int_0^\infty |G(x' - y', x_n, t)| dt \right) dS_{y'} \\ &\leq C \int_{|x'-y'|=a} \frac{dS_{y'}}{|x' - y'|^{n-2}} \leq C. \end{aligned}$$

If $n = 2$, then we denote by \mathcal{C} a semicircle $\mathcal{C} = \{z = (z_1, z_2) \in \mathbb{R}^2 : (z_1^2 + (z_2 - x_2)^2)^{1/2} = a, z_2 > x_2\}$. By (36), ($|l| = 1$) we have

$$I_a = \int_0^\infty |G(-a, x_2, t) - G(a, x_2, t)| dt = \int_0^\infty \left| \int_{\mathcal{C}} \nabla G(z, t) \cdot dl_z \right| dt \leq C \int_{\mathcal{C}} \frac{|dl_z|}{|z|} \leq C$$

The proof is finished. \square

3.4. Main estimates of w

LEMMA 5. *Suppose that assumption (12) is valid and $\varphi \in C_{\alpha,0}^{1+\theta}(\mathbb{R}_T^{n-1})$. Then the following estimates hold ($i = 1, \dots, n$)*

$$\langle w_{x_i} \rangle_{x, \mathbb{R}_{+,T}^n}^{(\theta)} \leq C(T) \langle \phi \rangle_{\alpha, \mathbb{R}_T^{n-1}}^{(\theta)}, \tag{49}$$

$$\langle w_{x_i} \rangle_{t, \mathbb{R}_{+,T}^n}^{(\theta \frac{\alpha}{2})} \leq C(T) \langle \phi \rangle_{\alpha, \mathbb{R}_T^{n-1}}^{(\theta)}, \tag{50}$$

$$\langle D_{*i}^\alpha w \rangle_{t, \mathbb{R}_{+,T}^n}^{(\gamma)} \leq C(T) \langle \phi \rangle_{t, \mathbb{R}_T^{n-1}}^{(\gamma)}, \quad \gamma \in (0, 1). \tag{51}$$

Proof. We construct the solution w of (7)–(10) in the form (25).

We get

$$w_{x_j} = (G_{x_j} * \varphi), \quad j = 1, \dots, n. \tag{52}$$

First we estimate the Hölder constant $\langle w_{x_k} \rangle_{x, \mathbb{R}_{+,T}^n}^{(\theta)}$. We consider the difference ($\rho = |x - z|$)

$$\begin{aligned} & w_{x_j}(x, t) - w_{z_j}(z, t) \\ &= \int_0^t d\tau \int_{|x' - z'| \leq 2\rho} G_{x_j}(x' - y', x_n, t - \tau) \varphi(y', \tau) dy' \\ &\quad - \int_0^t d\tau \int_{|x' - z'| \leq 2\rho} G_{z_j}(z' - y', z_n, t - \tau) \varphi(y', \tau) dy' \\ &= \int_0^t d\tau \int_{|x' - z'| \leq 2\rho} G_{x_j}(x' - y', x_n, t - \tau) (\varphi(y', \tau) - \varphi(x', \tau)) dy' \\ &\quad - \int_0^t d\tau \int_{|x' - z'| \leq 2\rho} G_{z_j}(z' - y', z_n, t - \tau) (\varphi(y', \tau) - \varphi(z', \tau)) dy' \\ &\quad + \int_0^t (\varphi(x', \tau) - \varphi(z', \tau)) d\tau \int_{|x' - z'| \leq 2\rho} G_{z_j}(x' - y', x_n, t - \tau) dy' \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t d\tau \int_{|x-z| \geq 2\rho} (G_{x_j}(x' - y', x_n, t - \tau) - G_{z_j}(z' - y', z_n, t - \tau))(\varphi(y', \tau) - \varphi(z', \tau)) dy' \\
 & + \int_0^t \varphi(z', \tau) d\tau \int_{\mathbb{R}^{n-1}} (G_{x_j}(x' - y', x_n, t - \tau) - G_{z_j}(z' - y', z_n, t - \tau)) dy' \\
 & = \sum_{i=1}^5 I_i.
 \end{aligned}$$

By (32), we obtain

$$\begin{aligned}
 |I_1| + |I_2| & \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^{(\theta)} \left(\int_{|x'-y'| \leq 2\rho} |x' - y'|^{\theta-n+1} dy' + \int_{|x'-y'| \leq 3\rho} |x' - y'|^{\theta-n+1} dy' \right) \\
 & \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^{(\theta)} \rho^\theta.
 \end{aligned} \tag{53}$$

If $j \neq n$ then we use (48)

$$|I_3| \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^{(\theta)} \rho^\theta. \tag{54}$$

Let $j = n$ in I_3 . By (28) we have

$$G_{x_n}(x, t) = -2\Gamma_\alpha(x, t) - 2h_n G(x, t) + 2 \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \tilde{G}_i(x, t) \tag{55}$$

here

$$\begin{aligned}
 \tilde{G}_i(x, t) & = \int_0^t d\tau \int_0^\infty \left(h_i \Gamma_\alpha(x - h\eta, \tau) + \frac{\partial \Gamma_\alpha}{\partial x_i}(x - h\eta, \tau) \right) \\
 & \quad \times (t - \tau)^{-1} \phi(-\alpha, 0, -\eta(t - \tau)^{-\alpha}) d\eta,
 \end{aligned}$$

(compare the last expression with (24)). It is easy to show

$$\int_0^\infty |\tilde{G}_i(x, t)| dt \leq C|x|^{2-n}, \quad n > 2, \tag{56}$$

$$\int_0^\infty |\tilde{G}_{i,x_m}(x, t)| dt \leq C|x|^{-1}, \quad n = 2, \quad m < n \tag{57}$$

in a similar manner as estimate (32).

The estimate

$$\int_0^t \left| \int_{|y'| \leq 2\rho} G_{x_n}(y', x_n, t) dy' \right| d\tau \leq C(T) \tag{58}$$

follows from (48). We consider each term in (55) separately. From (47) and (33) we have

$$\int_0^t d\tau \int_{|y'| \leq 2\rho} |\Gamma_\alpha(y', x_n, t)| dy' \leq C \int_0^t \tau^{\alpha/2-1} d\tau \leq CT^{\alpha/2} \tag{59}$$

and

$$\int_0^t d\tau \int_{|y'| \leq 2\rho} |G(y', x_n, t)| dy' \leq C \int_0^t \tau^{\alpha-1} d\tau \leq CT^\alpha. \tag{60}$$

The proof of inequality

$$\int_0^t \left| \int_{|y'| \leq 2\rho} \left| \frac{\partial \tilde{G}_i}{\partial x_m}(y', x_n, t) \right| dy' \right| d\tau \leq C, \quad m \neq n \tag{61}$$

parallels that of integral (48) in view (56), (57). Inequalities (59), (60), (61) follows from (58) and consequently (54) for $j = n$.

For I_4 we have

$$\begin{aligned} I_4 &= \int_0^t d\tau \int_{|x-z| \geq 2\rho} (\varphi(y', \tau) - \varphi(z', \tau)) dy' \\ &\quad \times \int_0^\infty \int_0^1 \sum_{m=1}^n G_{x_j x_m}(z' + \lambda(x' - z') - y', z_n + \lambda(x_n - z_n), t - \tau)(z_m - x_m) d\lambda. \end{aligned}$$

Estimate (32) gives

$$|I_4| \leq C|x-z| \int_0^1 d\lambda \int_{|x-z| \geq 2\rho} |y' - z'|^\theta \left((z' - y' + \lambda(x' - z'))^2 + (z_n + \lambda(x_n - z_n))^2 \right)^{-\frac{n}{2}} dy' \tag{62}$$

Routine calculations show

$$|x - z| \leq |z' + \lambda(x' - z') - y'|, \quad |z' - y'| \leq 2|z' + \lambda(x' - z') - y'|. \tag{63}$$

We denote $\zeta = z + \lambda(x - z)$. With the help of (63) we can continue (62)

$$\begin{aligned} |I_4| &\leq C|x-z| \int_0^1 d\lambda \int_{|\zeta' - y'| \geq |x-z|} \frac{(2|\zeta' - y'|)^\theta}{|\zeta' - y'|^n} dy' \\ &\leq C|x-z| \int_0^1 d\lambda \int_{|\zeta' - y'| \geq |x-z|} |\zeta' - y'|^{\theta-n} dy' \leq C|x-z|\rho^{\theta-1} \leq C\rho^\theta. \end{aligned} \tag{64}$$

One can easily observe that

$$I_5 = 0, \quad \text{if } j \neq n. \tag{65}$$

If $j = n$, then we have

$$|I_5| = \left| \int_0^t \varphi(z', \tau) (K(x_n, t - \tau) - K(z_n, t - \tau)) d\tau \right| \leq C(T) T^{\theta \frac{\alpha}{2}} \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^{\theta \frac{\alpha}{2}} |x_n - z_n|^\theta, \tag{66}$$

here we used (45).

Summing up (53), (54), (64), (66) we establish (49).

Next we estimate ($h = t - \bar{t} > 0$)

$$\begin{aligned} w_{x_j}(x, t) - w_{x_i}(x, \bar{t}) &= \int_{t-2h}^t d\tau \int_{\mathbb{R}^{n-1}} G_{x_j}(x' - y', x_n, t - \tau) (\varphi(y', \tau) - \varphi(x', \tau)) dy' \\ &\quad - \int_{t-2h}^{t-h} d\tau \int_{\mathbb{R}^{n-1}} G_{x_j}(x' - y', x_n, t - h - \tau) (\varphi(y', \tau) - \varphi(x', \tau)) dy' \\ &\quad + \int_0^t d\tau \int_{\mathbb{R}^{n-1}} (G_{x_j}(x' - y', x_n, t - \tau) - G_{x_j}(x' - y', x_n, t - h - \tau)) \\ &\quad \times (\varphi(y', \tau) - \varphi(x', \tau)) dy' \\ &\quad + \left[\int_0^t \varphi(x', \tau) d\tau \int_{\mathbb{R}^{n-1}} G_{x_j}(x' - y', x_n, t - \tau) dy' \right. \\ &\quad \left. - \int_0^t \varphi(x', \tau) d\tau \int_{\mathbb{R}^{n-1}} G_{x_j}(x' - y', x_n, t - h - \tau) dy' \right] = \sum_{i=1}^4 I_i. \end{aligned}$$

Estimate (35) gives

$$|I_1| + |I_2| \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta \left(\int_{t-2h}^t (t - \tau)^{\theta \frac{\alpha}{2} - 1} d\tau + \int_{t-2h}^{t-h} (t - h - \tau)^{\theta \frac{\alpha}{2} - 1} d\tau \right) \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta h^{\theta \frac{\alpha}{2}} \tag{67}$$

By (35) we get

$$\begin{aligned} |I_3| &\leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta \int_{-\infty}^{t-2h} d\tau \int_{t-h-\tau}^{t-\tau} \eta^{\theta \frac{\alpha}{2} - 2} d\eta = C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta \int_{-\infty}^{t-2h} d\tau \int_0^h (t - \tau - s)^{\theta \frac{\alpha}{2} - 2} ds \\ &= C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta \int_0^h ds \int_{-\infty}^{t-2h} (t - \tau - s)^{\theta \frac{\alpha}{2} - 2} d\tau \tag{68} \\ &\leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta \int_0^h (2h - s)^{\theta \frac{\alpha}{2} - 1} ds \leq C \langle \varphi \rangle_{x, \mathbb{R}_T^{n-1}}^\theta h^{\theta \frac{\alpha}{2}}. \end{aligned}$$

It is obvious that

$$I_4 = 0, \quad j \neq n \tag{69}$$

If $j = n$, then

$$\begin{aligned} I_4 &= \int_0^t \varphi(x', \tau)K(x_n, t - \tau)d\tau - \int_0^{t-h} \varphi(x', \tau)K(x_n, t - h - \tau)d\tau \\ &= \int_0^t \varphi(x', \tau)K(x_n, t - \tau)d\tau - \int_{-h}^{t-h} \varphi(x', \tau)K(x_n, t - h - \tau)d\tau \\ &= \int_0^t (\varphi(x', \tau) - \varphi(x', \tau - h))K(x_n, t - \tau)d\tau. \end{aligned}$$

So from (44) we see

$$|I_4| \leq C \langle \varphi \rangle_{\mathbb{R}_T^{n-1}}^{(\frac{\theta}{2})} h^{\frac{\theta}{2}}. \tag{70}$$

Inequalities (67), (68), (69), (70) prove estimate (50).

The derivative $D_{*,t}^\alpha w(x, t)$ is represented as

$$\begin{aligned} D_{*,t}^\alpha w(x, t) &= \int_0^t d\tau \int_{\mathbb{R}^{n-1}} D_\tau^\alpha G(x' - y', x_n, \tau) (\varphi(y', t - \tau) - \varphi(y', t)) dy' \\ &\quad + \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) \varphi(y', t) dy' \equiv v_1(x, t) + v_2(x, t). \end{aligned} \tag{71}$$

In fact we have

$$\widehat{w}_0(x, t) \equiv J_t^{1-\alpha} (G * \varphi)(x, t) = \int_0^t d\tau \int_{\mathbb{R}^{n-1}} J_\sigma^{1-\alpha} G(x' - y', x_n, \sigma) |_{\sigma=t-\tau} \varphi(y', \tau) dy'.$$

We use (17). Then we differentiate with respect to t the function

$$\widehat{w}_\delta(x, t) = \int_0^{t-\delta} d\tau \int_{\mathbb{R}^{n-1}} J_\sigma^{1-\alpha} G(x' - y', x_n, \sigma) |_{\sigma=t-\tau} \varphi(y', \tau) dy'$$

and get

$$\begin{aligned} \frac{\partial}{\partial t} \widehat{w}_\delta(x, t) &= \int_0^{t-\delta} d\tau \int_{\mathbb{R}^{n-1}} D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-\tau} \varphi(y', \tau) dy' \\ &\quad + \int_{\mathbb{R}^{n-1}} J_\sigma^{1-\alpha} G(x' - y', x_n, \sigma) |_{\sigma=\delta} \varphi(y', t - \delta) dy' \end{aligned}$$

$$\begin{aligned}
 &= \int_{\delta}^t d\tau \int_{\mathbb{R}^{n-1}} D_{\tau}^{\alpha} G(x' - y', x_n, \tau) \varphi(y', t - \tau) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} (J_{\sigma}^{1-\alpha} G(x' - y', x_n, \sigma)|_{\sigma=\delta} - J_t^{1-\alpha} G(x' - y', x_n, t)) \varphi(y', t - \delta) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) \varphi(y', t - \delta) dy' \\
 &= \int_{\delta}^t d\tau \int_{\mathbb{R}^{n-1}} D_{\tau}^{\alpha} G(x' - y', x_n, \tau) \varphi(y', t - \tau) dy' \\
 &\quad - \int_{\delta}^t d\tau \int_{\mathbb{R}^{n-1}} D_{\tau}^{\alpha} G(x' - y', x_n, \tau) \varphi(y', t - \delta) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) \varphi(y', t - \delta) dy' \\
 &= \int_{\delta}^t d\tau \int_{\mathbb{R}^{n-1}} D_{\tau}^{\alpha} G(x' - y', x_n, \tau) (\varphi(y', t - \tau) - \varphi(y', t - \delta)) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) \varphi(y', t - \delta) dy'.
 \end{aligned}$$

Letting $\delta \rightarrow 0$, we derive (71).

Let $0 < \bar{t} < t$. Denote $h = t - \bar{t}$. We consider the cases a) $h \geq \frac{t}{4}$; b) $h < \frac{t}{4}$.

First we estimate $v_2(x, t) - v_2(x, \bar{t})$. In the case a) we apply estimate (34)

$$|v_2(x, t) - v_2(x, \bar{t})| \leq |v_2(x, t)| + |v_2(x, \bar{t})| \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^{\gamma} (t^{\gamma} + \bar{t}^{\gamma}) \leq \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^{\gamma} h^{\gamma}. \quad (72)$$

In the case b) we have

$$\begin{aligned}
 v_2(x, t) - v_2(x, \bar{t}) &= \int_{\mathbb{R}^{n-1}} (J_t^{1-\alpha} G(x' - y', x_n, t) - J_{\bar{t}}^{1-\alpha} G(x' - y', x_n, \bar{t})) \varphi(y', t) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_{\bar{t}}^{1-\alpha} G(x' - y', x_n, \bar{t}) (\varphi(y', t) - \varphi(y', \bar{t})) dy' \\
 &= \int_{\bar{t}}^t d\tau \int_{\mathbb{R}^{n-1}} D_{\tau}^{\alpha} G(x' - y', x_n, \tau) \varphi(y', t) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_{\bar{t}}^{1-\alpha} G(x' - y', x_n, \bar{t}) (\varphi(y', t) - \varphi(y', \bar{t})) dy'.
 \end{aligned}$$

By (34) we get

$$\begin{aligned}
 |v_2(x, t) - v_2(x, \bar{t})| &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma \left(t^\gamma \int_{\bar{t}}^t \frac{d\tau}{\tau} + |t - \bar{t}|^\gamma \right) \\
 &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma \left(t^\gamma \frac{h}{\bar{t}} + h^\gamma \right) \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h^\gamma,
 \end{aligned} \tag{73}$$

since $4h = 4(t - \bar{t}) < t$ and $\frac{1}{\bar{t}} < \frac{4}{3t} \leq \frac{4}{3t^\gamma(4h)^{1-\gamma}}$.

In the case a) first term v_1 are estimated as follows (see (34))

$$|v_1(x, t)| \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma \int_0^t (t - \tau)^{\gamma-1} d\tau \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma t^\gamma$$

and

$$|v_1(x, t) - v_1(x, \bar{t})| \leq |v_1(x, t)| + |v_1(x, \bar{t})| \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma (t^\gamma + \bar{t}^\gamma) \leq \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h^\gamma. \tag{74}$$

In the case b) we evaluate the difference $v_1(x, t) - v_1(x, \bar{t})$ in this way

$$\begin{aligned}
 v_1(x, t) - v_1(x, \bar{t}) &= \int_{t-2h}^t d\tau \int_{\mathbb{R}^{n-1}} D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-\tau} (\varphi(y', \tau) - \varphi(y', t)) dy' \\
 &\quad - \int_{t-2h}^{t-h} d\tau \int_{\mathbb{R}^{n-1}} D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-h-\tau} (\varphi(y', \tau) - \varphi(y', t-h)) dy' \\
 &\quad + \int_0^{t-2h} d\tau \int_{\mathbb{R}^{n-1}} D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-h-\tau} (\varphi(y', t) - \varphi(y', t-h)) dy' \\
 &\quad + \int_0^{t-2h} d\tau \int_{\mathbb{R}^{n-1}} (D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-\tau} - D_\sigma^\alpha G(x' - y', x_n, \sigma) |_{\sigma=t-h-\tau}) \\
 &\quad \times (\varphi(y', \tau) - \varphi(y', t-h)) dy' = \sum_{i=1}^4 I_i.
 \end{aligned}$$

Besides in I_3 we integrate with respect to τ

$$\begin{aligned}
 I_3 &= \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) (\varphi(y', t) - \varphi(y', t-h)) dy' \\
 &\quad - \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) |_{t=h} (\varphi(y', t) - \varphi(y', t-h)) dy'
 \end{aligned}$$

Estimate (34) yields

$$|I_1| + |I_2| + |I_3| \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h^\gamma. \tag{75}$$

We observe that $t - 2h > \frac{t}{2}$ in the case b) and get

$$\begin{aligned} I_4 = I'_4 + I''_4 &= \int_0^{t/2} d\tau \int_{t-h-\tau}^{t-\tau} d\zeta \int_{\mathbb{R}^{n-1}} D_\zeta^{\alpha+1} G(x' - y', x_n, \zeta) (\varphi(y', \tau) - \varphi(y', t)) dy' \\ &\quad + \int_{t/2}^{t-2h} d\tau \int_{t-h-\tau}^{t-\tau} d\zeta \int_{\mathbb{R}^{n-1}} D_\zeta^{\alpha+1} G(x' - y', x_n, \zeta) (\varphi(y', \tau) - \varphi(y', t)) dy'. \end{aligned}$$

We employ (34) again

$$\begin{aligned} |I'_4| &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma \int_0^{t/2} d\tau \int_{t-h-\tau}^{t-\tau} \zeta^{-2} (t - \tau)^\gamma d\zeta \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma t^\gamma \int_0^{t/2} d\tau \int_{t-h-\tau}^{t-\tau} \zeta^{-2} d\zeta \\ &= C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma t^\gamma \int_0^{t/2} \frac{hd\tau}{(t - \tau)(t - h - \tau)} \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma t^\gamma h \frac{t/2}{t/2(t/2 - h)} \\ &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma t^{\gamma-1} h \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h^\gamma, \end{aligned} \tag{76}$$

since $h < t/4$ in the case b). Then we estimate I''_4 as follows

$$\begin{aligned} |I''_4| &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma \int_{t/2}^{t-2h} d\tau \int_{t-h-\tau}^{t-\tau} t - \tau^\gamma \zeta^{-2} d\zeta \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h \int_{t/2}^{t-2h} (t - \tau)^{\gamma-2} d\tau \\ &\leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h (h^{\gamma-1} + (t/2)^{\gamma-1}) \leq C \langle \varphi \rangle_{t, \mathbb{R}_T^{n-1}}^\gamma h^\gamma, \end{aligned} \tag{77}$$

Inequality (51) follows from (72), (73), (74), (75), (76), (77). \square

3.5. Proof of Theorem 1

We need estimates of the Hölder constants

$$\langle w \rangle_{\alpha, \mathbb{R}_{+,T}^n}^{(k+1+\theta)}, \quad \langle w \rangle_{t, \mathbb{R}_{+,T}^n}^{((k+1+\theta)\frac{\alpha}{2})}.$$

We claim that

$$\langle w \rangle_{t, \mathbb{R}_{+,T}^n}^{((k+1+\theta)\frac{\alpha}{2})} \leq C \langle w \rangle_{\alpha, \mathbb{R}_{+,T}^n}^{(k+1+\theta)}. \tag{78}$$

First we prove (78) for $k = 1$. We use the identity

$$w(x, t) = w(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} D_{*,\tau}^\alpha w(x, \tau) d\tau,$$

and Theorem 2.5 from [5] to obtain

$$\langle w(\cdot, t) - w(\cdot, \tau) \rangle_{x, \mathbb{R}_+^n}^{(\theta)} \leq C|t - \tau|^\alpha \langle D_{*t}^\alpha w \rangle_{x, \mathbb{R}_+^n}^{(\theta)} \tag{79}$$

Then the interpolation inequality (see [19])

$$|u_{x_j}|_Q \leq C \left([u]_Q^{(2+\theta)} \right)^{\frac{1-\theta}{2}} \left([u]_Q^{(\theta)} \right)^{\frac{1+\theta}{2}}, \quad j = 1, \dots, n,$$

(79) and Young’s inequality give

$$\langle w_{x_j} \rangle_{t, \mathbb{R}_+^n, T}^{((1+\theta)\frac{\alpha}{2})} \leq C \langle w \rangle_{\alpha, \mathbb{R}_+^n, T}^{(2+\theta)}, \quad j = 1, \dots, n \tag{80}$$

i.e. (78) for $k = 1$.

Now we consider derivatives $(D_{*t}^\alpha)^m D_x^l w(x, t)$, $2m + |l| = k$. Since w is solution of (7) we have $(D_{*t}^\alpha)^m D_x^l w(x, t) = \Delta^m D_x^l w(x, t)$. Denote $\bar{l}_j = (l_1, \dots, l_j - 1, \dots, l_n)$. We deduce

$$\begin{aligned} \langle w \rangle_{t, \mathbb{R}_+^n, T}^{((k+1+\theta)\frac{\alpha}{2})} &= \sum_{2m+|l|=k} \langle (D_{*t}^\alpha)^m D_x^l w \rangle_{t, \mathbb{R}_+^n, T}^{((1+\theta)\frac{\alpha}{2})} = \sum_{2m+|l|=k} \langle \Delta^m D_x^l w \rangle_{t, \mathbb{R}_+^n, T}^{(1+\theta)\frac{\alpha}{2}} \\ &\leq C(k) \sum_{|l|=k} \left\langle \frac{\partial}{\partial x_j} D_x^{\bar{l}_j} w \right\rangle_{t, \mathbb{R}_+^n, T}^{((1+\theta)\frac{\alpha}{2})} \leq C(k) \sum_{j=1}^n \sum_{|l|=k} \langle D_x^{\bar{l}_j} w \rangle_{\alpha, \mathbb{R}_+^n, T}^{(2+\theta)\frac{\alpha}{2}} \\ &\leq C(k) \langle w \rangle_{\alpha, \mathbb{R}_+^n, T}^{(k+1+\theta)}. \end{aligned}$$

Inequality (78) is proved.

We just estimate $\langle w \rangle_{\alpha, \mathbb{R}_+^n, T}^{(k+1+\theta)}$. We consider derivatives $(2m + |l| = k + 1)$

$$(D_{*t}^\alpha)^m D_x^l w(x, t) = (D_{*t}^\alpha)^m D_x^l (G * \varphi).$$

Since the function w satisfies equation (7), one can take $l_n = 0, 1$. If $l_j \neq 0$ for some $j \in \{1, \dots, n\}$, then we have

$$(D_{*t}^\alpha)^m D_x^l (G * \varphi) = \frac{\partial}{\partial x_j} (G * (D_{*t}^\alpha)^m D_x^{\bar{l}_j} \varphi).$$

By (49), (50) we get

$$\left[\frac{\partial}{\partial x_j} (G * (D_{*t}^\alpha)^m D_x^{\bar{l}_j} \varphi) \right]_{\alpha, \mathbb{R}_+^n, T}^{(\theta)} \leq C(T) [\varphi]_{\alpha, \mathbb{R}_T^{n-1}}^{(k+\theta)}. \tag{81}$$

If $|l| = 0$ then $2m = k + 1$

$$(D_{*t}^\alpha)^m (G * \varphi) = D_{*t}^\alpha (G * (D_{*t}^\alpha)^{m-1} \varphi)$$

By (51) we obtain

$$\langle (D_{*t}^\alpha)^m w \rangle_{t, \mathbb{R}_+^n, T}^{(\theta\frac{\alpha}{2})} \leq C(T) [\varphi]_{\alpha, \mathbb{R}_T^{n-1}}^{(k+\theta)}. \tag{82}$$

To this end we estimate the Hölder constant of $(D_{*,t}^\alpha)^m(G * \varphi)$ with respect to x . By analogy with (71) we obtain

$$\begin{aligned}
 (D_{*,t}^\alpha)^m(G * \varphi)(x,t) &= D_{*,t}^\alpha \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G(x' - y', x_n, t - \tau) (D_{*,\tau}^\alpha)^{m-1} \varphi(y', \tau) dy' \\
 &= \int_0^t d\tau \int_{\mathbb{R}^{n-1}} D_\tau^\alpha G(x' - y', x_n, \tau) (D_{*,\sigma}^\alpha)^{m-1} \varphi(y', \sigma)|_{\sigma=t-\tau} dy' \\
 &\quad - \int_0^t d\tau \int_{\mathbb{R}^{n-1}} D_\tau^\alpha G(x' - y', x_n, \tau) (D_{*,t}^\alpha)^{m-1} \varphi(y', t) dy' \\
 &\quad + \int_{\mathbb{R}^{n-1}} J_t^{1-\alpha} G(x' - y', x_n, t) (D_{*,t}^\alpha)^{m-1} \varphi(y', t) dy'.
 \end{aligned}
 \tag{83}$$

We represent $D_t^\alpha G$ from (27) as

$$D_t^\alpha G(x,t) = -2 \frac{\partial \Gamma^\alpha}{\partial x_n}(x,t) - \sum_{i=1}^n h_i \frac{\partial G}{\partial x_i}(x,t).$$

Lemma 3.7 from [15] and estimate (49) give bound for the first and second term in (83). Third term is estimated by (34). Thus we have

$$\langle (D_{*,t}^\alpha)^m w \rangle_{x, \mathbb{R}_+^n}^{(\theta)} \leq C(T) [\varphi]_{\alpha, \mathbb{R}_T^{n-1}}^{(k+\theta)}.
 \tag{84}$$

The inequalities (81), (82), (84) result in estimate (13).

4. Proof of Theorem 2

4.1. Local in time solvability

We set

$$u_1(x) = f(x, 0) + \mathcal{A} \left(x, 0, \frac{\partial}{\partial x} \right) u_0(x), \quad x \in \Omega,
 \tag{85}$$

$$u_2(x) = \psi(x, 0) - \mathcal{B} \left(x, 0, \frac{\partial}{\partial x} \right) u_0(x), \quad x \in \Sigma.
 \tag{86}$$

Assumptions of Theorem 2 imply (see (16))

$$u_1 \in C^\theta(\Omega), \quad u_2 \in C^{1+\theta}(\Sigma), \quad u_1 \in C^{1+\theta}(\Sigma).$$

We look for solution of the problem (1), (2), (3) as

$$u(x,t) = \tilde{V}(x,t) + v(x,t),$$

where

$$\begin{aligned} \tilde{V}(x, 0) &= u_0(x), \quad D_{*,t}^\alpha \tilde{V}(x, t)|_{t=0} = u_1(x), \quad x \in \Omega \\ \tilde{V} &\in C_{\alpha}^{2+\theta}(\Omega_T), \quad D_{*,t}^\alpha \tilde{V} \in C_{\alpha}^{1+\theta}(\Sigma_T) \end{aligned} \tag{87}$$

and v is the new unknown function such that

$$D_{*,t}^\alpha v - \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) v = f_1 \equiv f - D_{*,t}^\alpha \tilde{V} + \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{V}, \quad (x, t) \in \Omega_T, \tag{88}$$

$$v(x, 0) = 0, \quad x \in \Omega, \tag{89}$$

$$D_{*,t}^\alpha v + \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) v = \psi_1 \equiv \psi - D_{*,t}^\alpha \tilde{V} - \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{V}, \quad (x, t) \in \Sigma_T. \tag{90}$$

It is easy to see

$$f_1 \in C_{\alpha,0}^\theta(\Omega_T), \quad \psi_1 \in C_{\alpha,0}^{1+\theta}(\Sigma_T) \tag{91}$$

under conditions that relations (14), (87) are fulfilled.

THEOREM 3. *Suppose that assumptions (4), (5), (6), (14), (91) hold. Then for sufficiently small τ there exists a unique solution v of problem (88), (89), (90): $v \in C_{0,\alpha}^{2+\theta}(\Omega_\tau)$, $D_{*,t}^\alpha v \in C_{0,\alpha}^{2+\theta}(\Sigma_\tau)$, and*

$$|v|_{\alpha,\Omega_\tau}^{(2+\theta)} + |D_{*,t}^\alpha v|_{\alpha,\Sigma_\tau}^{(1+\theta)} \leq C(T) \left(|f_1|_{\alpha,\Omega_\tau}^{(\theta)} + |\psi_1|_{\alpha,\Sigma_\tau}^{(1+\theta)} \right). \tag{92}$$

This theorem can be established by the construction of a regularizer (see §4-§7 from Chapter IV [17]). This approach is based on freezing the coefficients of the operators \mathcal{A} , \mathcal{B} and study of two model problems: problem (7)–(10) and Cauchy problem for the equation $D_{*,t}^\alpha u - \Delta u(x, t) = f(x, t)$.

As it follows from what will be said below, in order to construct the function \tilde{V} we need the function V such that

$$\begin{aligned} V(x, 0) &= 0, \quad D_{*,t}^\alpha V(x, t)|_{t=0} = u_2(x), \quad x \in \Sigma \\ V &\in C_{\alpha}^{2+\theta}(\Sigma_T), \quad D_{*,t}^\alpha V \in C_{\alpha}^{1+\theta}(\Sigma_T). \end{aligned} \tag{93}$$

We use the approach similar to [3] and cover the boundary Σ with balls $B_m = \{|x - x_m| \leq d\}$, ($m = 1, \dots, N$) for sufficiently small d . Let Φ_m be $C^{2+\theta}$ -mapping of the set $\sigma_m = \Sigma \cap B_m$ on a domain in \mathbb{R}^{n-1} . Let $\tilde{\eta}_m(x)$, $\eta_m(x)$ be the sets of smooth functions such that

$$\overline{\text{supp } \tilde{\eta}_m} \subseteq \sigma_m, \quad \tilde{\eta}_m(x) \eta_m(x) = \eta_m(x), \quad \sum_{m=1}^N \eta_m(x) = 1 \quad x \in \Sigma. \tag{94}$$

We define functions $V_m(y, t)$ as the solutions to Cauchy problem

$$\begin{aligned} D_{*,t}^\alpha V_m(y, t) - \Delta V_m(y, t) &= F_m(y, t), \quad (y, t) \in \mathbb{R}_T^{n-1}, \\ V_m(y, 0) &= 0, \quad y \in \mathbb{R}^{n-1}, \end{aligned} \tag{95}$$

where

$$F_m(y) = (\eta_m u_2) \circ \Phi_m^{-1}(y) \in C_{\alpha}^{1+\theta}(\mathbb{R}_T^{n-1}).$$

By virtue of results [15] we have

$$|V_m|_{\alpha, \mathbb{R}_T^{n-1}}^{3+\theta} \leq C(T, \Sigma) |u_2|_{\alpha, \Sigma}^{1+\theta}. \tag{96}$$

Now we define V as follows

$$V(x, t) = \sum_{m=1}^N \tilde{\eta}(V_m \circ \Phi_m)(x, t). \tag{97}$$

In view of (94), (95) the function V satisfies (93). Then we define \tilde{V} as the solution of Dirichlet problem

$$\begin{aligned} D_{*,t}^{\alpha} \tilde{V} - \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{V} &= f, \quad (x, t) \in \Omega_T, \\ \tilde{V}(x, 0) &= u_0, \quad x \in \Omega, \\ \tilde{V} &= g(x, t) \equiv u_0 + V, \quad (x, t) \in \Sigma_T. \end{aligned} \tag{98}$$

From (93), (86),(6) we see that compatibility conditions

$$\begin{aligned} g(x, 0) &= u_0(x), \quad x \in \Sigma, \\ D_{*,t}^{\alpha} g(x, t)|_{t=0} &= \mathcal{A} \left(x, 0, \frac{\partial}{\partial x} \right) u_0(x) + f(x, 0), \quad x \in \Sigma \end{aligned} \tag{99}$$

are fulfilled. Hence by virtue of results [15] there exists unique solution of (98): $\tilde{V} \in C_{\alpha}^{2+\theta}(\Omega_T)$. By (98) we deduce $D_{*,t}^{\alpha} \tilde{V} = D_{*,t}^{\alpha} V$ on Γ_T , so that $D_{*,t}^{\alpha} \tilde{V} \in C_{\alpha}^{1+\theta}(\Sigma_T)$ and

$$|\tilde{V}|_{\alpha, \Omega_T}^{(2+\theta)} + |D_{*,t}^{\alpha} \tilde{V}|_{\alpha, \Sigma_T}^{(1+\theta)} \leq C(T, \Sigma) (|u_0|_{\Omega}^{(2+\theta)} + |u_1|_{\Omega}^{(\theta)} + |u_2|_{\alpha, \Sigma}^{(1+\theta)}). \tag{100}$$

To this end the statements of Theorem 2 for sufficiently small T follows from estimate (100) and Theorem 3.

4.2. Solvability on interval $(0, T]$

Now we extend the obtained solution on interval (τ, T) .

LEMMA 6. *There exist functions W, \tilde{W} such that*

$$W \in C_{\alpha}^{2+\theta}(\Sigma_T), \quad D_{*,t}^{\alpha} W \in C_{\alpha}^{1+\theta}(\Sigma_T), \quad W(x, t) = v(x, t) \quad (x, t) \in \Sigma_{\tau}, \tag{101}$$

$$\tilde{W} \in C_{\alpha}^{2+\theta}(\Omega_T), \quad D_{*,t}^{\alpha} \tilde{W} \in C_{\alpha}^{1+\theta}(\Sigma_T), \quad \tilde{W}(x, t) = v(x, t) \quad (x, t) \in \Omega_{\tau}, \tag{102}$$

here v is the solution of (88), (89), (90).

Proof. We use the functions $\Phi_m, \eta_m, \tilde{\eta}_m$ again. Define

$$w_m = (\eta_m v) \circ \Phi_m^{-1} \tag{103}$$

and s_m as a solution to Dirichlet problem

$$\begin{aligned} \Delta s_m &= 0, & y_n &> 0, \\ s_m &= w_m, & y_n &= 0. \end{aligned} \tag{104}$$

Then we set

$$g_m = D_{*,t}^\alpha w_m - s_{m,y_n} \tag{105}$$

and

$$g'_m(y, t) = \begin{cases} g_m(y, t) & \text{for } t \in [0, \tau], \\ g_m(y, \tau) & \text{for } t \in [\tau, T]. \end{cases} \tag{106}$$

To this end we define (s'_m, w'_m) as a solution of the problem

$$\begin{aligned} \Delta s_m &= 0, & y_n &> 0, \\ s'_m &= w'_m, & D_{*,t}^\alpha w'_m - s'_{m,y_n} &= g'_m(y', t), & y_n &= 0, \\ w'_m &= 0, & t &= 0, & y_n &= 0. \end{aligned} \tag{107}$$

Following [33], we conclude that there exists a unique solution of (107) such that $w'_m \in C_\alpha^{2+\theta}(\mathbb{R}_T^{n-1}), D_{*,t}^\alpha w'_m \in C_\alpha^{1+\theta}(\mathbb{R}_T^{n-1})$. By uniqueness of solution of (107) we see from (103), (104), (105), (106) that

$$w'_m(y', t) = w_m(y', t), \quad t \in [0, \tau]. \tag{108}$$

It can be shown in a standard way that the function

$$W = \sum_{m=1}^N \tilde{\eta}(w'_m \circ \Phi_m)(x)$$

satisfies (101). The function \tilde{W} is defined as a solution of the problem

$$\begin{aligned} D_{*,t}^\alpha \tilde{W} - \mathcal{L} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{W} &= f_1(x, t), & (x, t) &\in \Omega_T \\ \tilde{W}|_{t=0} &= 0, & x &\in \Omega, \\ \tilde{W} &= W, & (x, t) &\in \Sigma_T. \end{aligned}$$

We conclude from results of [15] that \tilde{W} exists and satisfies the estimate

$$|\tilde{W}|_{\alpha, \Omega_T}^{(2+\theta)} + |D_{*,t}^\alpha \tilde{W}|_{\alpha, \Sigma_T}^{(1+\theta)} \leq C(T) \left(|f_1|_{\alpha, \Omega_T}^{(\theta)} + |v|_{\alpha, \Omega_T}^{(2+\theta)} + |D_{*,t}^\alpha v|_{\alpha, \Sigma_T}^{(1+\theta)} \right). \tag{109}$$

□

Once the function \tilde{W} is constructed, we look for the solution of (1), (2), (3), as

$$u(x, t) = \tilde{V}(x, t) + v(x, t),$$

with

$$v(x, t) = \tilde{W}(x, t) + w(x, t),$$

where $w(x, t)$ is the new unknown function

$$D_{*,t}^\alpha w - \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) w = f_2 \equiv f_1 - D_{*,t}^\alpha \tilde{W} + \mathcal{A} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{W}, \quad (x, t) \in \Omega_T, \\ w(x, 0) = 0, \quad x \in \Omega, \quad (110)$$

$$D_{*,t}^\alpha w + \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) w = \psi_2 \equiv \psi_1 - D_{*,t}^\alpha \tilde{W} - \mathcal{B} \left(x, t, \frac{\partial}{\partial x} \right) \tilde{W}, \quad (x, t) \in \Sigma_T.$$

Routine calculations shows

$$f_2 \in C_\alpha^\theta(\Omega_T), \quad \psi_2 \in C_\alpha^{1+\theta}(\Sigma_T), \quad (111)$$

$$f_2, \psi_2 = 0, \quad t \in [0, \tau]. \quad (112)$$

We apply Theorem 3 to problem (110). In view of (112) and the uniqueness of solution to (110)

$$w(x, t) = 0, \quad t \in [0, \tau].$$

This identity allows us as in [16] shift variable $t \rightarrow t - \tau$ and apply Theorem 3 on the segment $[\tau, 2\tau]$. Then we repeat this procedure to get the solution of (1), (2), (3) on any interval $[0, m\tau]$, $m \in \mathbb{N}$. This ends the proof of Theorem 2.

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