

SOLVABILITY AND POSITIVE SOLUTIONS OF A SYSTEM OF HIGHER ORDER FRACTIONAL BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS

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(Communicated by A. Ashyralyev)

Abstract. The main purpose of this paper is to study the problem of the existence, uniqueness and positivity of solutions of a system of higher order fractional differential equations with boundary value problem expressed by fractional and integral conditions. Using fixed point theorems, we discuss the existence and the uniqueness of solutions of this problem, and we apply Guo-Krasnoselskii's fixed point theorem in cone to study the existence of positive solutions. We give some examples to illustrate our results.

1. Introduction

Fractional differential equations have been of great interest. This is because of the big numbers of applications in various fields of science and engineering [4, 5, 6, 7]. Recently, many books about fractional calculus and fractional differential equations have been appeared [1, 2, 3]. The main purpose of the present paper is to investigate sufficient conditions for the existence, uniqueness and positivity solution of the following higher-order fractional differential equation:

$${}^c D_{0+}^{\alpha_i} u_i(t) = f_i(t, \mathbf{u}(t), {}^c D_{0+}^{\beta_1} \mathbf{u}(t), \dots, {}^c D_{0+}^{\beta_{n_i-2}} \mathbf{u}(t)), \quad 0 < t < T \quad (1)$$

with fractional integral and integral conditions:

$$\begin{cases} u_i^{(k)}(0) = \int_0^T h_{i,k}(s, \mathbf{u}(s)) ds, & 0 \leq k \leq n_i - 2 \\ u_i^{(n_i-1)}(0) = \gamma_i I_{0+}^{m_i} u_i(T) + \int_0^T h_{i,n_i-1}(s, \mathbf{u}(s)) ds, \end{cases} \quad (2)$$

where $p \in \mathbb{N}^*$, $i \in \{1, \dots, p\}$, $n_i \in \mathbb{N}^* \setminus \{1\}$, $m_i \in \mathbb{R}_+$, $0 \leq \gamma_i < \frac{\Gamma(m_i+n_i)}{T^{n_i+m_i-1}}$, $\mathbf{u}(t) = (u_1(t), \dots, u_p(t))$, $f_i : [0, T] \times \mathbb{R}^{(n_i-1) \times p} \rightarrow \mathbb{R}$ and for all $0 \leq k \leq n_i - 1$, $h_{i,k} : [0, T] \times \mathbb{R}^p \rightarrow \mathbb{R}$ are continuous functions. The non-local condition $I_{0+}^{m_i} u_i(T)$ has physical significations such as total mass, moment, etc. Sometimes it is better to impose integral conditions to get a more accurate measure than a local condition (see [21]).

Mathematics subject classification (2010): 26A33, 34B15, 34B18, 34B27.

Keywords and phrases: Higher order fractional boundary value problem with fractional and integral conditions, Guo-Krasnosel'skii fixed point in cone, positive solution.

Various type of boundary value problems involving fractional derivative were studied by many authors using fixed point theorems on cones, fixed point index theory, Adomian decomposition method, fixed point theorem combined with the technique of measures of weak non-compactness, Leray-Schauder nonlinear alternative, Leggett-Williams fixed point theorem, upper and lower solutions method (see [13, 12, 14, 15, 16, 17, 18, 19, 20]).

In [15], V. Daftardar-Gejji et al. discussed existence, uniqueness and stability of solutions of the system of nonlinear fractional differential equations:

$${}^c D_{0+}^{\alpha_i} u_i(t) = f_i(t, u_1(t), \dots, u_n(t)) = 0, \quad 0 < t < 1 \tag{3}$$

$$u_i^{(k)}(0) = c_k^i, \tag{4}$$

where $1 \leq i \leq n, 1 \leq k \leq m_i, m_i < \alpha_i \leq m_i + 1$.

In [14], K. Diethelm considered the following ordinary fractional differential equations with Caputo-type differential operators:

$${}^c D_{0+}^{\alpha} u(t) = g(t, u(t)), \quad 0 < t < 1 \tag{5}$$

$$u^{(j)}(0) = u_0^{(j)}. \tag{6}$$

K. Diethelm give a full characterization of the situations where smooth solutions exist of (5)–(6). The results can be extended to a class of weakly singular Volterra integral equations.

In [17] J. R. Graef et al. investigated the existence of positive solution of nonlinear fractional boundary value problems of the form:

$$-D_{0+}^{\alpha} u(t) + aD_{0+}^{\gamma} u(t) = f(t, u(t)), \quad 0 < t < 1 \tag{7}$$

$$D_{0+}^{\beta} u(0) = 0, \quad D_{0+}^{\alpha-\gamma} u(1) = au(1) \tag{8}$$

where D_{0+}^{α} is the α -th Riemann-Liouville fractional derivative and $1 < \gamma < \alpha \leq 2, 0 \leq \beta < \alpha - \gamma, 0 \leq a < \Gamma(\alpha - \gamma + 1)$ and $f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R}_+)$. The existence of positive solutions of a (7)–(8) can be established by finding fixed points of an associated operator. The construction of such operators often involves the derivation of the Green’s functions and is a key step in this approach.

In [12], J. Deng et al. studied the existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations:

$${}^c D_{0+}^{\alpha} u(t) = g(t, {}^c D_{0+}^{\beta} u(t)), \quad 0 < t < 1 \tag{9}$$

$$u^{(k)}(0) = \eta_k, \quad k = 0, 1, \dots, m - 1, \tag{10}$$

with $m - 1 < \alpha < m, n - 1 < \beta < n, (m, n \in \mathbb{N}, m - 1 \geq n), g \in C([0, 1] \times \mathbb{R})$. Using of the Schauder fixed point theorem, J. Deng et al. obtained some new results for the existence and uniqueness of solutions of (9)–(10).

In [16], A. M. A. El-Sayed et al. studied the following fractional boundary value problem:

$$u^{(n)}(t) = f(t, u(t), {}^c D_{0+}^{\alpha_1} u(t), {}^c D_{0+}^{\alpha_2} u(t), \dots, {}^c D_{0+}^{\alpha_n} u(t)) \tag{11}$$

$$u^{(j)}(0) = c_j, \quad j = 0, 1, \dots, n - 1. \tag{12}$$

Two methods are used to solve this type of equations. The first is an analytical method called Adomian decomposition method. Convergence analysis of this method is discussed. This analysis is used to estimate the maximum absolute truncated error of Adomian’s series solution. The second method is a proposed numerical method. A comparison between the results of the two methods is given.

For some recent contributions on fractional differential equations, we refer the reader to (see [22, 23, 24, 25, 26, 27, 28, 29]). Our aim is to use Banach contraction principle and Leray-Schauder nonlinear alternative to prove the existence and uniqueness solutions of our problem. For this, we formulate the boundary value problem as the fixed point problem. However, the Schauder fixed point theorem cannot ensure the solutions to be positive. Since only positive solutions are useful for many applications, motivated by the above works, the existence of positive solution are obtained by the Guo-Krasnosels’kii fixed point theorem. The particularity of our equation (1)–(2) is that the nonlinear term contain the fractional order derivative $D_{0^+}^{\beta_{n_i}-2} \mathbf{u}(t)$, and boundary condition involving fractional integral condition $I_{0^+}^{m_i} u_i(1)$ which leads to extra difficulties. To the best of our knowledge, no one has studied the existence and positivity of solutions for nonlinear differential fractional equation (1) jointly with fractional and integral conditions (2). For $p = 1, \alpha_1 = 2, m_1 = 0, \gamma_1 = 1$ we have the second order problem $u_1''(t) = f_1(t, u_1(t)), t \in [0, 1]$ with classical boundary conditions $u(0) = 0, u'(0) = u(1)$ and for $p = 1, \alpha_1 = 2, \gamma_1 = 0$ we have the second order problem $u_1''(t) = f_1(t, u_1(t)), t \in [0, 1]$ with the initial conditions $u(0) = u'(0) = 0$. In special cases our problem reduces to (3)–(4), (5)–(6), (9)–(10) and (11)–(12).

1.1. Lemmas

In this section we present the necessary definitions and lemmas from fractional calculus theory. For details, see [8, 9, 10, 11].

DEFINITION 1. Let $f \in L^1([a, b])$ and $\alpha \geq 0$. The Riemann-Liouville fractional integral is defined by

$$I_{a^+}^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds & \text{if } \alpha > 0 \\ f(t) & \text{if } \alpha = 0 \end{cases} \tag{13}$$

where Γ is the gamma function.

DEFINITION 2. Let $f \in C^n([a, b])$, the Caputo fractional derivative of order $\alpha > 0$ of f is defined by

$${}^c D_{a^+}^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds & \text{if } n-1 < \alpha < n \\ f^{(n)}(t) & \text{if } \alpha = n. \end{cases} \tag{14}$$

For $\alpha = 0$ we have ${}^c D_{a^+}^\alpha f(t) = f(t)$.

LEMMA 1. [9] Let $\alpha, \beta \geq 0, f \in L^1([a, b])$. Then

$$I_{a^+}^\alpha I_{a^+}^\beta f(t) = I_{a^+}^{\alpha+\beta} f(t) = I_{a^+}^\beta I_{a^+}^\alpha f(t) \tag{15}$$

is satisfied almost everywhere on $[a, b]$.

If additionally $f \in C([a, b])$ or $\alpha + \beta \geq 1$, then (15) is true for all $t \in [a, b]$.

LEMMA 2. Let $\beta \geq \alpha > 0, f \in L^1([a, b])$. Then almost everywhere on $[a, b]$

$${}^c D_{a^+}^\alpha I_{a^+}^\beta f(t) = I_{a^+}^{\beta-\alpha} f(t). \tag{16}$$

LEMMA 3. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$${}^c D_{0^+}^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$$

where $\beta > n$ and ${}^c D_{0^+}^\alpha t^k = 0, k = 0, 1, 2, \dots, n - 1$.

LEMMA 4. For $\alpha > 0, g \in C(0, 1) \cap L^1(0, 1)$, the homogeneous fractional differential equation

$${}^c D_{0^+}^\alpha g(t) = 0 \tag{17}$$

has a solution

$$g(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1} \tag{18}$$

where, $c_i \in \mathbb{R}, i = 0, \dots, n - 1$ and $n = [\alpha] + 1$.

LEMMA 5. Assume that $g \in C(0, 1) \cap L^1(0, 1)$, with derivative of order n that belongs to $C(0, 1) \cap L^1(0, 1)$, then

$$I_{0^+}^\alpha {}^c D_{0^+}^\alpha g(t) = g(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{19}$$

where, $c_i \in \mathbb{R}, i = 0, \dots, n - 1$ and $n = [\alpha] + 1$.

LEMMA 6. Let $\alpha > 0, f \in L^1([0, T], \mathbb{R}_+)$. Then, for all $t \in [0, T]$ we have

$$I_{0^+}^{\alpha+1} f(t) \leq \|I_{0^+}^\alpha f\|_{L^1}. \tag{20}$$

Proof. Let $f \in L^1([0, T], \mathbb{R}_+)$, Lemma 1 we get

$$\|I_{0^+}^\alpha f\|_{L^1} = \int_0^T |{}^c D_{0^+}^1 I_{0^+}^{\alpha+1} f(s)| ds \geq \int_0^t |{}^c D_{0^+}^1 I_{0^+}^{\alpha+1} f(s)| ds = I_{0^+}^{\alpha+1} f(t). \quad \square$$

Let $p \in \mathbb{N}^*, i \in \{1, \dots, p\}, n_i \in \mathbb{N}, n_i \geq 3, n_i - 1 < \alpha_i \leq n_i$, for all $j \in \{1, \dots, n_i - 2\}, j - 1 < \beta_j \leq j$ and $\beta_0 = 0$.

$$E = \{(u_1, \dots, u_p) \in \prod_{i=1}^p C^{n_i}([0, T]; \mathbb{R}), {}^c D_{0^+}^{\beta_j}(u_i) \in C([0, T]; \mathbb{R}), j \in \{0, \dots, n_i - 2\}, i \in \{1, \dots, p\}\}.$$

The space E is a Banach space equipped with the norm

$$\|\mathbf{u}\| = \sum_{i=1}^p \max_{j \in \{0, \dots, n_i - 2\}} \|{}^c D_{0^+}^{\beta_j}(u_i)\|_\infty \tag{21}$$

where

$$\|{}^c D_{0^+}^{\beta_j}(u_i)\|_\infty = \sup_{t \in [0, 1]} |{}^c D_{t, 0^+}^{\beta_j}(u_i)(t)| \tag{22}$$

and ${}^c D_{t, 0^+}^{\beta_j}(u_i)(t)$ fractional derivatives with respect to the variable t .

DEFINITION 3.

1. The function $\mathbf{u} = (u_1, \dots, u_p)$ is called a nonnegative solution of the system (1)–(2) if and only if \mathbf{u} satisfies (1)–(2) and for all $i \in \{1, \dots, p\}$, $u_i(t) \geq 0$ for $t \in [0, T]$.
2. The function $\mathbf{u} = (u_1, \dots, u_p)$ is called a positive solution of the system (1)–(2) if and only if \mathbf{u} satisfies (1)–(2) and for all $i \in \{1, \dots, p\}$, $u_i(t) > 0$ for $t \in (0, T)$.

LEMMA 7. Let $i \in \{1, \dots, p\}$, $0 \leq k \leq n_i - 1$. h_i and $g_{i,k}$ are continuous functions. Then the fractional differential problem:

$$\begin{cases} {}^c D_{0^+}^{\alpha_i} u_i(t) = h_i(t) & 0 < t < T \\ u_i^{(k)}(0) = \int_0^T g_{i,k}(s) ds & 0 \leq k \leq n_i - 2 \\ u_i^{(n_i-1)}(0) = \gamma_i I_{0^+}^{m_i} u_i(T) + \int_0^T g_{i, n_i-1}(s) ds \end{cases} \tag{23}$$

has a unique solution $u_i(t) = \int_0^T G_i(t, s)h_i(s) ds + \varphi_i(t)$ where G_i is the Green's function defined by

$$G_i(t, s) = \begin{cases} \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} + \frac{\gamma_i t^{n_i-1} (T-s)^{m_i+\alpha_i-1}}{M(m_i, n_i, \gamma_i) \Gamma(m_i+\alpha_i)} & \text{if } 0 \leq s \leq t \\ \frac{\gamma_i t^{n_i-1} (T-s)^{m_i+\alpha_i-1}}{M(m_i, n_i, \gamma_i) \Gamma(m_i+\alpha_i)} & \text{if } t \leq s \leq T \end{cases} \tag{24}$$

and $M(m_i, n_i, \gamma_i) = (n_i - 1)! \left(1 - \frac{\gamma_i T^{n_i+m_i-1}}{\Gamma(n_i+m_i)}\right) > 0$.

$$\begin{aligned} \varphi_i(t) = & \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T g_{i,k}(s) ds \right. \\ & \left. + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T g_{i, n_i-1}(s) ds \right] t^{n_i-1} + \sum_{k=0}^{n_i-2} \frac{t^k}{k!} \int_0^T g_{i,k}(s) ds. \end{aligned} \tag{25}$$

Proof. Applying Lemma 5 we reduce equation ${}^c D_{0^+}^{\alpha_i} u_i(t) = h_i(t)$ to an equivalent integral equation

$$u_i(t) = I_{0^+}^{\alpha_i}(h_i)(t) + c_{0,i} + c_{1,i}t + \dots + c_{n_i-2,i}t^{n_i-2} + c_{n_i-1,i}t^{n_i-1}. \tag{26}$$

From the boundary condition $u_i^{(k)}(0) = \int_0^T g_{i,k}(s) ds$, we deduce that

$$c_{k,i} = \frac{1}{k!} \int_0^T g_{i,k}(s) ds.$$

Therefore, we have successively

$$\begin{aligned} u_i(t) &= I_{0^+}^{\alpha_i}(h_i)(t) + \sum_{k=0}^{n_i-2} \frac{t^k}{k!} \int_0^T g_{i,k}(s) ds + c_{n_i-1,i}t^{n_i-1} \\ I_{0^+}^{m_i} u_i(t) &= I_{0^+}^{m_i+\alpha_i}(h_i)(t) + \sum_{k=0}^{n_i-2} \frac{1}{k!} I_{0^+}^{m_i}((\cdot)^k)(t) \int_0^T g_{i,k}(s) ds + c_{n_i-1,i} I_{0^+}^{m_i}((\cdot)^{n_i-1})(t), \\ I_{0^+}^{m_i} u_i(t) &= I_{0^+}^{m_i} u_i(t) = I_{0^+}^{m_i+\alpha_i}(h_i)(t) + \sum_{k=0}^{n_i-2} \frac{t^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T g_{i,k}(s) ds \\ &\quad + c_{n_i-1,i} \frac{\Gamma(n_i)}{\Gamma(n_i+m_i)} t^{n_i+m_i-1}, \\ I_{0^+}^{m_i} u_i(T) &= I_{0^+}^{m_i+\alpha_i}(h_i)(T) + \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T g_{i,k}(s) ds \\ &\quad + c_{n_i-1,i} \frac{\Gamma(n_i)}{\Gamma(n_i+m_i)} T^{n_i+m_i-1}, \end{aligned} \tag{27}$$

$$u^{(n_i-1)}(0) = \gamma_i I_{0^+}^{m_i} u_i(T) + \int_0^T g_{i,n_i-1}(s) ds,$$

$$\begin{aligned} (n_i-1)!c_{n_i-1,i} &= \gamma_i I_{0^+}^{m_i+\alpha_i}(h_i)(T) + \gamma_i \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T g_{i,k}(s) ds \\ &\quad + c_{n_i-1,i} \frac{\gamma_i \Gamma(n_i) T^{n_i+m_i-1}}{\Gamma(n_i+m_i)} + \int_0^T g_{i,n_i-1}(s) ds. \end{aligned}$$

We denote by

$$M(m_i, n_i, \gamma_i) = (n_i-1)! \left(1 - \frac{\gamma_i T^{n_i+m_i-1}}{\Gamma(n_i+m_i)} \right).$$

We obtain that

$$\begin{aligned} c_{n_i-1,i} &= \frac{\gamma_i I_{0^+}^{m_i+\alpha_i}(h_i)(T)}{M(m_i, n_i, \gamma_i)} + \frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T g_{i,k}(s) ds \\ &\quad + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T g_{i,n_i-1}(s) ds. \end{aligned} \tag{28}$$

Then

$$u_i(t) = I_{0+}^{\alpha_i}(h_i)(T) + \frac{\gamma_i I_{0+}^{m_i+\alpha_i}(h_i)(T)}{M(\gamma_i, m_i, \gamma_i)} t^{n_i-1} + \varphi_i(t), \tag{29}$$

that can be written as $u_i(t) = \int_0^T G_i(t,s)h_i(s) ds + \varphi_i(t)$, where G_i and φ_i are defined by 24 and 25. The proof is complete. \square

Let T_i the operator defined by

$$\begin{aligned} T_i : E &\rightarrow C([0, T]; \mathbb{R}) \\ \mathbf{u} &\mapsto T_i(\mathbf{u}) \end{aligned}$$

where for all $t \in [0, T]$

$$T_i(\mathbf{u})(t) = P_i(\mathbf{u})(t) + Q_i(\mathbf{u})(t).$$

$P_i(\mathbf{u})$ and $Q_i(\mathbf{u})$ are given by

$$P_i(\mathbf{u})(t) = \int_0^T G_i(t,s)f_i(s, \mathbf{u}(s), {}^c D_{0+}^{\beta_1} \mathbf{u}(s), \dots, D_{0+}^{\beta_{n_i-2}} \mathbf{u}(s)) ds \tag{30}$$

and

$$\begin{aligned} Q_i(\mathbf{u})(t) = & \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T h_{i,k}(s, \mathbf{u}(s)) ds \right. \\ & \left. + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T h_{i,n_i-1}(s, \mathbf{u}(s)) ds \right] t^{n_i-1} + \sum_{k=0}^{n_i-2} \frac{t^k}{k!} \int_0^T h_{i,k}(s, \mathbf{u}(s)) ds. \end{aligned} \tag{31}$$

Let \mathbf{T} , \mathbf{P} and \mathbf{Q} the operators defined by

$$\begin{aligned} \mathbf{T} : E &\rightarrow E \\ \mathbf{u} &\mapsto (T_1(\mathbf{u}), \dots, T_n(\mathbf{u})), \end{aligned}$$

$$\begin{aligned} \mathbf{P} : E &\rightarrow E \\ \mathbf{u} &\mapsto (P_1(\mathbf{u}), \dots, P_n(\mathbf{u})), \end{aligned}$$

$$\begin{aligned} \mathbf{Q} : E &\rightarrow E \\ \mathbf{u} &\mapsto (Q_1(\mathbf{u}), \dots, Q_n(\mathbf{u})) \end{aligned}$$

and

$$\mathbf{T}(\mathbf{u}) = \mathbf{P}(\mathbf{u}) + \mathbf{Q}(\mathbf{u}). \tag{32}$$

LEMMA 8. Let $i \in \{1, \dots, p\}$, $0 \leq k \leq n_i - 1$, $f_i \in C([0, T] \times \mathbb{R}^{(n_i-1) \times p}, \mathbb{R})$, $h_{i,k} \in C([0, T] \times \mathbb{R}^p, \mathbb{R})$ then $\mathbf{u} \in E$ is a solution of the fractional differential boundary value problem (1)–(2) if and only if $\mathbf{T}(\mathbf{u})(t) = \mathbf{u}(t)$ for all $t \in [0, T]$.

2. Existence and uniqueness results

LEMMA 9. For all $i \in \{1, \dots, n\}$, $j \in \{0, \dots, n_i - 2\}$, $(t, s) \in [0, T] \times [0, T]$,

$$0 \leq {}^c D_{t,0^+}^{\beta_j} G_i(t, s) \leq G_{j,i}(s) \tag{33}$$

where ${}^c D_{t,0^+}^{\beta_j} G_i(t, s)$ the Caputo derivative with respect to t of the function $G_i(t, s)$ and

$$G_{j,i}(s) = \frac{(T-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} + \frac{\gamma_i \Gamma(n_i) (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)}. \tag{34}$$

(Note that ${}^c D_{t,0^+}^{\beta_0} G_i(t, s) = G_i(t, s)$).

Proof. For all $i \in \{1, \dots, p\}$, $j \in \{0, \dots, n_i - 2\}$ we have

$${}^c D_{t,0^+}^{\beta_j} G_i(t, s) = \begin{cases} \frac{(t-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} + \frac{\gamma_i \Gamma(n_i) t^{n_i - \beta_j - 1} (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)} & \text{if } 0 \leq s \leq t \\ \frac{\gamma_i \Gamma(n_i) t^{n_i - \beta_j - 1} (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)} & \text{if } t \leq s \leq T. \end{cases} \tag{35}$$

It's clear that ${}^c D_{t,0^+}^{\beta_j} G_i(t, s) \geq 0$. Let $t \in [0, T]$. If $0 \leq s \leq t$, we have

$$\begin{aligned} & \frac{(t-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} + \frac{\gamma_i \Gamma(n_i) t^{n_i - \beta_j - 1} (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)} \\ & \leq \frac{(T-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} + \frac{\gamma_i \Gamma(n_i) T^{n_i - \beta_j - 1} (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)}. \end{aligned} \tag{36}$$

If $0 \leq t \leq s \leq T$, we have

$$\frac{(t-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} \leq \frac{(T-s)^{\alpha_i - \beta_j - 1}}{\Gamma(\alpha_i - \beta_j)} + \frac{\gamma_i \Gamma(n_i) T^{n_i - \beta_j - 1} (T-s)^{m_i + \alpha_i - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i) \Gamma(m_i + \alpha_i)}. \tag{37}$$

This achieves the proof. \square

Now, we prove the existence and uniqueness of solutions in the Banach space E . The uniqueness result is based on the Banach's contraction principle Theorem [30].

LEMMA 10. Let $i \in \{1, \dots, p\}$, assume that the functions f_i and $h_{i,k}$, $k \in \{0, \dots, n_i - 1\}$ are continuous and there exist nonnegative functions $\varphi_{l,i,k} \in L^1([0, T], \mathbb{R}_+)$, $k \in \{0, \dots, n_i - 2\}$, $\psi_{l,i,k} \in L^1([0, T], \mathbb{R}_+)$, $k \in \{0, \dots, n_i - 1\}$ such that for all $k \in \{0, \dots, n_i - 2\}$, $x_k = (x_{1,k}, \dots, x_{p,k})$, $y_k = (y_{1,k}, \dots, y_{p,k}) \in \mathbb{R}^p$ and for all $t \in [0, T]$.

$$I. \quad |f_i(t, x_0, \dots, x_{n_i-2}) - f_i(t, y_0, \dots, y_{n_i-2})| \leq \sum_{k=0}^{n_i-2} \sum_{l=1}^p \varphi_{l,i,k}(t) |x_{l,k} - y_{l,k}|$$

$$2. |h_{i,k}(t, x_0) - h_{i,k}(t, y_0)| \leq \sum_{l=1}^p \psi_{l,i,k}(t) |x_{l,0} - y_{l,0}|$$

$$3. \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} [A_{i,j}(P) + A_{i,j}(Q)] < 1.$$

where

$$A_{i,j}(Q) = \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \sum_{l=1}^p \|\psi_{l,i,k}\|_{L^1} + \frac{1}{M(m_i, n_i, \gamma_i)} \|\psi_{l,i,n_i-1}\|_{L^1} \right] \\ \times \frac{\Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(n_i-\beta_j)} + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k}}{\Gamma(k-\beta_k+1)} \sum_{l=1}^p \|\psi_{l,i,k}\|_{L^1} \tag{38}$$

and

$$A_{i,j}(P) = \sum_{k=0}^{n_i-2} \sum_{l=1}^p \left[\|I^{\alpha_i-\beta_j-1}(\varphi_{l,i,k})\|_{L^1} + \frac{\gamma_i \Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(n_i-\beta_j) M(m_i, n_i, \gamma_i)} \|I^{m_i+\alpha_j-1}(\varphi_{l,i,k})\|_{L^1} \right]. \tag{39}$$

Then the problem (1)–(2) has a unique solution \mathbf{u} in E .

Proof. We need to verify that \mathbf{T} is a contraction function.

For all $i \in \{1, \dots, p\}$, $j \in \{0, \dots, n_i-2\}$, $\mathbf{u} = (u_1, \dots, u_p)$, $\mathbf{v} = (v_1, \dots, v_p) \in E$, for each $t \in [0, T]$ we get

$$|{}^c D_{t,0+}^{\beta_j} Q_i(\mathbf{u})(t) - {}^c D_{t,0+}^{\beta_j} Q_i(\mathbf{v})(t)| \\ \leq \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T |h_{i,k}(s, \mathbf{u}(s)) - h_{i,k}(s, \mathbf{v}(s))| ds \right. \\ \left. + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T |h_{i,n_i-1}(s, \mathbf{u}(s)) - h_{i,n_i-1}(s, \mathbf{v}(s))| ds \right]^c D_{0+}^{\beta_j} t^{n_i-1} \\ + \sum_{k=0}^{n_i-2} \frac{{}^c D_{0+}^{\beta_j} t^k}{k!} \int_0^T |h_{i,k}(s, \mathbf{u}(s)) - h_{i,k}(s, \mathbf{v}(s))| ds \\ \leq \left[\left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \sum_{l=1}^p \int_0^T \psi_{l,i,k}(s) ds + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T \psi_{l,i,n_i-1}(s) ds \right] \right. \\ \left. \times \frac{\Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(n_i-\beta_j)} + \sum_{k=0}^{n_i-2} \frac{\Gamma(k+1)}{k! \Gamma(k-\beta_j+1)} \sum_{l=1}^p \int_0^T \psi_{l,i,k}(s) ds \right] \|\mathbf{u} - \mathbf{v}\| \\ \leq \left[\left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \sum_{l=1}^p \|\psi_{l,i,k}\|_{L^1} + \frac{1}{M(m_i, n_i, \gamma_i)} \|\psi_{l,i,n_i-1}\|_{L^1} \right] \right. \\ \left. \times \frac{\Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(n_i-\beta_j)} + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k}}{\Gamma(k-\beta_k+1)} \sum_{l=1}^p \|\psi_{l,i,k}\|_{L^1} \right] \|\mathbf{u} - \mathbf{v}\|.$$

Then

$$\|{}^c D_{0^+}^{\beta_j} Q_i(\mathbf{u}) - {}^c D_{0^+}^{\beta_j} Q_i(\mathbf{v})\|_{\infty} \leq A_{i,j}(Q) \|\mathbf{u} - \mathbf{v}\| \tag{40}$$

where $A_{i,j}(Q)$ is given in (38). From (40) we deduce that

$$\|\mathbf{Q}(\mathbf{u}) - \mathbf{Q}(\mathbf{v})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} A_{i,j}(Q) \|\mathbf{u} - \mathbf{v}\|. \tag{41}$$

On the other hand. For all $i \in \{1, \dots, p\}$, $j \in \{0, \dots, n_i - 2\}$ we have

$$\begin{aligned} & |{}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t) - {}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{v})(t)| \\ & \leq \int_0^T {}^c D_{t,0^+}^{\beta_j} G_i(t,s) |f_i(s, \mathbf{u}(s), {}^c D_{0^+}^{\beta_1} \mathbf{u}(s), \dots, D_{0^+}^{\beta_{n_i-2}} \mathbf{u}(s)) \\ & \quad - f_i(s, \mathbf{v}(s), {}^c D_{0^+}^{\beta_1} \mathbf{v}(s), \dots, D_{0^+}^{\beta_{n_i-2}} \mathbf{v}(s))| ds \\ & \leq \int_0^T {}^c D_{t,0^+}^{\beta_j} G_i(t,s) \left[\sum_{k=0}^{n_i-2} \sum_{l=1}^p \varphi_{l,i,k}(s) |{}^c D_{0^+}^{\beta_k} u_l(s) - {}^c D_{0^+}^{\beta_k} v_l(s)| \right] ds \\ & \leq \left[\sum_{k=0}^{n_i-2} \sum_{l=1}^p I^{\alpha_i - \beta_j}(\varphi_{l,i,k})(t) + \sum_{k=0}^{n_i-2} \sum_{l=1}^p \frac{\gamma_i \Gamma(n_i) T^{n_i - \beta_j - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i)} I^{m_i + \alpha_j}(\varphi_{l,i,k})(T) \right] \|\mathbf{u} - \mathbf{v}\| \\ & \leq \sum_{k=0}^{n_i-2} \sum_{l=1}^p \left[\|I^{\alpha_i - \beta_j - 1}(\varphi_{l,i,k})\|_{L^1} + \frac{\gamma_i \Gamma(n_i) T^{n_i - \beta_j - 1}}{\Gamma(n_i - \beta_j) M(m_i, n_i, \gamma_i)} \|I^{m_i + \alpha_j - 1}(\varphi_{l,i,k})\|_{L^1} \right] \|\mathbf{u} - \mathbf{v}\|. \end{aligned}$$

Then

$$\|{}^c D_{0^+}^{\beta_j} P_i(\mathbf{u}) - {}^c D_{0^+}^{\beta_j} P_i(\mathbf{v})\|_{\infty} \leq A_{i,j}(P) \|\mathbf{u} - \mathbf{v}\|, \tag{42}$$

where $A_{i,j}(P)$ is given in (39). From (42) we deduce that

$$\|\mathbf{P}(\mathbf{u}) - \mathbf{P}(\mathbf{v})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} A_{i,j}(P) \|\mathbf{u} - \mathbf{v}\|. \tag{43}$$

From (41) and (43) we deduce that

$$\|\mathbf{T}(\mathbf{u}) - \mathbf{T}(\mathbf{v})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} [A_{i,j}(P) + A_{i,j}(Q)] \|\mathbf{u} - \mathbf{v}\|. \tag{44}$$

Since

$$\sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} [A_{i,j}(P) + A_{i,j}(Q)] < 1. \tag{45}$$

Then \mathbf{T} is contraction, hence it has a unique fixed point which is the unique solution of (1)–(2). The proof is complete. \square

We establish an existence result using the nonlinear alternative of Learay–Schauder type.

LEMMA 11. (Leray–Schauder nonlinear alternative [30]) *Let F be a Banach space and Ω be a bounded open subset of F , $0 \in \Omega$. $T : \overline{\Omega} \rightarrow F$ be a completely continuous operator. Then there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

From nonlinear alternative of Leray–Schauder type we have the following result.

THEOREM 1. *Let $i \in \{1, \dots, p\}$ and $l \in \{1, \dots, p\}$ assume that the functions $f_i \in C([0, T] \times \mathbb{R}^{(n_i-1) \times p}, \mathbb{R})$ and $h_{i,k} \in C([0, T] \times \mathbb{R}^p, \mathbb{R})$, $k \in \{0, \dots, n_i - 1\}$ are continuous such that there exist nonnegative functions $\phi_{l,i,k}^h \in L^1([0, T], \mathbb{R}_+)$, $k \in \{0, \dots, n_i - 1\}$, $\phi_{l,i,j}^f \in L^2([0, T], \mathbb{R}_+)$, $j \in \{0, \dots, n_i - 2\}$, $\theta_i^f \in L^2([0, T], \mathbb{R}_+)$ and nondecreasing $\psi_{i,k}^h, \psi_i^f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $k \in \{0, \dots, n_i - 1\}$ and there exist $r > 0$ such that for all $x_k = (x_{1,k}, \dots, x_{p,k}) \in \mathbb{R}^p$, $k \in \{0, \dots, n_i - 2\}$ for all $t \in [0, T]$*

$$|f_i(t, x_0, \dots, x_{n_i-2})| \leq \sum_{l=1}^p \sum_{j=0}^{n_i-2} \phi_{l,i,j}^f(t) \psi_i^f(|x_{l,j}|) + \theta_i^f(t), \tag{46}$$

$$|h_{i,k}(t, x_0)| \leq \sum_{l=1}^p \phi_{l,i,k}^h(t) \psi_{i,k}^h(|x_{l,0}|), \tag{47}$$

$$\sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} [B_{i,j}(Q)(r) + B_{i,j}(P)(r)] < r. \tag{48}$$

where

$$B_{i,j}(P)(r) = \|G_{j,i}\|_{L^2} \|\theta_i^f\|_{L^2} + \psi_i^f(r) \sum_{l=1}^p \sum_{j=0}^{n_i-2} \|\phi_{l,i,j}^f\|_{L^2} \|G_{j,i}\|_{L^2} \tag{49}$$

and

$$\begin{aligned} B_{i,j}(Q)(r) = & \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \psi_{i,k}^h(r) \sum_{l=1}^p \|\phi_{l,i,k}^h\|_{L^1} \right. \\ & + \frac{1}{M(m_i, n_i, \gamma_i)} \psi_{i,n_i-1}^h(r) \sum_{l=1}^p \|\phi_{l,i,n_i-1}^h\|_{L^1} \left. \right] \frac{\Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(\alpha_i - \beta_j)} \\ & + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k}}{\Gamma(k-\beta_k+1)} \psi_{i,k}^h(r) \sum_{l=1}^p \|\phi_{l,i,k}^h\|_{L^1}. \end{aligned} \tag{50}$$

Then the boundary value problem (1)–(2) has at least one nontrivial solution $\mathbf{u}^* \in E$.

Proof. The proof will be done in some steps. First let us prove that \mathbf{T} is complete continuous.

Step 1. It is easy to see that \mathbf{T} is continuous since f_i , $h_{i,k}$ and G_i are continuous. Let $B_\eta = \{\mathbf{u} \in E; \|\mathbf{u}\| \leq \eta\}$ be a bounded subset in E . We shall prove that $\mathbf{T}(B_\eta)$ is relatively compact.

Step 2. For $\mathbf{u} \in B_\eta$ and using Lemma 9 we get

$$\begin{aligned} |{}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t)| &\leq \int_0^T G_{j,i}(s) \theta_i^f(s) ds + \int_0^T G_{j,i}(s) \sum_{l=1}^p \sum_{j=0}^{n_i-2} \varphi_{l,i,j}^f(s) \psi_i^f(|{}^c D_{0^+}^{\beta_j} u_l(s)|) ds \\ &\leq \int_0^T G_{j,i}(s) \theta_i^f(s) ds + \psi_i^f(\|\mathbf{u}\|) \sum_{l=1}^p \sum_{j=0}^{n_i-2} \int_0^T \varphi_{l,i,j}^f(s) G_{j,i}(s) ds \\ &\leq \int_0^T G_{j,i}(s) \theta_i^f(s) ds + \psi_i^f(\eta) \sum_{l=1}^p \sum_{j=0}^{n_i-2} \int_0^T \varphi_{l,i,j}^f(s) G_{j,i}(s) ds. \end{aligned} \quad (51)$$

Using Cauchy–Schwarz inequality we have for all $i \in \{1, \dots, p\}$, for all $t \in [0, T]$

$$|{}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t)| \leq \|G_{j,i}\|_{L^2} \|\theta_i^f\|_{L^2} + \psi_i^f(\eta) \sum_{l=1}^p \sum_{j=0}^{n_i-2} \|\varphi_{l,i,j}^f\|_{L^2} \|G_{j,i}\|_{L^2}. \quad (52)$$

Then for all $i \in \{1, \dots, p\}$

$$\|{}^c D_{0^+}^{\beta_j} P_i(\mathbf{u})\|_\infty \leq B_{i,j}(P)(\eta). \quad (53)$$

We deduce that

$$\|\mathbf{P}(\mathbf{u})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} B_{i,j}(P)(\eta) \quad (54)$$

Using a similar technique, we get

$$\begin{aligned} |{}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t)| &\leq \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \int_0^T \sum_{l=1}^p \varphi_{l,i,k}^h(t) \psi_{i,k}^h(|u_l(s)|) ds \right. \\ &\quad \left. + \frac{1}{M(m_i, n_i, \gamma_i)} \int_0^T \sum_{l=1}^p \varphi_{l,i,k}^h(t) \psi_{i,k}^h(|u_l(s)|) ds \right] {}^c D_{0^+}^{\beta_j} t^{n_i-1} \\ &\quad + \sum_{k=0}^{n_i-2} \frac{1}{\Gamma(k-\beta_j+1)} \int_0^T \sum_{l=1}^p \varphi_{l,i,k}^h(s) \psi_{i,k}^h(|u_l(s)|) ds \\ &\leq \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+m_i}}{\Gamma(k+m_i+1)} \psi_{i,k}^h(\eta) \sum_{l=1}^p \|\varphi_{l,i,k}^h\|_{L^1} \right. \\ &\quad \left. + \frac{1}{M(m_i, n_i, \gamma_i)} \times \psi_{i,n_i-1}^h(\eta) \sum_{l=1}^p \|\varphi_{l,i,n_i-1}^h\|_{L^1} \right] \frac{\Gamma(n_i) T^{n_i-\beta_j-1}}{\Gamma(\alpha_i - \beta_j)} \\ &\quad + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k}}{\Gamma(k-\beta_k+1)} \psi_{i,k}^h(\eta) \sum_{l=1}^p \|\varphi_{l,i,k}^h\|_{L^1}. \end{aligned} \quad (55)$$

Then

$$\|{}^c D_{0^+}^{\beta_j} Q_i(\mathbf{u})\|_\infty \leq B_{i,j}(Q)(\eta). \quad (56)$$

We deduce that

$$\|\mathbf{Q}(\mathbf{u})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} B_{i,j}(\mathbf{Q})(\eta). \tag{57}$$

Consequently

$$\|\mathbf{T}(\mathbf{u})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} B_{i,j}(\mathbf{Q})(\eta) + \max_{j \in \{0, \dots, n_i-2\}} B_{i,j}(\mathbf{P})(\eta) \tag{58}$$

hence $\mathbf{T}(B_\eta)$ is uniformly bounded.

Step 3 In the end we show that $T(B_\eta)$ is equicontinuous. In fact, we denote by:

$$C(f_i) = \max \left\{ |f_i(t, \mathbf{u}(t), {}^c D_{0+}^{\beta_1} \mathbf{u}(t), \dots, D_{0+}^{\beta_{n_i-2}} \mathbf{u}(t))|, 0 \leq t \leq T, \|\mathbf{u}\| \leq \eta \right\} \tag{59}$$

and

$$C(h_{i,k}) = \max \left\{ |h_{i,k}(t, \mathbf{u}(t))|, 0 \leq t \leq T, \|\mathbf{u}\| \leq \eta \right\}. \tag{60}$$

Let $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$ and $\mathbf{u} \in B_\eta$ we have

$$\begin{aligned} & |{}^c D_{t,0+}^{\beta_j} P_i(\mathbf{u})(t_1) - {}^c D_{t,0+}^{\beta_j} P_i(\mathbf{u})(t_2)| \\ & \leq C(f_i) \left[\int_0^{t_1} |{}^c D_{t,0+}^{\beta_j} G_i(t_1, s) - {}^c D_{t,0+}^{\beta_j} G_i(t_2, s)| ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} |{}^c D_{t,0+}^{\beta_j} G_i(t_1, s) - {}^c D_{t,0+}^{\beta_j} G_i(t_2, s)| ds + \int_{t_2}^T |{}^c D_{t,0+}^{\beta_j} G_i(t_1, s) - {}^c D_{t,0+}^{\beta_j} G_i(t_2, s)| ds \right] \\ & \leq C(f_i) \left[\int_0^{t_1} \left[\frac{(t_2-s)^{\alpha_i-\beta_j-1}}{\Gamma(\alpha_i-\beta_j)} - \frac{(t_1-s)^{\alpha_i-\beta_j-1}}{\Gamma(\alpha_i-\beta_j)} \right. \right. \\ & \quad \left. \left. + (t_2^{n_i-\beta_j-1} - t_1^{n_i-\beta_j-1}) \frac{\gamma_i \Gamma(n_i)(T-s)^{m_i+\alpha_i-1}}{\Gamma(n_i-\beta_j)M(m_i, n_i, \gamma_i)\Gamma(m_i+\alpha_i)} \right] ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} \left[(t_2^{n_i-\beta_j-1} - t_1^{n_i-\beta_j-1}) \frac{\gamma_i \Gamma(n_i)(T-s)^{m_i+\alpha_i-1}}{\Gamma(\alpha_i-\beta_j)M(m_i, n_i, \gamma_i)\Gamma(m_i+\alpha_i)} + \frac{(t_2-s)^{\alpha_i-\beta_j-1}}{\Gamma(\alpha_i-\beta_j)} \right] ds \right. \\ & \quad \left. + \int_{t_2}^T (t_2^{n_i-\beta_j-1} - t_1^{n_i-\beta_j-1}) \frac{\gamma_i \Gamma(n_i)(T-s)^{m_i+\alpha_i-1}}{\Gamma(n_i-\beta_j)M(m_i, n_i, \gamma_i)\Gamma(m_i+\alpha_i)} ds \right] \\ & \leq C(f_i) \left[C_1(t_2^{\alpha_i-\beta_j} - t_1^{\alpha_i-\beta_j}) + 3C_2(t_2^{n_i-\beta_j-1} - t_1^{n_i-\beta_j-1}) \right] \tag{61} \end{aligned}$$

where

$$C_1 = \frac{1}{\Gamma(\alpha_i - \beta_j + 1)} \tag{62}$$

and

$$C_2 = \frac{\gamma_i \Gamma(n_i)}{\Gamma(n_i - \beta_j)M(m_i, n_i, \gamma_i)\Gamma(m_i + \alpha_i)} \int_0^T (T-s)^{m_i+\alpha_i-1} ds. \tag{63}$$

Similarly, we have

$$\begin{aligned}
 & |{}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t_2) - {}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t_1)| \\
 & \leq \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{C(h_{i,k})}{\Gamma(k + m_i + 1)} + \frac{C(h_{i,n_i-1})}{M(m_i, n_i, \gamma_i)} \right] \frac{\Gamma(n_i)}{\Gamma(n_i - \beta_j)} \\
 & \quad \times (t_2^{n_i-\beta_j-1} - t_1^{n_i-\beta_j-1}) + \sum_{k=0}^{n_i-2} \frac{(t_2^{k-\beta_k} - t_1^{k-\beta_k})C(h_{i,k})}{\Gamma(k - \beta_k + 1)}. \tag{64}
 \end{aligned}$$

As $t_1 \rightarrow t_2$ then $|{}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t_2) - {}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t_1)|$ and $|{}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t_2) - {}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t_1)|$ tend to 0, consequently $T(B_\eta)$ is equicontinuous. By means of Arzela-Ascoli theorem [30] we deduce that \mathbf{T} is complete continuous. Now, we apply Leray-Schauder non-linear alternative to prove that \mathbf{T} has at least a nontrivial solution in E . We denote by $\Omega = \{\mathbf{u} \in E; \|\mathbf{u}\| < r\}$. Then for $\mathbf{u} \in \partial\Omega$, such that $\mathbf{u} = \lambda \mathbf{T}(\mathbf{u})$, $0 < \lambda < 1$, we have

$$\|\mathbf{u}\| = \lambda \|\mathbf{T}(\mathbf{u})\| < \|\mathbf{T}(\mathbf{u})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} B_{j,i}(Q)(r) + \max_{j \in \{0, \dots, n_i-2\}} B_{j,i}(P)(r).$$

Then

$$\|\mathbf{u}\| < \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} [B_{j,i}(Q)(r) + B_{j,i}(P)(r)] < r \tag{65}$$

which is a contradiction to the fact that $\mathbf{u} \in \partial\Omega$. Theorem 1 allows us to conclude that \mathbf{T} has a fixed point $\mathbf{u}^* \in \overline{\Omega}$, and then problem (1)–(2) has a nontrivial solution $\mathbf{u}^* \in E$. This achieves the proof. \square

3. Existence of positive solutions

In this section, we will give some preliminary considerations and some lemmas which are essential to establish a sufficient conditions for the existence of at least one positive solutions for our problem. We make the following additional assumption.

- (H1) For all $i \in \{1, \dots, p\}$, $0 \leq k \leq n_i - 1$, the functions $f_i : [0, T] \times \mathbb{R}_+^{(n_i-1) \times p} \rightarrow \mathbb{R}_+$, $h_{i,k} : [0, T] \times \mathbb{R}_+^p \rightarrow \mathbb{R}_+$ are continuous.
- (H2) For all $i \in \{1, \dots, p\}$ there exist $[\sigma_i, \tau_i] \subset (0, T)$ and $m(f_i) > 0$ such that $f_i(t, u) \geq m(f_i)$ for all $t \in [\sigma_i, \tau_i]$, $u \in \mathbb{R}_+^{(n_i-1) \times p}$.
- (H3) For all $i \in \{1, \dots, p\}$, $0 \leq k \leq n_i - 1$ there exist $M(f_i) > 0$ and $M(h_{i,k}) > 0$ such that $f_i(t, u) \leq M(f_i)$, and $h_{i,k}(t, u) \leq M(h_{i,k})$ for all $u \in \mathbb{R}_+^p$.

THEOREM 2. (Guo-Krasnosel’skii fixed point theorem [31]) *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 be two bounded open subsets in E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that:*

1. $\|A(u)\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|A(u)\| \geq \|u\|, u \in K \cap \partial\Omega_2$ or
2. $\|A(u)\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|A(u)\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

REMARK 1.

$$\mathbf{C} = \{ \mathbf{u} = (u_1, \dots, u_p) \in E, u_i(t) \geq 0, t \in [0, T], i \in \{1, \dots, p\} \}$$

is a cone of E .

We employ Guo–Krasnoselskii’s fixed point theorem in cone to prove the existence of positive solutions of our problem, we have the following theorem.

THEOREM 3. Assume that the conditions (H1), (H2) and (H3) hold. Then the equation (1)–(2) has at least one positive solution.

Proof. Remark 1 shows that \mathbf{C} is a cone subset of E . Lemma 9 and (H1) show that $T : \mathbf{C} \rightarrow \mathbf{C}$. In addition, a standard argument involving the Arzela-Ascoli theorem [30] implies that \mathbf{T} is a completely continuous operator. For all $i \in \{1, \dots, p\}$

$$\begin{aligned} T_i(\mathbf{u})(\tau_i) &\geq \int_0^{\tau_i} \frac{(\tau_i - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(s, \mathbf{u}(s), {}^c D_{0^+}^{\beta_1} \mathbf{u}(s), \dots, {}^c D_{0^+}^{\beta_{n_i-2}} \mathbf{u}(s)) ds \\ &\geq m(f_i) \int_{\sigma_i}^{\tau_i} \frac{(\tau_i - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} ds = m(f_i) \frac{(\tau_i - \sigma_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}. \end{aligned} \tag{66}$$

Hence, for all $i \in \{1, \dots, p\}$

$$\|T_i(\mathbf{u})\| \geq m(f_i) \frac{(\tau_i - \sigma_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

Thus

$$\|\mathbf{T}(\mathbf{u})\| \geq \sum_{i=1}^p m(f_i) \frac{(\tau_i - \sigma_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

Now, we choose a positive constant R_1 such that $R_1 \leq \sum_{i=1}^p m(f_i) \frac{(\tau_i - \sigma_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}$ and define $\Omega_1 = \{ \mathbf{u} \in E : \|\mathbf{u}\| < R_1 \}$. For any $\mathbf{u} \in \mathbf{C} \cap \partial\Omega_1$, we find that

$$\|\mathbf{T}(\mathbf{u})\| \geq \|\mathbf{u}\|. \tag{67}$$

Now, we prove the second inequality. For all $i \in \{1, \dots, p\}, j \in \{0, \dots, n_i - 2\}, t \in [0, T], \mathbf{u} \in \mathbf{C}$

$$|{}^c D_{t,0^+}^{\beta_j} P_i(\mathbf{u})(t)| \leq M(f_i) \|G_{j,i}\|_{L^1}.$$

We obtain

$$\|\mathbf{P}(\mathbf{u})\| \leq \sum_{i=1}^p M(f_i) \max_{j \in \{0, \dots, n_i-2\}} \|G_{j,i}\|_{L^1}.$$

On the other hand, for all $i \in \{1, \dots, p\}$, $j \in \{0, \dots, n_i-2\}$, $t \in [0, T]$, $\mathbf{u} \in \mathbf{C}$

$$\begin{aligned} |{}^c D_{t,0^+}^{\beta_j} Q_i(\mathbf{u})(t)| &\leq \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+1+m_i} M(h_{i,k})}{\Gamma(k+m_i+1)} + \frac{M(h_{i,n_i-1})}{M(m_i, n_i, \gamma_i)} \right] \\ &\times \frac{\Gamma(n_i) T^{n_i-1}}{\Gamma(\alpha_i - \beta_j)} + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k} M(h_{i,k})}{\Gamma(k-\beta_k+1)}. \end{aligned}$$

We obtain

$$\|\mathbf{Q}(\mathbf{u})\| \leq \sum_{i=1}^p \max_{j \in \{0, \dots, n_i-2\}} C_{j,i}$$

where

$$\begin{aligned} C_{j,i} &= \left[\frac{\gamma_i}{M(m_i, n_i, \gamma_i)} \sum_{k=0}^{n_i-2} \frac{T^{k+1+m_i} M(h_{i,k})}{\Gamma(k+m_i+1)} + \frac{M(h_{i,n_i-1})}{M(m_i, n_i, \gamma_i)} \right] \\ &\times \frac{\Gamma(n_i) T^{n_i-1}}{\Gamma(\alpha_i - \beta_j)} + \sum_{k=0}^{n_i-2} \frac{T^{k-\beta_k} M(h_{i,k})}{\Gamma(k-\beta_k+1)}. \end{aligned} \tag{68}$$

we deduce that

$$\|\mathbf{T}(\mathbf{u})\| \leq \sum_{i=1}^p M(f_i) \max_{j \in \{0, \dots, n_i-2\}} \{ \|G_{j,i}\|_{L^1} + C_{j,i} \}.$$

Let

$$R_2 = \max \left\{ \sum_{i=1}^p M(f_i) \max_{j \in \{0, \dots, n_i-2\}} \{ \|G_{j,i}\|_{L^1} + C_{j,i} \}, 2R_1 \right\}$$

and we define $\Omega_2 = \{\mathbf{u} \in E : \|\mathbf{u}\| < R_2\}$. Clearly, $\overline{\Omega_1} \subset \Omega_2$ and for any $\mathbf{u} \in \mathbf{C} \cap \partial\Omega_2$, we obtain $\|\mathbf{T}(\mathbf{u})\| \leq R_2 = \|\mathbf{u}\|$. Thus, for any $\mathbf{u} \in \mathbf{C} \cap \partial\Omega_2$, it implies that

$$\|\mathbf{T}(\mathbf{u})\| \leq \|\mathbf{u}\|. \tag{69}$$

Based on Theorem 2, we get from (67) and (69) that the operator \mathbf{T} has at least one fixed point. Thus, it follows (1)–(2) has at least one nonnegative solution and from (H1) and (H2), (1)–(2) has at least one positive solution. \square

EXAMPLE 1. Consider the following system of boundary value problem.

$$\left\{ \begin{aligned} & {}^c D_{0+}^{\frac{5}{4}} u_1(t) = \frac{1}{2} \cos \left[\frac{(1+t)^2}{60000} u_1(t) + \frac{(1+t)^3}{60000} u_2(t) \right] \\ & \quad + \frac{1}{2} \sin \left[\frac{(1+t)^2}{60000} u_1(t) + \frac{(1+t)^3}{60000} u_2(t) \right] + e^t, \quad 0 < t < 1 \\ & {}^c D_{0+}^{\frac{5}{2}} u_2(t) = \frac{1}{2} \cos \left[\frac{(1+t)^3}{60000} u_1(t) + \frac{(1+t)^4}{60000} u_2(t) \right] \\ & \quad + \frac{1}{2} \sin \left[\frac{(1+t)^4}{60000} {}^c D_{0+}^{\frac{1}{6}} u_1(t) + \frac{(1+t)^5}{60000} {}^c D_{0+}^{\frac{1}{6}} u_2(s) \right] \\ & u_1(0) = \int_0^1 \frac{s}{1 + \frac{|u_1(s)|}{20000(s+1)} + \frac{|u_2(s)|}{20000(s+2)}} ds \\ & u_1'(0) = \frac{15\sqrt{\pi}}{32} I_{0+}^{\frac{3}{4}} u_1(1) + \int_0^1 \frac{s}{1 + \frac{|u_1(s)|}{20000(s+1)^2} + \frac{|u_2(s)|}{20000(s+2)^2}} ds \\ & u_2(0) = \int_0^1 \frac{s}{1 + \frac{|u_1(s)|}{20000(s+1)^2} + \frac{|u_2(s)|}{20000(s+2)^2}} ds \\ & u_2'(0) = \int_0^1 \frac{|\sin[\frac{u_1(s)}{20000(s+1)^3}]| + |\cos[\frac{u_2(s)}{20000(s+2)^3}]|}{\sqrt{1 + u_1(s)^2 + |u_2(s)|}} ds \\ & u_2''(0) = \frac{15\sqrt{\pi}}{16} I_{0+}^{\frac{1}{2}} u_2(1) + \int_0^1 \frac{s}{1 + \frac{|u_1(s)|}{20000(s+1)^4} + \frac{|u_2(s)|}{20000(s+2)^4}} ds. \end{aligned} \right. \tag{70}$$

The following notations can be easily specified as $x_0 = (x_{1,0}, x_{2,0})$, $y_0 = (y_{1,0}, y_{2,0})$, $x_1 = (x_{1,1}, x_{2,1})$, $y_1 = (y_{1,1}, y_{2,1})$, $T = 1$, $p = 2$, $n_1 = 2$, $n_2 = 3$, $\alpha_1 = \frac{5}{4}$, $\alpha_2 = \frac{5}{2}$, $\beta_0 = 0$, $\beta_1 = \frac{1}{6}$, $m_1 = \frac{3}{4}$, $m_2 = \frac{1}{2}$, $\gamma_1 = \frac{15\sqrt{\pi}}{32}$, $\gamma_2 = \frac{15\sqrt{\pi}}{16}$. We denote by

$$\begin{aligned} f_1(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) &= \frac{1}{2} \cos \left[\frac{(1+t)^2}{60000} x_{1,0} + \frac{(1+t)^3}{60000} x_{2,0} \right] \\ & \quad + \frac{1}{2} \sin \left[\frac{(1+t)^2}{60000} x_{1,0} + \frac{(1+t)^3}{60000} x_{2,0} \right] + e^t \end{aligned}$$

$$\begin{aligned} f_2(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) &= \frac{1}{2} \cos \left[\frac{(1+t)^3}{60000} x_{1,0} + \frac{(1+t)^4}{60000} x_{2,0} \right] \\ & \quad + \frac{1}{2} \sin \left[\frac{(1+t)^4}{60000} x_{1,1} + \frac{(1+t)^5}{60000} x_{2,1} \right], \end{aligned}$$

$$h_{1,0}(t, x_{1,0}, x_{2,0}) = \frac{t}{1 + \frac{|x_{1,0}|}{20000(t+1)} + \frac{|x_{2,0}|}{20000(t+2)}}$$

$$h_{1,1}(t, x_{1,0}, x_{2,0}) = \frac{t}{1 + \frac{|x_{1,0}|}{20000(t+1)^2} + \frac{|x_{2,0}|}{20000(t+2)^3}}$$

$$h_{2,0}(t, x_{1,0}, x_{2,0}) = \frac{t}{1 + \frac{|x_{1,0}|}{20000(t+1)^2} + \frac{|x_{2,0}|}{20000(t+2)^2}},$$

$$h_{2,1}(t, x_{1,0}, x_{2,0}) = \frac{|\sin[\frac{x_{1,0}}{20000(t+1)^3}]| + |\cos[\frac{x_{2,0}}{20000(t+2)^3}]|}{\sqrt{1 + x_{1,0}^2 + |x_{2,0}|}},$$

$$h_{2,2}(t, x_{1,0}, x_{2,0}) = \frac{t}{1 + \frac{|x_{1,0}|}{20000(t+1)^4} + \frac{|x_{2,0}|}{20000(t+2)^4}},$$

$$\varphi_{1,1,0}(t) = \frac{(1+t)^2}{60000}, \quad \varphi_{1,2,0}(t) = \frac{(1+t)^3}{60000}, \quad \varphi_{2,1,0}(t) = \frac{(1+t)^3}{60000},$$

$$\varphi_{2,2,0}(t) = \frac{(1+t)^4}{60000}, \quad \varphi_{2,1,1}(t) = \frac{(1+t)^4}{60000} \quad \text{and} \quad \varphi_{2,2,1}(t) = \frac{(1+t)^5}{60000}.$$

We have

$$\psi_{1,1,0}(t) = \frac{1}{20000(t+1)}, \quad \psi_{2,1,0}(t) = \frac{1}{20000(t+2)}, \quad \psi_{1,2,0}(t) = \frac{1}{20000(t+1)^2},$$

$$\psi_{2,2,0}(t) = \frac{1}{20000(t+2)^2}, \quad \psi_{1,2,1}(t) = \frac{1}{20000(t+1)^3}, \quad \psi_{2,2,1}(t) = \frac{1}{20000(t+2)^3},$$

$$\psi_{1,2,1}(t) = \frac{1}{20000(t+1)^3}, \quad \psi_{2,2,1}(t) = \frac{1}{20000(t+2)^3}, \quad \psi_{1,2,2}(t) = \frac{1}{20000(t+1)^4}$$

and

$$\psi_{2,2,2}(t) = \frac{1}{20000(t+2)^4}.$$

We have

$$|f_i(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) - f_1(t, y_{1,0}, y_{2,0}, y_{1,1}, y_{2,1})| \leq \sum_{k=0}^0 \sum_{l=1}^2 \varphi_{l,1,k}(t) |x_{l,k} - y_{l,k}|$$

$$|f_2(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) - f_2(t, y_{1,0}, y_{2,0}, y_{1,1}, y_{2,1})| \leq \sum_{k=0}^1 \sum_{l=1}^2 \varphi_{l,2,k}(t) |x_{l,k} - y_{l,k}|$$

$$|h_{i,k}(t, x_0) - h_{i,k}(t, y_0)| \leq \sum_{l=1}^2 \psi_{l,i,k}(t) |x_{l,0} - y_{l,0}|$$

$$A_{1,0}(Q) = 0.000131418, A_{2,0}(Q) = 0.000159482, A_{2,1}(Q) = 0.000176465, A_{1,0}(P) = 0.000100849, A_{2,0}(P) = 0.000371124, A_{2,1}(P) = 0.000147162.$$

$$\sum_{i=1}^2 \max_{j \in \{0, \dots, n_i-2\}} [A_{i,j}(P) + A_{i,j}(Q)] = 0.000779857 < 1.$$

Then from Theorem 10 we conclude that the fractional differential boundary value problem (70) has a unique solution $u^* = (u_1, u_2) \in E$.

EXAMPLE 2. Consider the following system of boundary value problem.

$$\begin{cases} u^{(6)}(t) = \sum_{k=0}^4 \frac{t^2}{10} (u^{(k)}(t))^2, & 0 < t < 1 \\ u(0) = u'(0) = \dots = u^{(4)}(0) = 0, & u^{(5)}(0) = I_{0+}^5 u(1) \end{cases} \quad (71)$$

In this example $T = 1, p = 1, i = 1, n_1 = 6, \alpha_1 = 6, m_1 = 5, \gamma_1 = 1, j \in \{0, \dots, 4\}, \beta_j = j, \varphi_{1,1,j}^f(t) = \frac{t^2}{10}, \psi_1^f(x) = x^2, f_i : [0, 1] \times \mathbb{R}^5 \rightarrow \mathbb{R}$, with

$$f_1(t, x_0, \dots, x_4) = \sum_{i=0}^4 \frac{t^2}{10} x_i^2$$

and

$$|f_1(t, x_0, \dots, x_4)| \leq \sum_{j=0}^4 \frac{t^2}{10} |x_j|^2 = \sum_{j=0}^4 \varphi_{1,1,j}^f(t) \psi_1^f(|x_{1,j}|)$$

where $\varphi_{1,1,j}^f$ and ψ_1^f are nondecreasing functions on \mathbb{R}_+ $0 \leq k \leq 5$ and $h_{1,k} \equiv 0$.

$$\max_{j \in \{0, \dots, 4\}} [B_{1,j}(Q)(2) + B_{1,j}(P)(2)] = 0.893631 < 2.$$

By Theorem 1 the problem (71) has at least one nontrivial solution in E .

EXAMPLE 3. Consider the following system of boundary value problem.

$$\begin{cases} {}^c D_{0+}^{\frac{3}{2}} u_1(t) = \frac{\cos^2[u_1(t)+u_2(t)]}{\sqrt{1+t^2}} + \frac{t^2+e^t}{\sqrt[3]{1+t+t^2+t^3}}, & 0 < t < 1 \\ {}^c D_{0+}^{\frac{5}{2}} u_2(t) = t e^{\sin[u_1(t)+{}^c D_{0+}^{\frac{1}{2}} u_1(t)+u_2(t)+{}^c D_{0+}^{\frac{1}{2}} u_2(t)]} \\ u_1(0) = u_1'(0) = u_2(0) = u_2'(0) = \int_0^1 \frac{ds}{1+|u_1(s)|+|u_2(s)|} \\ u_1''(0) = \frac{3\sqrt{\pi}}{8} I_{0+}^{\frac{3}{2}} u_1(1) + \int_0^1 \frac{s ds}{1+|u_1(s)|+|u_2(s)|} \\ u_2''(0) = \frac{3\sqrt{\pi}}{8} I_{0+}^{\frac{3}{2}} u_2(1) + \int_0^1 \frac{|\sin[u_1(s)]|}{\sqrt{1+u_1^2(s)+|u_2(s)|}} ds. \end{cases} \quad (72)$$

In this case we have $x_0 = (x_{1,0}, x_{2,0}), x_1 = (x_{1,1}, x_{2,1}), T = 1, p = 2, n_1 = 2, n_2 = 3, \alpha_1 = \frac{3}{2}, \alpha_2 = \frac{5}{2}, \beta_{n_1-2} = \beta_0 = 0, \beta_{n_2-2} = \beta_1 = \frac{1}{2}, m_1 = m_2 = \frac{3}{2}, \gamma_1 = \gamma_2 = \frac{3\sqrt{\pi}}{8}$,

$$f_1(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) = \frac{\cos^2[x_{1,0}+x_{2,0}]}{\sqrt{1+t^2}} + \frac{t^2+e^t}{\sqrt[3]{1+t+t^2+t^3}},$$

$$f_2(t, x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1}) = t e^{\sin[x_{1,0}+x_{1,1}+x_{2,0}+x_{2,1}]},$$

$$h_{1,0}(t, x_{1,0}, x_{2,0}) = h_{1,1}(t, x_{1,0}, x_{2,0}) = h_{2,0}(t, x_{1,0}, x_{2,0}) = h_{2,1}(t, x_{1,0}, x_{2,0}) = \frac{1}{1 + |x_{1,0}| + |x_{2,0}|},$$

$$h_{1,2}(t, x_{1,0}, x_{2,0}) = \frac{t}{1 + |x_{1,0}| + |x_{2,0}|},$$

and

$$h_{2,2}(t, x_{1,0}, x_{2,0}) = \frac{|\sin[x_{1,0}]|}{\sqrt{1 + x_{1,0}^2 + |x_{2,0}|}}.$$

Hence, from Theorem 3 we conclude that the problem (72) has at least one positive solution.

REFERENCES

- [1] S. G. SAMKO, A. A. KILBAS, AND O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland, 1993.
- [2] S. ABBAS, M. BENCHOHRA, AND G. M. N'GUÉRÉKATA, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [3] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, Longman, Ser Pitman Press Note in Math., Harlow, 1994.
- [4] R. L. BAGLEY, *A theoretical for the application of fractional calculus to viscoelasticity*, J. of Rheol. **27**, 201–210 (1983).
- [5] R. HILFER, *Application of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
- [6] R. L. MAGIN, *Fractional calculus in bioengineering*, Crit. Rev. Biomed. Eng. **32**, 1–104 (2004).
- [7] A. GUEZANE-LAKOUD, R. KHALDI, *Solvability of a fractional boundary value problem with fractional integral condition*, Nonlinear Anal. **75**, 2692–2700 (2012).
- [8] A. A. KILBAS, H. M. SRIVASTAVA, AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, vol. **204**, Elsevier, Amsterdam, The Netherlands, 2006.
- [9] G. A. ANASTASSIOU, *On right fractional calculus*, Chaos, Soliton. Fract. **42**, 365–376 (2009).
- [10] R. A. KHAN, M. U. REHMAN, AND J. HENDERSON, *Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions*, Fractional Differ. Calc. **1**, 29–43 (2011).
- [11] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [12] J. DENG, Z. DENG, *Existence of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett. **32**, 6–12 (2014).
- [13] A. YAKAR, M. E. KOKSAL, *Existences Results for Solutions of Nonlinear Fractional Differential Equations*, Abstract and Applied Analysis, **2012** (2012) Article Number: 267108, 1–12, (2012).
- [14] K. DIETHELM, *Smoothness properties of solutions of Caputo-type fractional differential equations*, Fract. Calc. Appl. Anal. **10**, 151–161 (2007).
- [15] V. DAFTARDAR-GEJJI, H. JAFARI, *Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives*, J. Math. Anal. Appl. **328**, 1026–1033 (2007).
- [16] A. M. A. EL-SAYED, I. L. EL-KALLAB, AND E. A. A. ZIADA, *Analytical and numerical solutions of multi-term nonlinear fractional orders differential equations*, Appl. Numer. Math. **60**, 788–797 (2010).
- [17] J. R. GRAEF, L. KONG, Q. KONG, AND M. WANG, *Positive solutions of nonlocal fractional boundary value problems*, Discrete Contin. Dyn. Syst. suppl., 283–290 (2013).
- [18] M. BENCHOHRA, F. MOSTAFAI, *Weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces*, Opuscula Math. **32**, 31–40 (2012).
- [19] Z. BAI AND H. LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311**, 495–505 (2005).
- [20] H. JAFARI, V. DAFTARDAR-GEJJI, *Positive solutions of nonlinear fractional boundary value problems using adomian decomposition method*, Appl. Math. Comput. **180**, 700–706 (2006).
- [21] A. GUEZANE-LAKOUD, D. BELAKROUM, *Rothe's method for telegraph equation with integral conditions*, Nonlinear Anal. **70**, 3842–3853 (2009).

- [22] M. JLELI AND B. SAMET, *Existence of positive solutions to a coupled system of fractional differential equations*, *Math. Method. Appl. Sci.* **38**, 1014–1031 (2015).
- [23] E. R. KAUFMANN, *Existence and Nonexistence of positive solutions for a nonlinear fractional boundary value problem*, *Discrete Contin. Dyn. Syst. suppl.*, 416–423 (2009).
- [24] Z. B. BAI, T. T. QIU, *Existence of positive solution for singular fractional differential equation*, *Appl. Math. Comput* **215**, 2761–2767 (2009).
- [25] C. F. LI, X. N. LUO, AND Y. ZHOU, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, *Comput. Math. Appl.* **59**, 1363–1375 (2010).
- [26] Y. Q. WANG, L. S. LIU, AND Y. H. WU, *Positive solutions for a nonlocal fractional differential equation*, *Nonlinear Anal.* **74**, 3599–3605 (2011).
- [27] R. P. AGARWAL, D. O'REGAN, AND S. STANEK, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, *J. Math. Anal. Appl.* **371**, 57–68 (2010).
- [28] B. AHMAD, J. J. NIETO, A. ALSAEDI, AND N. MOHAMAD, *On a new class of antiperiodic fractional boundary value problems*, *Abstr. Appl. Anal. Art. ID 606454* **2013**, page 7 (2013).
- [29] G. WANG, R. P. AGARWAL, AND A. CABADA, *Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations*, *Appl. Math. Lett.* **25**, 1019–1024 (2012).
- [30] K. DEIMLING, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [31] D. GUO, V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

(Received November 14, 2015)

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