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INEQUALITIES OF JENSEN'S TYPE FOR GENERALIZED k–g–FRACTIONAL INTEGRALS OF FUNCTION f FOR WHICH THE COMPOSITE $f\circ g^{-1}$ IS CONVEX

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Abstract. In this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of function f for which the composite function $f\circ g^{-1}$ is convex. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ are also given. Applications for Hadamard fractional integrals are provided as well.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(t)\,dt\leqslant\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [27], the recent survey paper [19] and the references therein.

In order to extend these type of inequalities for general fractional integrals we need the following preparations.

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds & \text{if } 0<t,\\ 0 & \text{if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geqslant1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

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Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.2}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.3}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt \\ &=: I^{\alpha}_{a+,g}f(x),\ a<x\leqslant b \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} S_{k,g,b-}f(x)&=\frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1}g'(t)\,f(t)\,dt \\ &=: I^{\alpha}_{b-,g}f(x),\ a\leqslant x<b, \end{aligned} \tag{1.5}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [30, p. 100].

For $g(t)=t$ in (1.5) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [30, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x\leqslant b \tag{1.6}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x<b. \tag{1.7}$$

One can consider the function $g(t)=-t^{-1}$ and define the *Harmonic fractional integrals* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\leqslant a<x\leqslant b \tag{1.8}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\leqslant a<x<b. \tag{1.9}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *β-Exponential fractional integrals*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.10}$$

for $a < x \leqslant b$ and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{1.11}$$

for $a \leqslant x < b$.

If we take $g(t) = t$ in (1.2) and (1.3), then we can consider the following k-*fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt, \; x \in (a,b] \tag{1.12}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt, \; x \in [a,b). \tag{1.13}$$

In [33], Raina studied a class of functions defined formally by

$$\mathscr{F}_{\rho,\lambda}^{\sigma}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \; |x| < R, \text{ with } R > 0 \tag{1.14}$$

for $\rho,\ \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.14), Raina defined the following left-sided fractional integral operator

$$\mathscr{J}_{\rho,\lambda,a+;w}^{\sigma} f(x) := \int_a^x (x-t)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^{\sigma}\left(w(x-t)^{\rho}\right) f(t)\, dt, \; x > a \tag{1.15}$$

where $\rho,\ \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathscr{J}_{\rho,\lambda,b-;w}^{\sigma} f(x) := \int_x^b (t-x)^{\lambda-1} \mathscr{F}_{\rho,\lambda}^{\sigma}\left(w(t-x)^{\rho}\right) f(t)\, dt, \; x < b \tag{1.16}$$

where $\rho,\ \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathscr{F}_{\rho,\lambda}^{\sigma}(wt^{\rho})$ we re-obtain the definitions of (1.15) and (1.16) from (1.12) and (1.13).

In [31], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathscr{T}_{a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\, dt, \; x > a \tag{1.17}$$

and

$$\mathscr{T}_{b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\, dt, \; x < b \tag{1.18}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.17) and (1.18) from (1.12) and (1.13).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}_{g,a+}^{\alpha}f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a \tag{1.19}$$

and

$$\mathcal{T}_{g,b-}^{\alpha}f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b \tag{1.20}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}_{g,a+}^{\alpha}f(x) := \int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt, \tag{1.21}$$

for $0<a<x\leqslant b$ and

$$\mathcal{L}_{g,b-}^{\alpha}f(x) := \int_x^b (g(t)-g(x))^{\alpha-1}\ln(g(t)-g(x))\,g'(t)f(t)\,dt, \tag{1.22}$$

for $0<a\leqslant x<b$, where $\alpha>0$. These are obtained from (1.12) and (1.13) for the kernel $k(t)=t^{\alpha-1}\ln t$, $t>0$.

For $\alpha=1$ we get

$$\mathcal{L}_{g,a+}f(x) := \int_a^x \ln(g(x)-g(t))\,g'(t)f(t)\,dt,\ 0<a<x\leqslant b \tag{1.23}$$

and

$$\mathcal{L}_{g,b-}f(x) := \int_x^b \ln(g(t)-g(x))\,g'(t)f(t)\,dt,\ 0<a\leqslant x<b. \tag{1.24}$$

For $g(t)=t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha}f(x) := \int_a^x (x-t)^{\alpha-1}\ln(x-t)f(t)\,dt,\ 0<a<x\leqslant b, \tag{1.25}$$

$$\mathcal{L}_{b-}^{\alpha}f(x) := \int_x^b (t-x)^{\alpha-1}\ln(t-x)f(t)\,dt,\ 0<a\leqslant x<b, \tag{1.26}$$

$$\mathcal{L}_{a+}f(x) := \int_a^x \ln(x-t)f(t)\,dt,\ 0<a<x\leqslant b \tag{1.27}$$

and

$$\mathcal{L}_{b-}f(x) := \int_x^b \ln(t-x)f(t)\,dt,\ 0<a\leqslant x<b. \tag{1.28}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]–[22], [28]–[41] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} & S_{k,g,a+,b-}f(x) \\ :=& \frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right] \\ =& \frac{1}{2}\left[\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt+\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt\right] \end{aligned} \tag{1.29}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt,\ x\in[a,b)$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt,\ x\in(a,b].$$

We can define also the dual mixed operator

$$\begin{aligned} & \breve{S}_{k,g,a+,b-}f(x) \\ :=& \frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right] \\ =& \frac{1}{2}\left[\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt+\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt\right] \end{aligned}$$

for any $x\in(a,b)$.

In the recent paper [26] we obtained the following inequalities for convex functions $f:[a,b]\to\mathbb{R}$:

THEOREM 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$\begin{aligned} & \frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right]f\left(\frac{K(g(x)-g(a))a+K(g(b)-g(x))b}{K(g(x)-g(a))+K(g(b)-g(x))}\right. \\ & \qquad \left.+\frac{\int_a^x K(g(x)-g(t))\,dt-\int_x^b K(g(t)-g(x))\,dt}{K(g(x)-g(a))+K(g(b)-g(x))}\right) \\ \leqslant & \frac{1}{2}\left[f\left(a+\frac{1}{K(g(x)-g(a))}\int_a^x K(g(x)-g(t))\,dt\right)K(g(x)-g(a))\right. \\ & \qquad \left.+f\left(b-\frac{1}{K(g(b)-g(x))}\int_x^b K(g(t)-g(x))\,dt\right)K(g(b)-g(x))\right] \\ \leqslant & S_{k,g,a+,b-}f(x) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\left[K(g(x)-g(a)) f(a)+K(g(b)-g(x)) f(b)\right] \\
&\quad+\frac{1}{2}\left[\frac{f(x)-f(a)}{x-a} \int_a^x K(g(x)-g(t)) d t-\frac{f(b)-f(x)}{b-x} \int_x^b K(g(t)-g(x)) d t\right]
\end{aligned} \tag{1.30}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right] \\
&\qquad \times f\left(x+\frac{\int_x^b K(g(b)-g(t)) d t-\int_a^x K(g(t)-g(a)) d t}{K(g(x)-g(a))+K(g(b)-g(x))}\right) \\
&\leqslant \frac{1}{2}\left[f\left(x-\frac{1}{K(g(x)-g(a))} \int_a^x K(g(t)-g(a)) d t\right) K(g(x)-g(a))\right. \\
&\quad\left.+f\left(x+\frac{1}{K(g(b)-g(x))} \int_x^b K(g(b)-g(t)) d t\right) K(g(b)-g(x))\right] \\
&\leqslant \breve{S}_{k, g, a+, b-} f(x) \\
&\leqslant \frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right] f(x) \\
&\quad+\frac{1}{2}\left[\frac{f(b)-f(x)}{b-x} \int_x^b K(g(b)-g(t)) d t-\frac{f(x)-f(a)}{x-a} \int_a^x K(g(t)-g(a)) d t\right]
\end{aligned} \tag{1.31}
$$

for $x \in (a,b)$.

Motivated by the above results, in this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of continuous functions $f:[a,b] \to \mathbb{R}$ for which the composite function $f \circ g^{-1}$ is convex on $[g(a), g(b)]$. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ are also given. Applications for Hadamard fractional integrals are provided as well.

2. General results

We have the following simple representation for the k-g-fractional integrals:

LEMMA 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f:[a,b] \to \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. Then*

$$
S_{k,g,a+} f(x) = \int_0^1 k((g(x)-g(a))s) f \circ g^{-1}(sg(a)+(1-s)g(x)) ds, \tag{2.1}
$$

for $x \in (a,b]$ *and*

$$S_{k,g,b-}f(x) = \int_0^1 k((g(b)-g(x))s) f\circ g^{-1}((1-s)g(x)+sg(b))\,ds, \tag{2.2}$$

for $x \in [a,b)$.

We also have

$$S_{k,g,x+}f(b) = \int_0^1 k((g(b)-g(x))s) f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds \tag{2.3}$$

for $x \in [a,b)$ *and*

$$S_{k,g,x-}f(a) = \int_0^1 k((g(x)-g(a))s) f\circ g^{-1}((1-s)g(a)+sg(x))\,ds \tag{2.4}$$

for $x \in (a,b]$.

Proof. Using the change of variable $u = g(t)$, then we have $du = g'(t)\,dt$, $t = g^{-1}(u)$ and

$$S_{k,g,a+}f(x) = \int_{g(a)}^{g(x)} k(g(x)-u) f\circ g^{-1}(u)\,du,\ x\in(a,b] \tag{2.5}$$

and

$$S_{k,g,b-}f(x) = \int_{g(x)}^{g(b)} k(u-g(x)) f\circ g^{-1}(u)\,du,\ x\in[a,b). \tag{2.6}$$

Further, if we change the variable $u = (1-s)g(a)+sg(x)$, with $s\in[0,1]$, then for $a < x \leqslant b$ we have

$$\begin{aligned} &S_{k,g,a+}f(x) \\ &= \int_0^1 k((g(x)-g(a))(1-s)) f\circ g^{-1}((1-s)g(a)+sg(x))\,ds,\ x\in(a,b] \\ &= \int_0^1 k((g(x)-g(a))s) f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds,\ x\in(a,b]. \end{aligned} \tag{2.7}$$

If we change the variable $u = (1-s)g(x)+sg(b)$, with $s\in[0,1]$, then for $a \leqslant x < b$ we also have

$$S_{k,g,b-}f(x) = \int_0^1 k((g(b)-g(x))s) f\circ g^{-1}((1-s)g(x)+sg(b))\,ds,\ x\in[a,b), \tag{2.8}$$

which proves the first part of the lemma.

Further, if we replace x with b and a with x in (2.7), then we get

$$S_{k,g,x+}f(b) = \int_0^1 k((g(b)-g(x))s) f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds$$

and if we replace x with a and b with x in (2.8), then we obtain

$$S_{k,g,x-}f(a)=\int_0^1 k((g(x)-g(a))s)\, f\circ g^{-1}((1-s)g(a)+sg(x))\,ds,$$

which proves the last part of the lemma. □

REMARK 1. From the above lemma, we have the representations

$$\begin{aligned}S_{k,g,a+,b-}f(x) &= \frac{1}{2}\int_0^1 k((g(x)-g(a))s)\, f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds \\ &+\frac{1}{2}\int_0^1 k((g(b)-g(x))s)\, f\circ g^{-1}((1-s)g(x)+sg(b))\,ds \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\int_0^1 k((g(b)-g(x))s)\, f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds \\ &+\frac{1}{2}\int_0^1 k((g(x)-g(a))s)\, f\circ g^{-1}((1-s)g(a)+sg(x))\,ds \end{aligned} \tag{2.10}$$

for $x\in(a,b)$.

We have:

THEOREM 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $f\circ g^{-1}$ is convex on $(g(a),g(b))$, then for any $x\in(a,b)$ we have the inequalities*

$$\begin{aligned}&\frac{1}{2}\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}\right]\\ &\quad\times f\circ g^{-1}\left(\frac{g(a)\frac{K(g(x)-g(a))}{g(x)-g(a)}+g(b)\frac{K(g(b)-g(x))}{g(b)-g(x)}}{\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right.\\ &\quad\left.+\frac{\int_0^1 K((g(x)-g(a))s)\,ds-\int_0^1 (K(g(b)-g(x))s)\,ds}{\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right)\\ &\leqslant\frac{1}{2}\left[f\circ g^{-1}\left(g(a)+\frac{g(x)-g(a)}{K(g(x)-g(a))}\int_0^1 K((g(x)-g(a))s)\,ds\right)\frac{K(g(x)-g(a))}{g(x)-g(a)}\right.\\ &\quad\left.+f\circ g^{-1}\left(g(b)-\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 (K(g(b)-g(x))s)\,ds\right)\frac{K(g(b)-g(x))}{g(b)-g(x)}\right]\\ &\leqslant S_{k,g,a+,b-}f(x)\end{aligned}$$

$$\leqslant \frac{1}{2}f(x)\left(\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)ds+\frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)ds\right)$$
$$+\frac{1}{2}\left[\frac{f(a)}{g(x)-g(a)}\int_0^1 \left[K(g(x)-g(a))-K((g(x)-g(a))s)\right]ds\right.$$
$$\left.+\frac{f(b)}{g(b)-g(x)}\int_0^1 \left[K(g(b)-g(x))-(K(g(b)-g(x))s)\right]ds\right] \tag{2.11}$$

Proof. By the convexity of $f\circ g^{-1}$ on $[g(a),g(b)]$ we have

$$\begin{aligned} f\circ g^{-1}(sg(a)+(1-s)g(x)) &\leqslant sf\circ g^{-1}(g(a))+(1-s)f\circ g^{-1}(g(x)) \\ &= sf(a)+(1-s)f(x) \end{aligned}$$

and

$$f\circ g^{-1}((1-s)g(x)+sg(b))\leqslant (1-s)f(x)+sf(b)$$

for $x\in(a,b)$.

Therefore

$$\begin{aligned} &S_{k,g,a+,b-}f(x) \\ \leqslant\ &\frac{1}{2}\int_0^1 k((g(x)-g(a))s)\left[sf(a)+(1-s)f(x)\right]ds \\ &+\frac{1}{2}\int_0^1 k((g(b)-g(x))s)\left[(1-s)f(x)+sf(b)\right]ds \\ =\ &\frac{1}{2}f(a)\int_0^1 k((g(x)-g(a))s)\,sds+\frac{1}{2}f(x)\int_0^1 k((g(x)-g(a))s)(1-s)ds \\ &+\frac{1}{2}f(x)\int_0^1 k((g(b)-g(x))s)(1-s)ds+\frac{1}{2}f(b)\int_0^1 k((g(b)-g(x))s)\,sds \\ =:\ &B(x) \end{aligned} \tag{2.12}$$

for $x\in(a,b)$.

Using the chain rule for the derivative over s we have

$$\begin{aligned} (K((g(x)-g(a))s))' &= K'((g(x)-g(a))s)(g(x)-g(a)) \\ &= k((g(x)-g(a))s)(g(x)-g(a)) \end{aligned}$$

and

$$(K(g(b)-g(x))s)'=k((g(b)-g(x))s)(g(b)-g(x))$$

for $x\in(a,b)$.

Therefore

$$\begin{aligned} &\int_0^1 k((g(x)-g(a))s)\,sds \\ &=\frac{1}{g(x)-g(a)}\int_0^1 (K((g(x)-g(a))s))'\,sds \\ &=\frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))s)\,s|_0^1-\int_0^1 K((g(x)-g(a))s)\,ds\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))) - \int_0^1 K((g(x)-g(a))s)\,ds\right]\\
&= \frac{K(g(x)-g(a))}{g(x)-g(a)} - \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 k((g(x)-g(a))s)(1-s)\,ds\\
&= \frac{1}{g(x)-g(a)}\int_0^1 (K((g(x)-g(a))s))'((1-s))\,ds\\
&= \frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))s)(1-s)|_0^1 + \int_0^1 K((g(x)-g(a))s)\,ds\right]\\
&= \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 k((g(b)-g(x))s)(1-s)\,ds\\
&= \frac{1}{g(b)-g(x)}\int_0^1 (K((g(b)-g(x))s))'((1-s))\,ds\\
&= \frac{1}{g(b)-g(x)}\left[K((g(b)-g(x))s)(1-s)|_0^1 + \int_0^1 K((g(b)-g(x))s)\,ds\right]\\
&= \frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 k((g(b)-g(x))s)\,sds\\
&= \frac{1}{g(b)-g(x)}\int_0^1 (K(g(b)-g(x))s)'\,sds\\
&= \frac{1}{g(b)-g(x)}\left[(K(g(b)-g(x))s)\,s|_0^1 - \int_0^1 (K(g(b)-g(x))s)\,ds\right]\\
&= \frac{1}{g(b)-g(x)}\left[K(g(b)-g(x)) - \int_0^1 (K(g(b)-g(x))s)\,ds\right]\\
&= \frac{K(g(b)-g(x))}{g(b)-g(x)} - \frac{1}{g(b)-g(x)}\int_0^1 (K(g(b)-g(x))s)\,ds
\end{aligned}
$$

for $x \in (a,b)$.

We have

$$
\begin{aligned}
B(x) &= \frac{1}{2}f(a)\left[\frac{K(g(x)-g(a))}{g(x)-g(a)} - \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]\\
&\quad + \frac{1}{2}f(x)\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds
\end{aligned}
$$

$$+\frac{1}{2}f(x)\frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds$$
$$+\frac{1}{2}f(b)\left[\frac{K(g(b)-g(x))}{g(b)-g(x)}-\frac{1}{g(b)-g(x)}\int_0^1 (K(g(b)-g(x))s)\,ds\right]$$

and by (2.12) we get the third inequality in (2.11).

We use Jensen's inequality to get

$$\begin{aligned}&\frac{\int_0^1 k((g(x)-g(a))s)\,f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds}{\int_0^1 k((g(x)-g(a))s)\,ds}\\ &\geqslant f\circ g^{-1}\left(\frac{\int_0^1 k((g(x)-g(a))s)\,[sg(a)+(1-s)g(x)]\,ds}{\int_0^1 k((g(x)-g(a))s)\,ds}\right)\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\frac{\int_0^1 k((g(b)-g(x))s)\,f\circ g^{-1}((1-s)g(x)+sg(b))\,ds}{\int_0^1 k((g(b)-g(x))s)\,ds}\\ &\geqslant f\circ g^{-1}\left(\frac{\int_0^1 k((g(b)-g(x))s)\,[(1-s)g(x)+sg(b)]\,ds}{\int_0^1 k((g(b)-g(x))s)\,ds}\right)\end{aligned} \tag{2.14}$$

for $x\in(a,b)$.

Observe that

$$\begin{aligned}\int_0^1 k((g(x)-g(a))s)\,ds &= \frac{1}{g(x)-g(a)}\int_0^1 (K((g(x)-g(a))s))'\,ds\\ &= \frac{K(g(x)-g(a))}{g(x)-g(a)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 k((g(x)-g(a))s)\,[sg(a)+(1-s)g(x)]\,ds\\ &= g(a)\int_0^1 k((g(x)-g(a))s)\,s\,ds+g(x)\int_0^1 k((g(x)-g(a))s)(1-s)\,ds\\ &= g(a)\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}-\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]\\ &\quad+g(x)\left[\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]\\ &= g(a)\frac{K(g(x)-g(a))}{g(x)-g(a)}+\int_0^1 K((g(x)-g(a))s)\,ds\end{aligned}$$

for $x\in(a,b)$.

Then

$$\begin{aligned}
&\frac{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\left[sg\left(a\right)+\left(1-s\right)g\left(x\right)\right]ds}{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds} \\
&=\frac{g\left(a\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}+\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds}{\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}} \\
&=g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds
\end{aligned}$$

and by (2.13) we get

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)ds \qquad (2.15)\\
&\geqslant f\circ g^{-1}\left(g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}
\end{aligned}$$

for $x\in\left(a,b\right)$.

Also

$$\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds=\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}$$

and

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left[\left(1-s\right)g\left(x\right)+sg\left(b\right)\right]ds \\
&=g\left(x\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)ds+g\left(b\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)sds \\
&=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds \\
&\quad+g\left(b\right)\left[\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right] \\
&=g\left(b\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds
\end{aligned}$$

for $x\in\left(a,b\right)$.

Then

$$\begin{aligned}
&\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left[\left(1-s\right)g\left(x\right)+sg\left(b\right)\right]ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds} \\
&=\frac{g\left(b\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}{\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}} \\
&=g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds
\end{aligned}$$

and by (2.14) we have

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right) f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)ds \\
&\geqslant f\circ g^{-1}\left(g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}
\end{aligned} \tag{2.16}$$

for $x\in(a,b)$.

Therefore, by (2.9) we have

$$\begin{aligned}
&S_{k,g,a+,b-}f\left(x\right) \\
&=\frac{1}{2}\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right) f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)ds \\
&\quad+\frac{1}{2}\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right) f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)ds \\
&\geqslant\frac{1}{2}\left[f\circ g^{-1}\left(g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}\right. \\
&\quad\left.+f\circ g^{-1}\left(g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}\right],
\end{aligned} \tag{2.17}$$

which proves the second inequality in (2.11).

The first inequality is obvious by the convexity of $f\circ g^{-1}$. □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b\in I$ as

$$M_g\left(a,b\right):=g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g\left(a,b\right)=LME\left(a,b\right):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

COROLLARY 1. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
&\frac{2K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)} f\left(M_g(a,b)\right)\\
&\leqslant \left[f\circ g^{-1}\left(g(a)+\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right.\\
&\quad \left. + f\circ g^{-1}\left(g(b)-\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right]\frac{K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)}\\
&\leqslant S_{k,g,a+,b-} f\left(M_g(a,b)\right)\\
&\leqslant f\left(M_g(a,b)\right)\frac{2}{g(b)-g(a)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\\
&\quad + \left(\frac{f(a)+f(b)}{2}\right)\frac{2}{g(b)-g(a)}\int_0^1 \left[K\left(\frac{g(b)-g(a)}{2}\right)-K\left(\left(\frac{g(b)-g(a)}{2}\right)s\right)\right]ds.
\end{aligned}
\tag{2.18}
$$

The following inequalities for the dual operator also hold:

THEOREM 3. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $f\circ g^{-1}$ is convex on $(g(a),g(b))$, then for any $x\in(a,b)$ we have the inequalities*

$$
\begin{aligned}
&\frac{1}{2}\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}\right]\\
&\qquad\times f\circ g^{-1}\left(g(x)+\frac{\int_0^1 K((g(b)-g(x))s)\,ds-\int_0^1 K((g(x)-g(a))s)\,ds}{\frac{K(g(b)-g(x))}{g(b)-g(x)}+\frac{K(g(x)-g(a))}{g(x)-g(a)}}\right)\\
&\leqslant \frac{1}{2}\left[f\circ g^{-1}\left(g(x)+\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 K((g(b)-g(x))s)\,ds\right)\frac{K(g(b)-g(x))}{g(b)-g(x)}\right.\\
&\quad\left.+f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K(g(x)-g(a))}\int_0^1 K((g(x)-g(a))s)\,ds\right)\frac{K(g(x)-g(a))}{g(x)-g(a)}\right]\\
&\leqslant \check{S}_{k,g,a+,b-} f(x)\\
&\leqslant \frac{1}{2} f(x)\left(\frac{1}{g(b)-g(x)}\int_0^1 \left[K(g(b)-g(x))-(K(g(b)-g(x))s)\right]ds\right.\\
&\quad\left.+\frac{1}{g(x)-g(a)}\int_0^1 \left[K(g(x)-g(a))-K((g(x)-g(a))s)\right]ds\right)
\end{aligned}
$$

$$+\frac{1}{2}\left[\frac{f(b)}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds+\frac{f(a)}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]. \tag{2.19}$$

Proof. From the convexity of $f\circ g^{-1}$ we have

$$\begin{aligned}
&\breve{S}_{k,g,a+,b-}f(x)\\
&= \frac{1}{2}\int_0^1 k((g(b)-g(x))s)\, f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds\\
&\quad+\frac{1}{2}\int_0^1 k((g(x)-g(a))s)\, f\circ g^{-1}((1-s)g(a)+sg(x))\,ds\\
&\leqslant \frac{1}{2}\int_0^1 k((g(b)-g(x))s)\,[sf(x)+(1-s)f(b)]\,ds\\
&\quad+\frac{1}{2}\int_0^1 k((g(x)-g(a))s)\,[(1-s)f(a)+sf(x)]\,ds\\
&= \frac{1}{2}f(x)\int_0^1 k((g(b)-g(x))s)\,sds+\frac{1}{2}f(b)\int_0^1 k((g(b)-g(x))s)(1-s)\,ds\\
&\quad+\frac{1}{2}f(a)\int_0^1 k((g(x)-g(a))s)(1-s)\,ds+f(x)\int_0^1 k((g(x)-g(a))s)\,sds\\
&=: C(x)
\end{aligned} \tag{2.20}$$

for any $x\in(a,b)$.

We have

$$\begin{aligned}
C(x) &= \frac{1}{2}f(x)\left[\frac{K(g(b)-g(x))}{g(b)-g(x)}-\frac{1}{g(b)-g(x)}\int_0^1 (K(g(b)-g(x))s)\,ds\right]\\
&\quad+\frac{1}{2}f(b)\frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds\\
&\quad+\frac{1}{2}f(a)\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\\
&\quad+\frac{1}{2}f(x)\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}-\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]\\
&= \frac{1}{2}f(x)\Bigg(\frac{1}{g(b)-g(x)}\int_0^1 [K(g(b)-g(x))-(K(g(b)-g(x))s)]\,ds\\
&\quad+\frac{1}{g(x)-g(a)}\int_0^1 [K(g(x)-g(a))-K((g(x)-g(a))s)]\,ds\Bigg)\\
&\quad+\frac{1}{2}\Bigg[\frac{f(b)}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds\\
&\quad+\frac{f(a)}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\Bigg]
\end{aligned}$$

for any $x\in(a,b)$ and by (2.19) we get the third inequality in (2.19).

By Jensen's inequality we have

$$\begin{aligned}
&\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right) f\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds} \qquad (2.21)\\
&\geqslant f\circ g^{-1}\left(\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}\right)\\
&= f\circ g^{-1}\left(\frac{g\left(x\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)sds+g\left(b\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}\right)
\end{aligned}$$

for any $x\in(a,b)$.

Since

$$\begin{aligned}
&g\left(x\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)sds+g\left(b\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)ds\\
&=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}\int_0^1\left(K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\right)' sds\\
&\quad+\frac{g\left(b\right)}{g\left(b\right)-g\left(x\right)}\int_0^1\left(K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\right)'\left(1-s\right)ds\\
&=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}\left[\left(K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\right)s\big|_0^1-\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]\\
&\quad+\frac{g\left(b\right)}{g\left(b\right)-g\left(x\right)}\left[\left(K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\right)\left(1-s\right)\big|_0^1+\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]\\
&=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}\left[K\left(g\left(b\right)-g\left(x\right)\right)-\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]\\
&\quad+\frac{g\left(b\right)}{g\left(b\right)-g\left(x\right)}\left[\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]\\
&=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}K\left(g\left(b\right)-g\left(x\right)\right)+\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds
\end{aligned}$$

and

$$\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds=\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)},$$

then by (2.21) we have

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right) f\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)ds \qquad (2.22)\\
&\geqslant f\circ g^{-1}\left(\frac{\frac{g(x)}{g(b)-g(x)}K\left(g\left(b\right)-g\left(x\right)\right)+\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}{\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}\\
&= f\circ g^{-1}\left(g\left(x\right)+\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}
\end{aligned}$$

for any $x \in (a,b)$.

By applying Jensen's inequality again, we get

$$\begin{aligned}
&\frac{\int_0^1 k\left(\left(g(x)-g(a)\right)s\right) f\circ g^{-1}\left((1-s)g(a)+sg(x)\right)ds}{\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)ds} \qquad (2.23)\\
&\geqslant f\circ g^{-1}\left(\frac{\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)\left((1-s)g(a)+sg(x)\right)ds}{\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)ds}\right)\\
&= f\circ g^{-1}\left(\frac{g(a)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)(1-s)\,ds+g(x)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)sds}{\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)ds}\right)
\end{aligned}$$

for any $x \in (a,b)$.

Since

$$\begin{aligned}
&g(a)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)(1-s)\,ds+g(x)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)sds\\
&=\frac{g(a)}{g(x)-g(a)}\int_0^1 \left(K\left(\left(g(x)-g(a)\right)s\right)\right)'(1-s)\,ds\\
&\quad+\frac{g(x)}{g(x)-g(a)}\int_0^1 \left(K\left(\left(g(x)-g(a)\right)s\right)\right)'sds\\
&=\frac{g(a)}{g(x)-g(a)}\left[K\left(\left(g(x)-g(a)\right)s\right)(1-s)|_0^1+\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&\quad+\frac{g(x)}{g(x)-g(a)}\left[\left(K\left(\left(g(x)-g(a)\right)s\right)\right)s|_0^1-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&=\frac{g(a)}{g(x)-g(a)}\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\\
&\quad+\frac{g(x)}{g(x)-g(a)}\left[K\left(\left(g(x)-g(a)\right)\right)-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&=\frac{g(x)}{g(x)-g(a)}K\left(g(x)-g(a)\right)-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds
\end{aligned}$$

and

$$\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)ds=\frac{K\left(g(x)-g(a)\right)}{g(x)-g(a)},$$

then by (2.23) we get

$$\begin{aligned}
&\int_0^1 k\left(\left(g(x)-g(a)\right)s\right) f\circ g^{-1}\left((1-s)g(a)+sg(x)\right)ds \qquad (2.24)\\
&\geqslant f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K\left(g(x)-g(a)\right)}\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right)\frac{K\left(g(x)-g(a)\right)}{g(x)-g(a)}
\end{aligned}$$

for $x \in (a,b)$.

Therefore,

$$
\begin{aligned}
&\breve{S}_{k,g,a+,b-}f(x) \\
&\geqslant \frac{1}{2} f\circ g^{-1}\left(g(x)+\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 K((g(b)-g(x))s)\,ds\right)\frac{K(g(b)-g(x))}{g(b)-g(x)} \\
&\quad+\frac{1}{2} f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K(g(x)-g(a))}\int_0^1 K((g(x)-g(a))s)\,ds\right)\frac{K(g(x)-g(a))}{g(x)-g(a)},
\end{aligned}
$$

which proves the second inequality in (2.19).

The first inequality is obvious by the convexity of $f\circ g^{-1}$. □

COROLLARY 2. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
&\frac{2K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)} f(M_g(a,b)) \\
&\leqslant \left[f\circ g^{-1}\left(\frac{g(a)+g(b)}{2}+\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right. \\
&\quad\left.+f\circ g^{-1}\left(\frac{g(a)+g(b)}{2}-\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right]\frac{K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)} \\
&\leqslant \breve{S}_{k,g,a+,b-}f(M_g(a,b)) \\
&\leqslant f(M_g(a,b))\frac{2}{g(b)-g(a)}\int_0^1\left[K\left(\frac{g(b)-g(a)}{2}\right)-\left(K\left(\frac{g(b)-g(a)}{2}\right)s\right)\right]ds \\
&\quad+\frac{f(b)+f(a)}{2}\frac{2}{g(b)-g(a)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds. \qquad (2.25)
\end{aligned}
$$

3. Applications for generalized Riemann-Liouville fractional integrals

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $\alpha>0$ where Γ is the *Gamma function*, then

$$
S_{k,g,a+}f(x)=I^{\alpha}_{a+,g}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1}g'(t)f(t)\,dt
$$

for $a<x\leqslant b$ and

$$
S_{k,g,b-}f(x)=I^{\alpha}_{b-,g}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1}g'(t)f(t)\,dt
$$

for $a\leqslant x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [30, p. 100].

We consider the mixed operators

$$I^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x)\right] \tag{3.1}$$

and

$$\breve{I}^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{x+,g}f(b) + I^{\alpha}_{x-,g}f(a)\right] \tag{3.2}$$

for $x \in (a,b)$.

We observe that for $\alpha > 0$ we have

$$K(t) = \frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds = \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{t^{\alpha}}{\Gamma(\alpha+1)},\ t \geqslant 0.$$

In what follows we assume that $f:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$. If $f\circ g^{-1}$ is convex on $[g(a),g(b)]$, then by using the inequality (2.11) we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right]\\
&\quad\times f\circ g^{-1}\left(\frac{\frac{g(x)+\alpha g(a)}{\alpha+1}(g(x)-g(a))^{\alpha-1}}{(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}}\right.\\
&\quad\left.+\frac{\frac{g(x)+\alpha g(b)}{\alpha+1}(g(b)-g(x))^{\alpha-1}}{\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right]}\right)\\
&\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[f\circ g^{-1}\left(\frac{g(x)+\alpha g(a)}{\alpha+1}\right)(g(x)-g(a))^{\alpha-1}\right.\\
&\quad\left.+f\circ g^{-1}\left(\frac{g(x)+\alpha g(b)}{\alpha+1}\right)(g(b)-g(x))^{\alpha-1}\right]\\
&\leqslant I^{\alpha}_{g,a+,b-}f(x)\\
&\leqslant \frac{1}{2\Gamma(\alpha+2)}f(x)\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right]\\
&\quad+\frac{\alpha}{2\Gamma(\alpha+2)}\left[f(a)(g(x)-g(a))^{\alpha-1}+f(b)(g(b)-g(x))^{\alpha-1}\right]
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right]\\
&\quad\times f\circ g^{-1}\left(\frac{\frac{g(b)+\alpha g(x)}{\alpha+1}(g(b)-g(x))^{\alpha-1}+\frac{g(a)+\alpha g(x)}{\alpha+1}(g(x)-g(a))^{\alpha-1}}{(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}}\right)\\
&\leqslant \frac{1}{2\Gamma(\alpha+1)}\left[f\circ g^{-1}\left(\frac{g(b)+\alpha g(x)}{\alpha+1}\right)(g(b)-g(x))^{\alpha-1}\right.\\
&\quad\left.+f\circ g^{-1}\left(\frac{g(a)+\alpha g(x)}{\alpha+1}\right)(g(x)-g(a))^{\alpha-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \breve{I}^{\alpha}_{g,a+,b-} f(x) \\
&\leqslant \frac{\alpha}{2\Gamma(\alpha+2)} f(x)\left[(g(b)-g(x))^{\alpha-1}+(g(x)-g(a))^{\alpha-1}\right] \\
&\quad +\frac{1}{2\Gamma(\alpha+2)}\left[f(b)(g(b)-g(x))^{\alpha-1}+f(a)(g(x)-g(a))^{\alpha-1}\right] \qquad (3.4)
\end{aligned}
$$

for $x \in (a,b)$.

From (2.18) we also have

$$
\begin{aligned}
&\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha-1} f(M_g(a,b)) \\
&\leqslant \left[f\circ g^{-1}\left(\frac{g(b)+(2\alpha+1)g(a)}{2(\alpha+1)}\right)+f\circ g^{-1}\left(\frac{g(a)+(2\alpha+1)g(b)}{2(\alpha+1)}\right)\right] \\
&\quad \times \frac{(g(b)-g(a))^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\leqslant I^{\alpha}_{g,a+,b-} f(M_g(a,b)) \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}(g(b)-g(a))^{\alpha-1}\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\alpha\right] \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha-1}\frac{f(a)+f(b)}{2} \qquad (3.5)
\end{aligned}
$$

while from (2.25) we have

$$
\begin{aligned}
&\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha-1} f(M_g(a,b)) \\
&\leqslant \left[f\circ g^{-1}\left(\frac{\alpha g(a)+(\alpha+2)g(b)}{2(\alpha+1)}\right)+f\circ g^{-1}\left(\frac{\alpha g(b)+(\alpha+2)g(a)}{2(\alpha+1)}\right)\right] \\
&\quad \times \frac{(g(b)-g(a))^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\leqslant \breve{I}^{\alpha}_{g,a+,b-} f(M_g(a,b)) \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}(g(b)-g(a))^{\alpha-1}\left[f(M_g(a,b))\alpha+\frac{f(a)+f(b)}{2}\right] \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha-1}\frac{f(a)+f(b)}{2}. \qquad (3.6)
\end{aligned}
$$

The last part is obvious by the fact that

$$
f(M_g(a,b)) \leqslant \frac{f(a)+f(b)}{2}.
$$

4. Applications for GA-convex functions

Let $I\subset(0,\infty)$ be an interval; a real-valued function $f:I\to\mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$f\left(x^{1-\lambda}y^{\lambda}\right)\leqslant(\geqslant)(1-\lambda)f(x)+\lambda f(y) \tag{4.1}$$

for all $x,\ y\in I$ and $\lambda\in[0,1]$.

Since the condition (4.1) can be written as

$$f\circ\exp((1-\lambda)\ln x+\lambda\ln y)\leqslant(\geqslant)(1-\lambda)f\circ\exp(\ln x)+\lambda f\circ\exp(\ln y), \tag{4.2}$$

then we observe that $f:I\to\mathbb{R}$ is *GA-convex* (concave) on I if and only if $f\circ\exp$ is convex (concave) on $\ln I:=\{\ln z,z\in I\}$. If $I=[a,b]$ then $\ln I=[\ln a,\ln b]$.

It is known that the function $f(x)=\ln(1+x)$ is GA-convex on $(0,\infty)$ [7].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x):=\int_0^{\infty}t^{x-1}e^{-t}dt \text{ and } \psi(x):=\frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [42] that the function $f:(0,\infty)\to\mathbb{R}$ defined by

$$f(x)=\psi(x)+\frac{1}{2x}$$

is GA-concave on $(0,\infty)$ while the function $g:(0,\infty)\to\mathbb{R}$ defined by

$$g(x)=\psi(x)+\frac{1}{2x}+\frac{1}{12x^2}$$

is GA-convex on $(0,\infty)$.

For some recent inequalities on GA-convex functions see [16]–[18].

Consider the *Hadamard fractional integrals* [30, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x\leqslant b$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leqslant a<x<b,$$

where $\alpha>0$.

We consider the mixed operators

$$H_{a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[H_{a+}^{\alpha}f(x)+H_{b-}^{\alpha}f(x)\right] \tag{4.3}$$

and

$$\breve{H}_{g,a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[H_{x+}^{\alpha}f(b)+H_{x-}^{\alpha}f(a)\right] \tag{4.4}$$

for $x \in (a,b)$.

If we write the inequalities (3.5) and (3.6) for $g(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$, then for any function $f : [a,b] \to \mathbb{R}$ that is *GA-convex* on $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} f(G(a,b)) \\
&\leqslant \left[f\left(a^{\frac{2\alpha+1}{2(\alpha+1)}} b^{\frac{1}{2(\alpha+1)}}\right) + f\left(a^{\frac{1}{2(\alpha+1)}} b^{\frac{2\alpha+1}{2(\alpha+1)}}\right)\right] \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\leqslant H^{\alpha}_{a+,b-} f(G(a,b)) \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}\left[f(G(a,b)) + \frac{f(a)+f(b)}{2}\alpha\right] \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} \frac{f(a)+f(b)}{2}
\end{aligned} \tag{4.5}$$

while from (2.25) we have

$$\begin{aligned}
&\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} f(G(a,b)) \\
&\leqslant \left[f\left(a^{\frac{\alpha}{2(\alpha+1)}} b^{\frac{\alpha+2}{2(\alpha+1)}}\right) + f\left(a^{\frac{\alpha+2}{2(\alpha+1)}} b^{\frac{\alpha}{2(\alpha+1)}}\right)\right] \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\leqslant \breve{H}^{\alpha}_{a+,b-} f(G(a,b)) \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}\left[f(G(a,b))\alpha + \frac{f(a)+f(b)}{2}\right] \\
&\leqslant \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} \frac{f(a)+f(b)}{2},
\end{aligned} \tag{4.6}$$

where $G(a,b) = \sqrt{ab}$.

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