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EXISTENCE OF POSITIVE SOLUTIONS FOR REGULAR FRACTIONAL STURM–LIOUVILLE PROBLEMS

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Abstract. In this article we investigate existence and nonexistence results for some regular fractional Sturm-Liouville problems. We find the eigenvalues intervals of this problem may or may not have a positive solution. Some sufficient conditions for existence and nonexistence of positive solutions are given. Further, we study some special properties of positive solutions. We give some examples at the end.

1. Introduction

Fractional calculus is one of the useful fields of applied mathematics which has applications in the areas such as engineering, economics, control theory, chemistry, biology, medicine and other fields, see [7, 15].

In [3], the existence of at least one positive solution of the following problem

$$\begin{cases} \mathscr{D}_{0^+}^{\alpha} x(t) + a(t) f(x) = 0, \\ x(0) = 0 \\ \mathscr{D}_{0^+}^{\beta} x(1) = 0 \end{cases}$$

was obtained under the assumption $0 \leqslant \beta \leqslant 1, 1 < \alpha \leqslant 2$ are real numbers, $\mathscr{D}_{0^+}^{\alpha}, \mathscr{D}_{0^+}^{\beta}$ is the Riemann-Liouville fractional derivatives of order α and β respectively, $f : [0,+\infty) \to [0,+\infty)$ is continuous. In this paper, we consider a fractional Sturm-Liouville problem (FSLP), given by

$$\begin{cases} {}^c\mathscr{D}_{1^-}^{\delta} p(t) {}^c\mathscr{D}_{0^+}^{\delta} y(t) = \eta h(y(t)), \quad 0 < t < 1, \\ y(0) = 0, \\ {}^c\mathscr{D}_{0^+}^{\delta} y(t)|_{t=1} = 0, \end{cases} \tag{1}$$

where $h : [0,+\infty) \to [0,+\infty)$ is continuous and p is an arbitrary positive function in $C[0,1]$, and $\frac{1}{2} < \delta < 2$. We provide sufficient conditions for the existence and nonexistence of positive solutions to (1) by using Guo-Krasnosel'skii fixed point theorem on cones.

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2. Preliminaries

In this section, we present notation and some preliminary lemmas that will be used in the proofs of the main results.

DEFINITION 1. [5] The left and the right Riemann-Liouville fractional integrals of order $\delta > 0$, of function $h : (0, +\infty) \to \mathbb{R}$ are defined as follows

$$\mathscr{I}_{a^+}^{\delta} h(t) = \frac{1}{\Gamma(\delta)} \int_a^t \frac{h(s)}{(t-s)^{1-\delta}} ds, \quad t \in [a,b],$$

$$\mathscr{I}_{b^-}^{\delta} h(t) = \frac{1}{\Gamma(\delta)} \int_t^b \frac{h(s)}{(s-t)^{1-\delta}} ds, \quad t \in [a,b].$$

DEFINITION 2. For $\delta \in (0,1)$ the left and right Riemann-Liouville fractional derivatives of order δ of a function h, defined by

$$\mathscr{D}_{a^+}^{\delta} h(t) := \mathscr{D}\mathscr{I}_{a^+}^{1-\delta} h(t), \ \forall t \in (a,b],$$

$$\mathscr{D}_{b^-}^{\delta} h(t) := -\mathscr{D}\mathscr{I}_{b^-}^{1-\delta} h(t), \ \forall t \in (a,b].$$

LEMMA 1. *Let $\delta \in (0,1)$ the left and the right Caputo fractional derivatives of order δ are given by*

$$\forall t \in (a,b], \ {}^c\mathscr{D}_{a^+}^{\delta} h(t) := \mathscr{D}_{a^+}^{\delta}[h(t) - h(a)],$$
$$\forall t \in [a,b), \ {}^c\mathscr{D}_{b^-}^{\delta} h(t) := \mathscr{D}_{b^-}^{\delta}[h(t) - h(b)],$$

for order $\delta \in (0,1)$ and $h \in AC[a,b]$, the Caputo fractional derivatives satisfy the following relations:

$${}^c\mathscr{D}_{a^+}^{\delta} h(t) := \mathscr{I}_{a^+}^{1-\delta} \mathscr{D} h(t), \quad {}^c\mathscr{D}_{b^-}^{\delta} h(t) := -\mathscr{I}_{b^-}^{1-\delta} \mathscr{D} h(t),$$

respectively.

DEFINITION 3. Assume that E is a real Banach space and $\mathscr{P} \subset E$ be a cone and for $e \in \mathscr{P} \setminus \{0\}$, we define

$$E_e = \{x \in E : \exists \lambda > 0 \textit{ such that } -\lambda e \leqslant x \leqslant \lambda e\},$$

with norm

$$\|x\|_e = \inf\{\lambda > 0 : -\lambda e \leqslant x \leqslant \lambda e\}, \quad \forall x \in E_e. \tag{2}$$

It is easy to see that E_e becomes a normed linear space under the norm $\|.\|_e$. The e-norm of the element $x \in E_e$ is denoted by $\|x\|_e$.

LEMMA 2. [2] *Suppose that the cone $\mathscr{P}$ be normal. Then*

(i) E_e *is a Banach space with e-norm, and there exists a constant* $k > 0$ *such that* $\|x\| \leqslant k\|x\|_e, \forall x \in E_e$;

(ii) $\mathscr{P}_e = E_e \cap \mathscr{P}$ *is a normal solid cone of* E_e*, and*

$$\mathscr{P}_e = \{x \in E_e : \exists \tau = \tau(x) > 0 \text{ such that } x \geqslant \tau e\}.$$

LEMMA 3. [2] *Suppose that* $\mathscr{P}$ *be a normal solid cone and*

$$\mathscr{P}^0 = \{x \in \mathscr{P} | x \text{ is an interior point of } \mathscr{P}\}$$

and the operator $\mathscr{A} : \mathscr{P}^0 \to \mathscr{P}^0$ *be increasing. Let there exists a constant* $0 < \gamma < 1$ *such that*

$$\mathscr{A}(tx) \geqslant t^\gamma \mathscr{A}x, \quad \forall x \in \mathscr{P}^0, \ 0 < t < 1. \tag{3}$$

If x_η *is the unique solution of the equation* $\mathscr{A}x = \eta x$ *in* $\mathscr{P}^0$*, then*

(i) If $0 < \eta_1 < \eta_2$ *then* $x_{\eta_1} \gg x_{\eta_2}$ *;*

(ii) If $\eta \to \eta_0 (\eta_0 > 0)$ *then* $\|x_\eta - x_{\eta_0}\| \to 0$ *;*

(iii) $\lim_{\eta \to +\infty} \|x_\eta\| = 0, \lim_{\eta \to 0^+} \|x_\eta\| = +\infty$.

3. Main results

We assume the operator

$$\mathscr{L}_\delta y(t) = {}^c\mathscr{D}^\delta_{1^-} p(t) {}^c\mathscr{D}^\delta_{0^+} y(t),$$

and consider the fractional differential equation

$$\mathscr{L}_\delta y(t) = h(y(t)),$$

$$y(t) = \mathscr{I}^\delta_{0^+} \frac{1}{p(t)} \mathscr{I}^\delta_{1^-} h(y(t)) = \mathscr{T} h(y(t)). \tag{4}$$

We define an operator $\mathscr{T}$ with kernel $\mathscr{K}$

$$\mathscr{T} h(y(t)) = \int_0^1 \mathscr{K}(t,s) h(y(s)) ds, \tag{5}$$

where kernel $\mathscr{K}$ given by

$$\mathscr{K}(t,s) = \frac{1}{\Gamma^2(\delta)} \begin{cases} k_1(t,s) = \int_0^s \dfrac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx, & s \leqslant t \\ k_2(t,s) = \int_0^t \dfrac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx, & s > t \end{cases} \tag{6}$$

see [4]. The following lemma and corollary describe the stationary functions of the Sturm-Liouville operator $\mathscr{L}_\delta$ in $C[0,1]$.

LEMMA 4. [5] *The function* $y(t) = C_1 + C_2 \mathscr{I}_{0^+}^{\delta} \dfrac{1}{p(t)}$ *is the solution of the equation*

$$ {}^{c}\mathscr{D}_{1^-}^{\delta} p(t) {}^{c}\mathscr{D}_{0^+}^{\delta} y(t) = 0, $$

where C_1 *and* C_2 *are arbitrary constants and* $y(t)$ *continuously differentiable in* $[0,1]$.

The fractional integral operator $\mathscr{T}$, defined for a positive continuous function p, is bounded on $L^2(0,1)$ see ([5], page 72, Lemma 2.1).

LEMMA 5. [4] *Let* $\delta \in (\frac{1}{2}, 1), p \in C[0,1]$. *Then,*
(i) The operator $\mathscr{T}$ *on the* $L^2(0,1)$*- space is a self-adjoint operator.*
(ii) The problem (1) has an infinite countable set of positive, simple eigenvalues.

LEMMA 6. *Let* $p \in C[0,1]$ *be a given function. The function* $\mathscr{K}(t,s)$ *defined by (6) satisfies the following conditions*
(R1) $\mathscr{K}(t,s) \in C([0,1] \times [0,1])$ *and* $\mathscr{K}(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$,
(R2) $\max_{t\in[0,1]} \mathscr{K}(t,s) \leqslant \dfrac{M}{(2\delta-1)\Gamma^2(\delta)}$, *for* $(t,s) \in (0,1) \times (0,1)$,
(R3) $\min_{\frac{1}{4}\leqslant t\leqslant \frac{3}{4}} \mathscr{K}(t,s) \geqslant \dfrac{m\gamma(s)}{(2\delta-1)\Gamma^2(\delta)}$, *for* $s \in (0,1)$,
where $\gamma(s) = \min\{(\frac{1}{4})^{2\delta-1}, s^{2\delta-1}\}, M = \max_{t\in[0,1]} \dfrac{1}{p(t)}, m = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \dfrac{1}{p(t)}$.

Proof. Obviously from definition of $\mathscr{K}(t,s)$ by (6) we conclude that $\mathscr{K}(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$. By definition of k_1 and k_2, we have

$$ \max_{t\in[0,1]} k_1(t,s) = \max_{t\in[0,1]} \int_0^s \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx \leqslant M \int_0^1 (1-x)^{(2\delta-2)} dx = \frac{M}{2\delta-1}, $$

$$ \max_{t\in[0,1]} k_2(t,s) = \max_{t\in[0,1]} \int_0^t \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx \leqslant M \int_0^1 (1-x)^{(2\delta-2)} dx = \frac{M}{2\delta-1}, $$

then we have

$$ \max_{t\in[0,1]} \mathscr{K}(t,s) \leqslant \frac{M}{(2\delta-1)\Gamma^2(\delta)}, $$

and

$$ \min_{\frac{1}{4}\leqslant t\leqslant \frac{3}{4}} k_1(t,s) = \min_{\frac{1}{4}\leqslant t\leqslant \frac{3}{4}} \int_0^s \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx \geqslant m \int_0^s (s-x)^{(2\delta-2)} dx = \frac{ms^{2\delta-1}}{2\delta-1}, $$

$$ \min_{\frac{1}{4}\leqslant t\leqslant \frac{3}{4}} k_2(t,s) = \min_{\frac{1}{4}\leqslant t\leqslant \frac{3}{4}} \int_0^t \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)} dx \geqslant m \int_0^{\frac{1}{4}} (t-x)^{(2\delta-2)} dx = \frac{m(\frac{1}{4})^{2\delta-1}}{2\delta-1}, $$

if $\gamma(s) = \min\{(\frac{1}{4})^{2\delta-1}, s^{2\delta-1}\}$ then

$$\min_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}} \mathscr{K}(t,s) \geqslant \frac{m\gamma(s)}{(2\delta-1)\Gamma^2(\delta)} = \gamma(s)\frac{m}{M}\max_{t\in[0,1]} \mathscr{K}(t,s).$$

The fractional integral operator $\mathscr{T}$ is bounded in $L^2(a,b)$ since for real order $\delta > 0$, we have the following relations

$$\|\mathscr{I}^{\delta}_{a^+} f\|_{L^2} \leqslant \mathscr{K}_\delta \|f\|_{L^2}, \quad \mathscr{K}_\delta = \frac{(b-a)^\delta}{\Gamma(\delta+1)}, \tag{7}$$

which follow from Lemma 2.1 [5].

LEMMA 7. *If $\frac{1}{2} < \delta < 1$, the operator $\mathscr{T}$ defined by (5) in $L^2(0,1)$ is a completely continuous operator.*

Proof. By (7), the operator $\mathscr{T}$ is well defined as a bounded operator mapping $L^2(a,b) \to L^2(a,b)$. We prove that

$$\int_0^1\int_0^1 \mathscr{K}^2(t,s)dtds < \infty. \tag{8}$$

We have

$$\int_0^1\int_0^1 \mathscr{K}^2(t,s)dtds = \int_0^1 \Big[\int_0^t \mathscr{K}^2(t,s)ds + \int_t^1 \mathscr{K}^2(t,s)ds\Big]dt. \tag{9}$$

For the first integral of the right hand side of (9), we have

$$\begin{aligned}\int_0^t \mathscr{K}^2(t,s)ds &= \frac{1}{\Gamma^4(\delta)}\int_0^t \Big[\int_0^s \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)}dx\Big]^2 ds \\ &\leqslant \frac{M}{\Gamma^4(\delta)}\int_0^t \Big[\int_0^s (s-x)^{2\delta-2}dx\Big]^2 ds = \frac{M}{\Gamma^4(\delta)}\int_0^t \frac{s^{4\delta-2}}{(2\delta-1)^2}ds \\ &= \frac{M}{\Gamma^4(\delta)}\frac{s^{4\delta-1}}{(2\delta-1)^2(4\delta-1)},\end{aligned}$$

where $M = \max_{t\in[0,1]} \dfrac{1}{p^2(t)}$ and $\dfrac{1}{2} < \delta < 1$. For the second integral of the right hand side of (9), we have

$$\begin{aligned}\int_t^1 \mathscr{K}^2(t,s)ds &= \frac{1}{\Gamma^4(\delta)}\int_t^1 \Big[\int_0^t \frac{(t-x)^{\delta-1}(s-x)^{\delta-1}}{p(x)}dx\Big]^2 ds \\ &\leqslant \frac{M}{\Gamma^4(\delta)}\int_t^1 \Big[\int_0^t (t-x)^{2\delta-2}dx\Big]^2 ds = \frac{M}{\Gamma^4(\delta)}\int_0^t \frac{t^{4\delta-2}}{(2\delta-1)^2}ds \\ &= \frac{M}{\Gamma^4(\delta)}\frac{t^{4\delta-2}(1-t)}{(2\delta-1)^2},\end{aligned}$$

when $\frac{1}{2} < \delta < 1$. By applying these estimations for integrals of (9), we obtain the following upper bound for the kernel of operator $\mathscr{T}$:

$$\int_0^1 \int_0^1 \mathscr{K}^2(t,s)dtds \leqslant \frac{M}{\Gamma^4(\delta)} \frac{2}{(2\delta-1)^2} < \infty. \tag{10}$$

Continuity of $\mathscr{K}$ in $[0,1]\times[0,1]$ and the condition (10) implies the operator $\mathscr{T}$, defined by kernel $\mathscr{K}$, is compact. Therefor it is well known that every compact operator is completely continuous [1].

THEOREM 1. *([6]) Suppose that $\mathscr{X}$ be a Banach space, and let $\mathscr{P} \subset \mathscr{X}$ be a cone in $\mathscr{X}$. Assume Ψ_1, Ψ_2 are open subsets of $\mathscr{X}$ with $0 \in \Psi_1 \subset \overline{\Psi_1} \subset \Psi_2$ and let $S : \mathscr{P} \to \mathscr{P}$ be a completely continuous operator such that, either*
(1) for $y \in \mathscr{P} \cap \partial\Psi_1$, we have $\|Sy\| \leqslant \|y\|$, and for $y \in \mathscr{P} \cap \partial\Psi_2$, we have $\|Sy\| \geqslant \|y\|$; or
(2) for $y \in \mathscr{P} \cap \partial\Psi_1$ we have $\|Sy\| \geqslant \|y\|$, and for $y \in \mathscr{P} \cap \partial\Psi_2$ we have $\|Sy\| \leqslant \|y\|$. Then S has a fixed point in $\mathscr{P} \cap (\overline{\Psi_2} \setminus \Psi_1)$.

3.1. Existence of a Positive Solution

Assume that the Banach space $E = C[0,1]$ be endowed the max norm $\|y\| = max_{0\leqslant t\leqslant 1}|y(t)|$. Define the cone $\mathscr{P} \subset E$ as follows

$$\mathscr{P} = \{y \in E : y(t) \geqslant 0, \min_{\frac{1}{4}<t<\frac{3}{4}} y(t) \geqslant \frac{1}{4}\|y\| \ \ t \in [0,1]\}. \tag{11}$$

We introduce an operator $A_\eta : \mathscr{P} \to E$ as follows

$$A_\eta y(t) = \eta \int_0^1 \mathscr{K}(t,s)h(y(s))ds, \quad t \in [0,1]. \tag{12}$$

Therefore, if u is a fixed point of the operator A_η, then u is a positive solution of boundary value problem (1). Obviously from definition (12) we conclude that $A_\eta(P) \subset P$.

We introduce the following notation:

$$H_0 = \limsup_{y\to 0^+} \frac{h(y)}{y}, \qquad H_\infty = \limsup_{y\to +\infty} \frac{h(y)}{y},$$

$$h_0 = \liminf_{y\to 0^+} \frac{h(y)}{y}, \qquad h_\infty = \liminf_{y\to +\infty} \frac{h(y)}{y},$$

$$c_1 = \frac{M}{\Gamma^2(\delta)(2\delta-1)}, \qquad c_2 = \frac{m}{4(2\delta-1)\Gamma^2(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)ds.$$

THEOREM 2. *If* $0 \leqslant h_\infty, H_0 \leqslant +\infty$, $h_\infty c_2 > H_0 c_1, h_\infty c_2 \neq 0$, *then for every* η *satisfying*

$$\frac{1}{h_\infty c_2} < \eta < \frac{1}{H_0 c_1}, \tag{13}$$

the problem (1) has at least one positive solution.

Proof. If η satisfies (13) and $\varepsilon > 0$ is chosen such that

$$\frac{1}{(h_\infty - \varepsilon)c_2} \leqslant \eta \leqslant \frac{1}{(H_0 + \varepsilon)c_1}, \tag{14}$$

then by the definition of H_0, there exist $r_1 > 0$ such that

$$h(y) \leqslant (H_0 + \varepsilon)y, \quad for \quad 0 < y \leqslant r_1. \tag{15}$$

If $y \in \partial\mathscr{P}$ whit $\|y\| = r_1$, then from (14) and (15), we have

$$\|A_\eta y\| \leqslant \frac{M\eta}{\Gamma^2(\delta)(2\delta - 1)} \int_0^1 h(y(s))ds \leqslant \eta(H_0 + \varepsilon)r_1 c_1 \leqslant r_1 = \|y\|.$$

Therefore, if we define the open set $\Psi_1 = \{y \in E : \|y\| < r_1\}$, then

$$\|A_\eta y\| \leqslant \|y\| \quad for \quad y \in \mathscr{P} \cap \partial\Psi_1. \tag{16}$$

Suppose $r_3 > 0$ such that

$$h(y) \geqslant (h_\infty - \varepsilon)y \quad for \quad y \geqslant r_3. \tag{17}$$

Let $\|y\| = r_2 = \max\{2r_1, r_3\}$ for $y \in \partial\mathscr{P}$, then using (14) and (17), we see that

$$\begin{aligned} \|A_\eta y\| &\geqslant A_\eta y(t) = \eta \int_0^1 \mathscr{K}(x,s)h(y(s))ds \\ &\geqslant \frac{m\eta}{4(2\delta - 1)\Gamma^2(\delta)}(h_\infty - \varepsilon)\|y\| \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)ds \geqslant \|y\|. \end{aligned}$$

Therefore, if we define

$$\Psi_2 = \{y \in E : \|y\| < r_2\}, \tag{18}$$

we have

$$\|A_\eta y\| \geqslant \|y\| \quad for \quad y \in \mathscr{P} \cap \partial\Psi_2. \tag{19}$$

Therefore, using the Theorem 1 and (16), (19) , we conclude that the problem (1) has a positive solution y in $\mathscr{P} \cap (\overline{\Psi_2} \setminus \Psi_1)$ with $r_1 \leqslant \|y\| \leqslant r_2$. This completes the proof.

THEOREM 3. *Let* $0 \leqslant h_0, H_\infty \leqslant +\infty$, $h_0c_2 > H_\infty c_1$, *and* $h_0c_2 \neq 0$. *For every* η *satisfying*

$$\frac{1}{h_0c_2} < \eta < \frac{1}{H_\infty c_1}, \tag{20}$$

the problem (1) has at least one positive solution.

Proof. If η satisfies (20) and $\varepsilon > 0$ is such that

$$\frac{1}{(h_0-\varepsilon)c_2} \leqslant \eta \leqslant \frac{1}{(F_\infty+\varepsilon)c_1}. \tag{21}$$

By h_0, we conclude that there exists $r_1 > 0$ such that

$$h(y) \geqslant (h_0-\varepsilon)y, \quad for \quad 0 < y \leqslant r_1. \tag{22}$$

For $y \in \partial\mathscr{P}$ with $\|y\| = r_1$, similar to the second part of Theorem 2, we deduce that

$$\|A_\eta y\| \geqslant \|y\|.$$

Define a set Ψ_1 as $\Psi_1 = \{y \in E : \|y\| < r_1\}$, then

$$\|A_\eta y\| \geqslant \|y\| \quad for \quad y \in \mathscr{P} \cap \partial\Psi_1. \tag{23}$$

We can choose a constant $R_1 > 0$ such that

$$h(y) \leqslant (H_\infty+\varepsilon)y \quad for \quad y \geqslant R_1. \tag{24}$$

Our proof will be divided into two Cases:

Case1. h is bounded. This implies that there exists some $N > 0$, such that

$$h(y) \leqslant N \quad for \quad y \in (0,+\infty).$$

We define $r_3 = \max\{2r_1, \eta N c_1\}$ and $y \in \mathscr{P}$ with $\|y\| = r_3$, then

$$\begin{aligned}\|A_\eta y\| &\leqslant \frac{M\eta}{\Gamma^2(\delta)(2\delta-1)}\int_0^1 h(y(s))ds \\ &\leqslant \frac{\eta MN}{\Gamma^2(\delta)(2\delta-1)} = \eta c_1 N \leqslant r_3 \leqslant \|y\|.\end{aligned}$$

If we define $P_{r_3} = \{y \in \mathscr{P} : \|y\| \leqslant r_3\}$, we fined

$$\|A_\eta y\| \leqslant \|y\| \quad for\, y \in \partial P_{r_3}. \tag{25}$$

Case2. h is unbounded. We deduce that there exists some $r_4 > \max\{2r_1, R_1\}$ such that

$$h(y) \leqslant h(r_4) \quad for\, 0 < y \leqslant r_4.$$

If y in $\mathscr{P}$ with $\|y\| = r_4$, hence by using (21) and (24), we have

$$\begin{aligned}\|A_\eta y\| &\leqslant \frac{\eta M}{\Gamma^2(\delta)(2\delta-1)}\int_0^1 (H_\infty+\varepsilon)\|y\|ds\\ &\leqslant \frac{\eta M}{\Gamma^2(\delta)(2\delta-1)}(H_\infty+\varepsilon)\|y\| = \eta c_1(H_\infty+\varepsilon)\|y\| \leqslant \|y\|.\end{aligned}$$

This shows that (25) is true.

In both Case 1 and Case 2, if we define $\Psi_2 = \{y \in E : \|y\| < r_2 = \max\{r_3, r_4\}\}$, we conclude that for $y \in \mathscr{P} \cap \partial\Psi_2$ we have

$$\|A_\eta y\| \leqslant \|y\|, \tag{26}$$

Hence, using the Theorem 1 and (26) , (23) we conclude that the problem (1) has a positive solution $y \in \mathscr{P} \cap (\overline{\Psi_2} \setminus \Psi_1)$ with $r_1 \leqslant \|y\| \leqslant r_2$. This completes the proof.

THEOREM 4. *Assume that for $r_2 > r_1 > 0$ and $\eta > 0$, we have*

$$\max_{0\leqslant y\leqslant r_2} h(y) \leqslant \frac{r_2}{\eta c_1}, \quad \min_{0\leqslant y\leqslant r_1} h(y) \geqslant \frac{r_1}{\eta c_2}.$$

Then the problem (1) has a positive solution $y \in \mathscr{P}$ with $r_1 \leqslant \|y\| \leqslant r_2$.

Proof. Let us set $\Psi_1 = \{y \in E : \|y\| < r_1\}$, then for $y \in \mathscr{P} \cap \partial\Psi_1$, we have

$$\begin{aligned}\|A_\eta y\| &\geqslant A_\eta y(t) = \eta\int_0^1 \mathscr{K}(t,s)h(y(s))ds\\ &\geqslant \frac{\eta m}{(2\delta-1)\Gamma^2(\delta)}\int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)h(y(s))ds\\ &\geqslant \frac{\eta m}{4(2\delta-1)\Gamma^2(\delta)}\int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s)\min_{0\leqslant y\leqslant r_1} h(y(s))ds\\ &\geqslant \eta c_2\frac{r_1}{\eta c_2} = r_1 = \|y\|.\end{aligned}$$

If we define $\Psi_2 = \{y \in E : \|y\| < r_2\}$ then for $y \in \mathscr{P} \cap \partial\Psi_2$, we conclude that

$$\begin{aligned}\|A_\eta y\| &\leqslant \frac{\eta M}{\Gamma^2(\delta)(2\delta-1)}\int_0^1 h(y(s))ds\\ &\leqslant \frac{\eta M}{\Gamma^2(\delta)(2\delta-1)}\int_0^1 \max_{0\leqslant y\leqslant r_2} h(y(s))ds\\ &\leqslant \eta c_1\frac{r_2}{\eta c_1} = r_2 = \|y\|.\end{aligned}$$

Therefore, using the Theorem 1, the problem (1) has a positive solution y in $\mathscr{P}$ with $r_1 \leqslant \|y\| \leqslant r_2$ .The proof is complete.

3.2. Nonexistence of Positive Solution

Assume that the following condition holds:
(B) $\sup_{r>0}\min_{y\in(0,r)} h(y) > 0$.

THEOREM 5. *Assume the condition* (B) *holds and* $H_\infty, H_0 < +\infty$, *then there exists a real number* $\eta_0 > 0$ *such that for every* $0 < \eta < \eta_0$ *the problem (1) has no positive solution.*

Proof. Because $H_0, H_\infty < +\infty$, thus

$$\begin{aligned} &\exists\, l_1,\, l_2,\, r_1,\, r_2 \,:\, r_1 < r_2, \\ &h(y) \leqslant l_1 y, \;\; for \;\; y \in [0, r_1], \\ &h(y) \leqslant l_2 y, \;\; for \;\; y \in [r_2, +\infty). \end{aligned}$$

Assume that

$$l = \max\Big\{l_1, l_2, \max_{r_1 \leqslant y \leqslant r_2}\{\frac{h(y)}{y}\}\Big\}.$$

Thus

$$h(y) \leqslant ly, \quad for \quad y \in [0, +\infty).$$

Let $w(t)$ is a positive solution of (1) . We will show that this leads to a contradiction for $0 < \eta < \eta_0 := \dfrac{1}{lc_1}$. In this case we have

$$A_\eta w(t) = w(t), \quad t \in [0,1].$$

Thus

$$\begin{aligned} \|w\| = \|A_\eta w(t)\| &\leqslant \frac{\eta M}{\Gamma^2(\delta)(2\delta-1)} \int_0^1 h(w(s))ds \\ &\leqslant \frac{\eta M l \|w\|}{\Gamma^2(\delta)(2\delta-1)} < \|w\|, \end{aligned}$$

which is a contradiction, therefore completes the proof and the problem (1) has no positive solution.

THEOREM 6. *Assume* (B) *holds. If* $h_0, h_\infty > 0$, *then there exists a real number* $\eta_0 > 0$ *such that for every* $\eta > \eta_0$ *the problem (1) has no positive solution in the cone* $\mathscr{P}$ *defined by (11).*

Proof. Since $h_0, h_\infty > 0$, thus we conclude that

$$\begin{aligned} &\exists\, n_1,\, n_2,\, r_1,\, r_2 \,:\, r_1 < r_2, \\ &h(y) \geqslant n_1 y, \;\; for \;\; y \in [0, r_1], \\ &h(y) \geqslant n_2 y, \;\; for \;\; y \in [r_2, +\infty). \end{aligned}$$

Assume that

$$n = \min\left\{n_1, n_2, \min_{r_1 \leqslant y \leqslant r_2}\{\frac{h(y)}{y}\}\right\} > 0.$$

Hence

$$h(y) \geqslant ny, \quad for \quad y \in [0, +\infty).$$

Let $w(t)$ be a positive solution of (1). We will show that this leads to a contradiction for $\eta > \eta_0 := \dfrac{1}{nc_2}$. Since $A_\eta w(t) = w(t)$ for $t \in [0,1]$, then

$$\|w\| = \|A_\eta w(t)\| \geqslant \frac{\eta m}{(2\delta - 1)\Gamma^2(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s) h(w(s)) ds$$

$$\geqslant \frac{\eta m}{4(2\delta - 1)\Gamma^2(\delta)} n\|w\| \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(s) ds > \|w\|, \tag{27}$$

which is a contradiction, therefore completes the proof and the problem (1) has no positive solution.

3.3. Uniqueness

THEOREM 7. *Let the Banach space $X = C[0,1]$ be endowed with the norm $\|.\|_\infty$ and $h : X \to X$, satisfying the Lipschitz condition*

$$\|h(y) - h(w)\| \leqslant L\|y - w\|, \quad y, w \in X, L > 0. \tag{28}$$

Then the problem (1) has exactly one positive solution provided

$$\frac{\eta LM}{\Gamma^2(\delta)(2\delta - 1)} < 1,$$

where M is defined in Lemma 6.

Proof. For any $y(t), w(t) \in X$, using the assumption (28), we have

$$\begin{aligned}
\|A_\eta y(t) - A_\eta w(t)\| &\leqslant \eta \int_0^1 K(t,s)\|h(y(t)) - f(w(t))\| ds \\
&\leqslant L\eta \|y - w\| \int_0^1 K(t,s) ds \\
&\leqslant \frac{\eta LM}{\Gamma^2(\delta)(2\delta - 1)} \|y - w\|.
\end{aligned}$$

Thus, when $\dfrac{\eta LM}{\Gamma^2(\delta)(2\delta - 1)} < 1$, the operator A_η is the contraction mapping. Therefore, the problem (1) has exactly one positive solution $y(t)$ in $C[0,1]$.

4. Properties of solutions

In this section, by using Lemmas 2 and 3 we got some properties of positive solutions of the problem (1).

Let $h:[0,+\infty)\to[0,+\infty)$ be continuous and increasing and assume that Banach space $\mathscr{E}=C[0,1]$ be endowed with the ordering $y\leqslant w$ if $y(t)\leqslant w(t)$ for all $t\in[0,1]$, and the max norm

$$\|y\|_{\mathscr{E}}=\max_{0\leqslant t\leqslant 1}|y(t)|.$$

Define the cone $\mathscr{P}\subset\mathscr{E}$ as follows

$$\mathscr{P}=\{y\in\mathscr{E}:y(t)\geqslant 0,\ \ t\in[0,1]\},$$

then it is easy to verify that $\mathscr{P}$ is a normal, solid cone in $\mathscr{E}=C[0,1]$. Also define $\mathscr{P}^0$ as follows

$$\mathscr{P}^0=\{y\in\mathscr{E}:y(t)>0,\ \ t\in[0,1]\}.$$

Let

$$e(t)=\int_0^1\mathscr{K}(t,s)ds,\quad t\in[0,1]. \tag{29}$$

From Lemma 6 we conclude that $\mathscr{K}(t,s)\geqslant 0$ and $\mathscr{K}(t,s)$ is nonzero. Therefore, $e(t)\geqslant 0$ and $e(t)$ is nonzero. We get $e\in\mathscr{P}\setminus\{0\}$.

Assume that

$$\mathscr{X}=\mathscr{E}_e=\big\{y\in\mathscr{E}:\exists\tau>0,s.t.\ -\tau e(t)\leqslant y(t)\leqslant\tau e(t),\forall t\in[0,1]\big\},$$

with endowed norm

$$\|y\|_{\mathscr{X}}=\inf\big\{\tau>0:-\tau e(t)\leqslant y(t)\leqslant\tau e(t),\forall t\in[0,1]\big\}.$$

Let $\tilde{\mathscr{P}}=\mathscr{X}\cap\mathscr{P}$. From Lemma 2, we know that $\mathscr{X}$ is a Banach space, $\tilde{\mathscr{P}}$ is a normal solid cone in $\mathscr{X}$ and

$$\tilde{\mathscr{P}}^0=\big\{y\in\mathscr{X}:\textit{there exists }\alpha>0\textit{ such that }y(t)\geqslant\alpha e(t),\forall t\in[0,1]\big\}.$$

Besides, there exists a constant $l>0$ such that

$$\|y\|_{\mathscr{E}}\leqslant l\|y\|_{\mathscr{X}},\forall y\in\mathscr{X}.$$

THEOREM 8. *Suppose that following conditions hold*
(1) *there exists a constant* $0<r<1$ *such that*

$$h(\theta y(t))\geqslant\theta^r h(y(t))\ \ \forall t\in[0,1],\ \ y\geqslant 0,\ \ \theta\in(0,1);$$

(2) *there exists a constant* $\gamma>0$ *such that* $h(1)\leqslant\gamma$;
(3) $\min_{t\in[0,1]}h(e(t))>0$, *where* $e(t)$ *defined by (29).*
Then
(*i*) *the problem (1) has exactly one positive solution* y_η *in* $\tilde{\mathscr{P}}^0$, *for any* $\eta>0$;

(ii) if $0<\eta_1<\eta_2$, *then* $y_{\eta_1}(t)\leqslant y_{\eta_2}(t), \quad \forall t\in[0,1]$ *and* $y_{\eta_1}(t)\neq y_{\eta_2}(t)$*;*
(iii) if $\eta\to\eta_0(\eta_0>0)$ *then* $\max_{t\in[0,1]}|y_\eta(t)-y_{\eta_0}(t)|\to 0$*;*
(iv) if $\eta\to+\infty$ *then* $\max_{t\in[0,1]}|y_\eta(t)|\to+\infty$*; if* $\eta\to 0^+$ *then*

$$\max_{t\in[0,1]}|y_\eta(t)|\to 0.$$

Proof. Defined the operator A as

$$(Ay)(t)=\int_0^1 \mathscr{K}(t,s)h(y(s))ds, \quad t\in[0,1].$$

For $y>1$, we have

$$h(1)=h(\frac{1}{y}.y)\geqslant(\frac{1}{y})^r h(y),$$

thus

$$h(y)\leqslant y^r h(1)\leqslant \gamma y^r.$$

Therefore

$$0\leqslant\int_0^1 \mathscr{K}(t,s)h(y(s))ds\leqslant\int_0^1 \mathscr{K}(t,s)h(\|y\|_{\mathscr{E}})ds\leqslant Me(t), \;\; y\in\mathscr{P}, \forall t\in[0,1],$$

where

$$M=\max_{t\in[0,1]}h(\|y\|_{\mathscr{E}})\leqslant\max_{t\in[0,1]}h(\|y\|_{\mathscr{E}}+1)\leqslant\gamma(\|y\|_{\mathscr{E}}+1)^r.$$

Since $0\leqslant Ay(t)\leqslant Me(t), t\in[0,1]$, $Ay\in\mathscr{X}$. Then Ay belongs to $\mathscr{X}\cap\mathscr{P}=\tilde{\mathscr{P}}$. Thus $A:\tilde{\mathscr{P}}\to\tilde{\mathscr{P}}$.

For $y\in\tilde{\mathscr{P}}^0$, there exists $\alpha>0$ such that $y(t)\geqslant\alpha e(t)\geqslant 0, t\in[1]$. So we can take $\tau\in(0,1)$ such that $\theta<\alpha$, then we have

$$\begin{aligned}(Ay)(t)&=\int_0^1 \mathscr{K}(t,s)h(y(s))ds\geqslant\int_0^1 \mathscr{K}(t,s)h(\alpha e(s))ds\\&\geqslant\theta^r\int_0^1 \mathscr{K}(t,s)h(e(s))ds.\end{aligned}$$

Let $m=\min_{t\in[0,1]}\{h(e(t))\}$. It is clear that $m>0$ and $Ay(t)\geqslant m\theta^r e(t), t\in[0,1]$. Therefore, $A:\tilde{\mathscr{P}}^0\to\tilde{\mathscr{P}}^0$. The increasing property of $h(y(t))$ implies that the operator A is increasing. If $y\in\tilde{\mathscr{P}}^0$ and $\theta\in(0,1)$, then

$$\begin{aligned}A(\theta y)(t)&=\int_0^1 \mathscr{K}(t,s)h(\theta y(s))ds\geqslant\int_0^1 \mathscr{K}(t,s)\theta^r h(y(s))ds\\&\geqslant\theta^r\int_0^1 \mathscr{K}(t,s)h(y(s))ds=\theta^r(Ay)(t).\end{aligned}$$

Thus, A satisfies (3). Consider the following equation

$$A(y)(t) = \mu y(t). \tag{30}$$

From Lemma 3, for any $\mu > 0$, (30) has a unique solution y_μ in $\tilde{\mathscr{P}}^0$, u_μ is strictly decreasing, i.e., $0 < \mu_1 < \mu_2$ implies $y_{\mu_1} \gg y_{\mu_2}$, y_μ is continuous, i.e. $\mu \to \mu_0 (\mu_0 > 0)$ implies $\|y_\mu - y_{\mu_0}\| \to 0, \lim_{\mu\to\infty} \|y_\mu\| = 0, \lim_{\mu\to 0^+} \|y_\mu\| = +\infty$.

Let $\eta = \dfrac{1}{\mu}, \eta_0 = \dfrac{1}{\mu_0}, \eta_1 = \dfrac{1}{\mu_1}, \eta_2 = \dfrac{1}{\mu_2}$. Then (30) is changed to $y(t) = \eta(Ay)(t)$ that y is the solution of the problem (1) if and only if $y = \eta Ay$. Then
(i) the problem (1) has exactly one positive solution y_η in $\tilde{\mathscr{P}}^0$, for any $\eta > 0$;
(ii) if $0 < \eta_1 < \eta_2$, then there exists $\alpha > 0$ such that $y_{\eta_1} - y_{\eta_2} \geqslant \alpha e(t), t \in [0,1]$ and thus $y_{\eta_1} \leqslant y_{\eta_2}, \forall t \in [0,1]$ and $y_{\eta_1} \neq y_{\eta_2}$;
(iii) if $\eta \to \eta_0 (\eta_0 > 0)$ then $\max_{t\in[0,1]} |y_\eta(t) - y_{\eta_0}(t)| \to 0$; (iv) if $\eta \to +\infty$ then $\max_{t\in[0,1]} |y_\eta(t)| \to +\infty$, if $\eta \to 0^+$ then $\max_{t\in[0,1]} |y_\eta(t)| \to 0$.

5. Illustrated Example

Example 5.1

Consider the fractional Sturm-Liouville problem

$$\begin{cases} {}^cD_{1^-}^{\frac{2}{3}}\, {}^cD_{0^+}^{\frac{2}{3}} y(t) = \eta \dfrac{(y^2(t) + y(t))(2 + \sin y(t))}{15y(t) + 1}, \quad 0 < t < 1, \\ y(0) = 0, \\ {}^cD_{0^+}^{\frac{2}{3}} y(t)|_{t=1} = 0. \end{cases} \tag{31}$$

Here $p(t) = 1$, $h(y) = \dfrac{(y^2 + y)(2 + \sin y)}{15y + 1}$, $\delta = \frac{2}{3}$. By simple calculation, we get $H_\infty = 0.2$, $h_0 = 2$, $c_1 = 1.636$, $c_2 = 0.643$, $f_0 c_2 = 1.286 > H_\infty c_1 = 0.327$. Thus, by Theorem 3, the problem (31) has a positive solution for each $\eta \in (0.7776, 3.0581)$.

Example 5.2

Consider the boundary value problem of fractional differential equation

$$\begin{cases} {}^cD_{1^-}^{\frac{2}{3}}\, {}^cD_{0^+}^{\frac{2}{3}} y(t) = \eta \dfrac{(10y^2(t) + y(t))(2 + \sin y(t))}{y(t) + 1}, \quad 0 < t < 1, \\ y(0) = 0, \\ {}^cD_{0^+}^{\frac{2}{3}} y(t)|_{t=1} = 0. \end{cases} \tag{32}$$

Here $p(t) = 1$, $h(y) = \dfrac{(10y^2 + y)(2 + \sin y)}{y + 1}$, $\delta = \frac{2}{3}$. By calculating, we get $H_\infty = 30$, $H_0 = h_0 = 2$, $h_\infty = 10$, $c_1 = 1.636$, $c_2 = 0.643$, $y < h < 30y$ or $y > 0$.

(i) Thus, by Theorem 2, the problem (32) has a positive solution for each $\eta \in (0.1555, 0.3057)$.

(ii) By Theorem 5, the problem (32) has no positive solution for $\eta \in (0, 0.02037)$.
(iii) By Theorem 6, the problem (32) has no positive solution for $\eta \in (1.5552, +\infty)$.

Example 5.3

Consider the boundary value problem of fractional differential equation

$$\begin{cases} {}^cD_{1^-}^{\frac{3}{5}}\left((t^2+1)\,{}^cD_{0^+}^{\frac{3}{5}}\right)y(t) = \eta(y(t)+1)^{\frac{1}{2}}, \quad 0<t<1, \\ y(0)=0, \\ {}^cD_{0^+}^{\frac{2}{3}}y(t)|_{t=1} = 0. \end{cases} \tag{33}$$

Where $p(t) = t^2+1, h(y) = (y+1)^{\frac{1}{2}}, \delta = \frac{3}{5}$. Obviously, $h : [0,+\infty) \to [0,+\infty)$ is continuous and increasing. For $0<\theta<1$,

$$h(\theta y) = (\theta y+1)^{\frac{1}{2}} > (\theta y+\theta)^{\frac{1}{2}} = \theta^{\frac{1}{2}}(y+1)^{\frac{1}{2}} = \theta^{\frac{1}{2}}h(y),$$

where $r = \frac{1}{2}$. We have $h(1) = \sqrt{2} \leqslant \gamma$ and $h(e(t)) = (e(t)+1)^{\frac{1}{2}} > 0$, and thus $\min_{t\in[0,1]} h(e(t)) > 0$. Thus all requirements of Theorem 8 hold. Therefore:

(i) the problem (33) has exactly one positive solution y_η in $\tilde{\mathscr{P}}^0$, for any $\eta > 0$;
(ii) if $0<\eta_1<\eta_2$, then $y_{\eta_1}(t) \leqslant y_{\eta_2}(t)$, $\forall t \in [0,1]$ and $y_{\eta_1}(t) \neq y_{\eta_2}(t)$;
(iii) if $\eta \to \eta_0, (\eta_0 > 0)$ then $\max_{t\in[0,1]} |y_\eta(t) - y_{\eta_0}(t)| \to 0$;
(iv) if $\eta \to +\infty$ then $\max_{t\in[0,1]} |y_\eta(t)| \to +\infty$; if $\eta \to 0^+$ then $\max_{t\in[0,1]} |y_\eta(t)| \to 0$.

Moreover $M = \max_{t\in[0,1]} \dfrac{1}{p(t)} = 1$ and

$$\|h(y)-h(w)\| = \|\sqrt{y+1}-\sqrt{w+1}\| \leqslant \|y+1-w-1\| = \|y-w\|.$$

So $L = 1$, Thus, by Theorem 7 for $\dfrac{\eta}{\frac{1}{5}\Gamma^2(\frac{3}{5})} < 1$ i.e. $\eta < \frac{1}{5}\Gamma^2(\frac{3}{5}) = 0.4435$, the problem (33) has exactly one positive solution.

Conclusion: In this paper the existence and nonexistence and uniqueness of positive solutions for the fractional initial value problem (1) are proved by transforming the problem in to an operator equation. Special properties of positive solutions are considered and under some assumption the uniqueness is obtained. A few examples are given at the end to illustrate the results.

REFERENCES

[1] J. B. CONWAY, *A Course in Functional Analysis*, Graduate texts in Mathematics, Springer-Verlag, New York, (1990).
[2] D. GUO, V. LAKSHMIKANTHAM *Nonlinear Problems in Abstract Cones*, Boston and New York: Academic Press, (1988).

[3] C. A. HOLLON, J. T. NEUGEBAUER, *Positive solutions of a fractional boundary value problem with a fractional derivative boundary condition*, Dynamical Systems, Diferential Equations and Applications, 615–620, (2015).
[4] M. KLIMEK AND M. BLASIK, *Regular Sturm-Liouville Problem with Riemann-Liouvill Derivatives of order in (1,2): Discrete Spectrum, Solutions and Applications*, Advances in Modelling and Control of Non-integer-Order Systems, (2015).
[5] A. A. KILBAS, H. H. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., (2006).
[6] M. A. KRASNOSELSKII, *Positive Solution of Operator Equation*, Noordhoff Groningen, (1964).
[7] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, John Wiley, New York, (1993).
[8] A. NEAMATY, R. DARZI, A. DABBAGHIAN, J. GOLIPOOR, *Introducing an Iterative Method for Solving a Special FDE*, International Mathematical Forum, **4**, 1449–1456, (2009).
[9] M. AL-REFAI, *Non-Existence Results and Analytical Bounds of Eigenvalues for a Class of Fractional Sturm-Liouville Eigenvalue Problems*, Fractional Differential Calculus, **8**, 43–55, (2018).
[10] M. AL-REFAI AND M. A. HAJJI, *Analysis of a fractional eigenvalue problem involving Atangana-Baleanu fractional derivative: A maximum principle and applications*, Chaos, **29**, 013135, (2019).
[11] M. AL-REFAI, K. PAL, *A Maximum Principle for a Fractional Boundary Value Problem with Convection Term and Applications*, Mathematical Modeling and Analysis, **24**, 62–71, (2019).
[12] M. AL-REFAI, T. ABDELJAWAD, *Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems*, Complexity, **2017**, 7 pages, (2017).
[13] M. AL-REFAI, *Basic Results on Nonlinear Eigenvalue Problems with Fractional Order*, Electronic Journal of Differential Equations, **191**, 1–12, (2012).
[14] M. RIVERO, J. J. TRUJILLO, VELASCO, M. P., *A fractional approach to the Sturm-Liouville problem*, Cent. Eur. J. Phys., (2013).
[15] Y. ZHOU, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, (2014).

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