

## BLOW UP OF NONAUTONOMOUS FRACTIONAL REACTION–DIFFUSION SYSTEMS

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*Abstract.* We provide a sufficient condition for finite time blow up of the positive mild solution to the nonautonomous Cauchy problem of a reaction-diffusion system with distinct fractional diffusions. The proof is based on the reduction to an ordinary differential system by means of a comparison between the transition densities of the semigroups generated by the different fractional Laplacians. Moreover, we prove that this condition is also a sufficient condition for the blow up of a related nonautonomous fractional diffusion-convection-reaction system.

### 1. Introduction

In this paper,  $i \in \{1, 2\}$  and  $i' = 3 - i$ . We consider blow up in finite time of positive solutions of the initial values problem for the nonautonomous reaction-diffusion system

$$\begin{aligned} \frac{\partial u_i(t, x)}{\partial t} &= k_i(t) \Delta_{\alpha_i} u_i(t, x) + h_i(t) u_{i'}^{\beta_i}(t, x), \quad t > 0, x \in \mathbb{R}^d, \\ u_i(0, x) &= \varphi_i(x) \geq 0, \quad x \in \mathbb{R}^d, \end{aligned} \quad (1)$$

where  $\Delta_{\alpha_i} := -(-\Delta)^{\frac{\alpha_i}{2}}$ ,  $0 < \alpha_i \leq 2$ , denotes the fractional power of the Laplacian,  $\beta_i$ ,  $i = 1, 2$  are positive constants such that  $\beta_1 \beta_2 > 1$ , the initial conditions  $\varphi_i$ ,  $i = 1, 2$  are bounded, continuous and not identically zero, and  $k_i, h_i : [0, \infty) \rightarrow [0, \infty)$ ,  $i = 1, 2$  are continuous functions such that for all  $t \geq 0$  large enough,

$$a_1^i t^{\rho_i} \leq \int_0^t k_i(r) dr \leq a_2^i t^{\rho_i}, \quad (2)$$

where  $a_1^i, a_2^i > 0$ ,  $\rho_i > 0$ , and

$$b_1^i t^{\sigma_i} \leq h_i(t) \leq b_2^i t^{\sigma_i}, \quad (3)$$

where  $b_1^i, b_2^i > 0$ ,  $\sigma_i \geq 0$ .

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The associated integral system of (1) is given by

$$u_i(t, x) = \int_{\mathbb{R}^d} p_i(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t, s), y - x) h_i(s) u_i^{\beta_i}(s, y) dy ds, \quad (4)$$

where  $p_i(t, x)$  is the fundamental solution of  $\frac{\partial}{\partial t} - \Delta_{\alpha_i}$  and

$$K_i(t, s) = \int_s^t k_i(r) dr, \quad t \geq s \geq 0. \quad (5)$$

A solution of the integral system (4) is called a mild solution of (1). In this paper solutions of (1) will be understood in this mild sense. If there exists a solution  $(u_1, u_2)$  of (4) defined in  $[0, \infty) \times \mathbb{R}^d$  such that  $\|u_1(t, \cdot)\|_\infty + \|u_2(t, \cdot)\|_\infty < \infty$  for any  $t \geq 0$ , we say that  $(u_1, u_2)$  is a global solution, and when there exists a positive number  $T_e < \infty$  such that (4) has a bounded solution  $(u_1, u_2)$  in  $[0, T] \times \mathbb{R}^d$  for all  $T < T_e$ , with  $\lim_{t \uparrow T_e} (\|u_1(t, \cdot)\|_\infty + \|u_2(t, \cdot)\|_\infty) = \infty$ , we say that  $(u_1, u_2)$  blows up in finite time.

The proof that we will give of the blow up in finite time of the positive mild solution of system (1) is based on a technique that go back to Sugitani [17] (for the case of a single equation), and which consists of reduce the blow up of system (1) to the blow up of an ordinary differential system. We will achieve such reduction through a comparison between the transition densities of the semigroup generated by the different fractional Laplacians (see (iv) of Lemma 1 below).

Reaction-diffusion systems of prototype (1) model a great number of molecular biology, physic and engineering problems (see [1, 14, 16]). The fractional Laplacians  $\Delta_{\alpha_i}$  with  $0 < \alpha_i < 2$ ,  $i = 1, 2$  count for the anomalous diffusion, and from a probabilistic perspective, correspond to stable Lévy processes [15]. The most common interpretation of system (1) is as a model to describe processes of heat diffusion and combustion in two-component continuous media with temporary-inhomogeneous thermal conductivity and volume energy release given by powers of  $u_i$ ,  $i = 1, 2$ . In this case  $u_i$ ,  $i = 1, 2$  represent the temperatures of the interacting components of some combustible mixture [14, 3].

For a single equation, with  $\alpha = 2$  and  $k = h \equiv 1$ , in his pioneering work, Fujita [5] showed the influence of spatial dimension on the finite time blow up versus global existence of solutions. Fujita's results was extended to the case of a system with  $\alpha_i = 2$  and  $k_i = h_i \equiv 1$ ,  $i = 1, 2$ , by Escobedo and Herrero [3]. They showed that the positive solution blows up in finite time if

$$d \leq \frac{2(\beta_1 \vee \beta_2 + 1)}{\beta_1 \beta_2 - 1}, \quad (6)$$

where  $\beta_1, \beta_2 > 1$  and  $\beta_1 \vee \beta_2 = \max\{\beta_1, \beta_2\}$ ,  $\beta_1 \wedge \beta_2 = \min\{\beta_1, \beta_2\}$ . A generalization of this result for the case  $\alpha_i = \alpha$ ,  $\alpha \in (0, 2]$  and  $k_i = h_i \equiv 1$ ,  $i = 1, 2$  was given by Takehi and Oshita [7]. In this case, they have proved that the positive mild solution blows up in finite time if

$$d \leq \frac{\alpha(\beta_1 \vee \beta_2 + 1)}{\beta_1 \beta_2 - 1}. \quad (7)$$

Note that this coincides with (6) for  $\alpha = 2$ . Other related cases involving Laplacians and fractional Laplacians for autonomous and nonautonomous systems with nonlinearities given by powers of  $u_i$ ,  $i = 1, 2$ , can be found for instance in [12, 20, 8, 6, 4, 11, 13, 21, 19].

This work can be considered as a generalization of the article [12]. In [12] system (1) was studied with  $k_i = h_i \equiv 1$ ,  $i = 1, 2$ ; it is shown that the positive mild solution of (1) blows up in finite time for any initial conditions  $\varphi_i$ ,  $i = 1, 2$ , bounded, continuous and not identically zero provided that

$$\frac{d(1 - \beta_1 \vee \beta_2)}{\alpha_2} + 1 \geq 0. \quad (8)$$

In this paper we will prove the next result.

**THEOREM 1.** *Suppose that  $0 < \alpha_i \leq 2$ ,  $i = 1, 2$  and let  $\alpha_1 \leq \alpha_2$ . If  $(u_1, u_2)$  is a positive mild solution of system (1) and*

$$\frac{d(\rho_2 - \beta_i)}{\alpha_2} + \sigma_i + 1 \geq 0, \quad i = 1, 2, \quad (9)$$

*then there exists a time  $t_0 \geq 1$  such that the mild solution  $(u_1, u_2)$  blows up for some time  $t_e > t_0 + t_0^{\frac{\alpha_2(\rho_1 \vee \rho_2)}{\alpha_1}}$ .*

Note that inequality (9) is reduced to inequality (8) when  $k_i = h_i \equiv 1$ , i.e.,  $\rho_i = 1$  and  $\sigma_i = 0$ ,  $i = 1, 2$ . Hence this result is a generalization of the corresponding result given in [12]. Also, it should be noted that under the assumption that  $\alpha_1 \leq \alpha_2$ , the positive mild solution of system (1) blows up in finite time regardless of the value of  $\rho_1$  (parameter  $\rho_1$  has only influence on the size of the blow up time). In this way, our result substantially improves that of Villa-Morales [20], because the blow up condition given in p. 2, Eq. (1.8) of [20] is not satisfied when  $\rho_1$  is much bigger than  $\rho_2$ . See also the comments in [12], p. 184 for comparisons with the blow up results of Villa-Morales [20] and Guedda and Kirane [6] for the autonomous case. Also, for the autonomous case, with  $k_i = h_i \equiv 1$  and  $\alpha_i = \alpha$ ,  $\alpha \in (0, 2]$ ,  $i = 1, 2$ , Wu and Tang [21] showed that the positive mild solution blows up in finite time if

$$d < \frac{\alpha(1 + \beta_1 \wedge \beta_2)}{2(\beta_1 \beta_2 - 1)}, \quad (10)$$

with  $\beta_i > 1$ ,  $i = 1, 2$ . In this particular case, our condition is better than that of Wu and Tang. However, in this particular case, the condition (7) of Kakehi and Oshita is the best.

For a single equation, with  $k = h \equiv 1$ , under one dimensional superdiffusive medium with advection, and a nonlinear source term, Kirk and Olmstead [9] showed that there exists a critical convection speed above which blow up is avoided and below which blow up is guaranteed (see also Tersenov [18] for the case  $\alpha = 2$ ). However, in Pérez [12] it was showed that when volumes energy release are given by powers greater

than one, the convection terms are of the form  $g_i(x) \cdot \nabla$  (see (11)) for  $g_i$  in the Kato class,  $i = 1, 2$  and  $\alpha_i \in (1, 2)$ ,  $i = 1, 2$ , the blow up in finite time of system (1) (with  $k_i = h_i \equiv 1$ ,  $i=1, 2$ ) implies blow up in finite time of the system with convection terms (11) (with  $k_i = h_i \equiv 1$ ,  $i=1, 2$ ), and vice versa. In this paper, we show that this result is still valid in the nonautonomous case considered here. More specifically, we consider the nonautonomous fractional reaction-diffusion system with convection terms

$$\begin{aligned} \frac{\partial u_i(t, x)}{\partial t} &= k_i(t) (\Delta_{\alpha_i} + g_i(x) \cdot \nabla) u_i(t, x) + h_i(t) u_i^{\beta_i}(t, x), \quad t > 0, x \in \mathbb{R}^d, \\ u_i(0, x) &= \varphi_i(x) \geq 0, \quad x \in \mathbb{R}^d, \end{aligned} \quad (11)$$

where  $\nabla$  is the gradient operator,  $\beta_i, \varphi_i, k_i, h_i$ ,  $i = 1, 2$  are as above,  $\alpha_i \in (1, 2)$  and  $g_i: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function in the Kato class  $\mathcal{K}_d^{\alpha_i-1}$  on  $\mathbb{R}^d$  (see Bogdan and Jakubowski [2], p. 185),  $i = 1, 2$ , and prove the next result.

**THEOREM 2.** *The positive mild solution of the nonautonomous reaction-convection-diffusion system (11) blows up in finite time if and only if the positive mild solution of the nonautonomous reaction-diffusion system (1) blows up in finite time.*

As a direct consequence of Theorems 1 and 2 we have the next corollary.

**COROLLARY 1.** *Suppose that  $1 < \alpha_i < 2$ ,  $i = 1, 2$  and let  $\alpha_1 \leq \alpha_2$ . Then the positive mild solution of the nonautonomous reaction-convection-diffusion system (11) blows up in finite time if (9) holds.*

We have organized this paper as follows. In Section 2 we give some lemmas that we will require for the proof of our main theorem (Theorem 1), in Section 3 we give the proof of Theorem 1, and finally, in Section 4 we give the proof of our second theorem.

## 2. Preliminary results

In the proof of our Theorem 1, we will require some preliminary results. The first one concerns about some properties of the fundamental solution,  $p_i(t, x)$ , of  $\frac{\partial}{\partial t} - \Delta_{\alpha_i}$ .

**LEMMA 1.** *Let  $s, t > 0$  and  $x, y \in \mathbb{R}^d$ , then*

- (i)  $p_i(ts, x) = t^{-\frac{d}{\alpha_i}} p_i\left(s, t^{-\frac{1}{\alpha_i}} x\right)$ ,
- (ii)  $p_i(t, x) \geq \left(\frac{s}{t}\right)^{\frac{d}{\alpha_i}} p_i(s, x)$  for  $t \geq s$ ,
- (iii)  $p_i\left(t, \frac{1}{\tau}(x - y)\right) \geq p_i(t, x) p_i(t, y)$  if  $p_i(t, 0) \leq 1$  and  $\tau \geq 2$ ,
- (iv)  $p_1(t, x) \geq c p_2\left(t^{\frac{\alpha_2}{\alpha_1}}, x\right)$  for some  $0 < c \leq 1$ , if  $\alpha_1 \leq \alpha_2$ .

*Proof.* For (i)-(iii) see Sugitani [17] pp. 46-47 and for (iv) see López-Mimbela and Villa-Morales [10] p. 1699.  $\square$

The second lemma that we will need establishes that there exists a time  $t_0$  such that the nonnegative mild solution of system (1) at time  $t_0$  is lower bounded by the

function  $c_0 p_2(\gamma_0, \cdot)$ , for some positive constants  $c_0$  and  $\gamma_0$  (here and in the sequel we assume that  $\alpha_1 \leq \alpha_2$ ). But first, note that (iv) of Lemma 1 applied to (4) implies

$$\begin{aligned} u_1(t, x) &\geq \int_{\mathbb{R}^d} c p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t, 0), y - x \right) \varphi_1(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} c p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t, s), y - x \right) h_1(s) u_2^{\beta_1}(s, y) dy ds, \\ u_2(t, x) &\geq \int_{\mathbb{R}^d} p_2(K_2(t, 0), y - x) \varphi_2(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p_2(K_2(t, s), y - x) h_2(s) u_1^{\beta_2}(s, y) dy ds. \end{aligned} \tag{12}$$

LEMMA 2. *If  $(u_1, u_2)$  is a nonnegative mild solution of (1), then there exist some positive constants  $c_0, \gamma_0$  and  $t_0 \geq 1$  such that*

$$\min \{u_1(t_0, x), u_2(t_0, x)\} \geq c_0 p_2(\gamma_0, x), \quad \forall x \in \mathbb{R}^d.$$

*Proof.* Let us fix  $t_0 \geq 1$  such that inequalities (2) and (3) are valid for all  $t \geq t_0$ , and  $K_i(t_0, 0) > 1$ ,  $i = 1, 2$ . Additionally, by (i) of Lemma 1, we can suppose that  $t_0$  is big enough such that

$$p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0), 0 \right) \leq 1 \quad \text{and} \quad p_2(K_2(t_0, 0), 0) \leq 1. \tag{13}$$

Hence, using (i) and (iii) of Lemma 1 we get

$$\begin{aligned} p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0), y - x \right) &\geq 2^{-d} p_2 \left( \frac{K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0)}{2^{\alpha_2}}, x \right) p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0), 2y \right), \\ p_2(K_2(t_0, 0), y - x) &\geq 2^{-d} p_2 \left( \frac{K_2(t_0, 0)}{2^{\alpha_2}}, x \right) p_2(K_2(t_0, 0), 2y). \end{aligned}$$

Using these inequalities, it follows from (12) that

$$\begin{aligned} u_1(t_0, x) &\geq 2^{-d} c p_2 \left( \frac{K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0)}{2^{\alpha_2}}, x \right) \int_{\mathbb{R}^d} p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0), 2y \right) \varphi_1(y) dy, \\ u_2(t_0, x) &\geq 2^{-d} p_2 \left( \frac{K_2(t_0, 0)}{2^{\alpha_2}}, x \right) \int_{\mathbb{R}^d} p_2(K_2(t_0, 0), 2y) \varphi_2(y) dy. \end{aligned}$$

Let us consider

$$a = \min \left\{ 2^{-d} c \int_{\mathbb{R}^d} p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0), 2y \right) \varphi_1(y) dy, 2^{-d} \int_{\mathbb{R}^d} p_2(K_2(t_0, 0), 2y) \varphi_2(y) dy \right\}$$

and

$$\gamma_0 = \min \left\{ \frac{K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0)}{2^{\alpha_2}}, \frac{K_2(t_0, 0)}{2^{\alpha_2}} \right\}.$$

Then from (ii) of Lemma 1 we get

$$u_1(t_0, x) \geq a \left( \frac{\gamma_0}{K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0) 2^{-\alpha_2}} \right)^{\frac{d}{\alpha_2}} p_2(\gamma_0, x),$$

$$u_2(t_0, x) \geq a \left( \frac{\gamma_0}{K_2(t_0, 0) 2^{-\alpha_2}} \right)^{\frac{d}{\alpha_2}} p_2(\gamma_0, x).$$

Finally, the desired result is obtained by taking

$$c_0 = \min \left\{ a \left( \frac{\gamma_0}{K_1^{\frac{\alpha_2}{\alpha_1}}(t_0, 0) 2^{-\alpha_2}} \right)^{\frac{d}{\alpha_2}}, a \left( \frac{\gamma_0}{K_2(t_0, 0) 2^{-\alpha_2}} \right)^{\frac{d}{\alpha_2}} \right\}. \quad \square$$

The next equalities system will be used in our last lemma. Let  $t_0 \geq 1$  be as in Lemma 2, the Chapman-Kolmogorov identity (see Sato [15], p. 54) implies

$$\begin{aligned} & u_i(t + t_0, x) \\ &= \int_{\mathbb{R}^d} p_i(K_i(t + t_0, 0), y - x) \varphi_i(y) dy + \int_0^{t_0} \int_{\mathbb{R}^d} p_i(K_i(t + t_0, s), y - x) h_i(s) u_i^{\beta_i}(s, x) dx ds \\ & \quad + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t + t_0, s + t_0), y - x) h_i(s + t_0) u_i^{\beta_i}(s + t_0, y) dy ds \quad (14) \\ &= \int_{\mathbb{R}^d} p_i(K_i(t + t_0, t_0), y - x) u_i(t_0, y) dy \\ & \quad + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t + t_0, s + t_0), y - x) h_i(s + t_0) u_i^{\beta_i}(s + t_0, y) dy ds. \end{aligned}$$

Now we shall derive an inequalities system that we will use in the proof of Theorem 1. Using (iv) of Lemma 1 we get

$$\begin{aligned} u_1(t + t_0, x) &\geq \int_{\mathbb{R}^d} c p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t + t_0, t_0), y - x \right) u_1(t_0, y) dy \\ & \quad + \int_0^t \int_{\mathbb{R}^d} c p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t + t_0, s + t_0), y - x \right) h_1(s + t_0) u_1^{\beta_1}(s + t_0, y) dy ds, \\ u_2(t + t_0, x) &\geq \int_{\mathbb{R}^d} p_2(K_2(t + t_0, t_0), y - x) u_2(t_0, y) dy \\ & \quad + \int_0^t \int_{\mathbb{R}^d} p_2(K_2(t + t_0, s + t_0), y - x) h_2(s + t_0) u_1^{\beta_2}(s + t_0, y) dy ds. \end{aligned}$$

Hence, Lemma 2 implies

$$\begin{aligned} u_1(t+t_0, x) &\geq c c_0 p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t+t_0, t_0) + \gamma_0, x \right) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} c p_2 \left( K_1^{\frac{\alpha_2}{\alpha_1}}(t+t_0, s+t_0), y-x \right) h_1(s+t_0) u_2^{\beta_1}(s+t_0, y) dy ds, \end{aligned} \quad (15)$$

$$\begin{aligned} u_2(t+t_0, x) &\geq c_0 p_2 (K_2(t+t_0, t_0) + \gamma_0, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} p_2 (K_2(t+t_0, s+t_0), y-x) h_2(s+t_0) u_1^{\beta_2}(s+t_0, y) dy ds. \end{aligned}$$

The result of the last lemma of this section allows us to reduce the blow up of system (1) to the blow up of an ordinary differential system. Let us define

$$\begin{aligned} \bar{u}_1(t) &= \int_{\mathbb{R}^d} p_2(K_2(t, 0), x) u_1(t, x) dx, \quad t \geq 0, \\ \bar{u}_2(t) &= \int_{\mathbb{R}^d} p_2(K_2(t, 0), x) u_2(t, x) dx, \quad t \geq 0. \end{aligned}$$

LEMMA 3. *If for some  $i$  ( $= 1, 2$ ), there exists  $t_e > 0$  such that  $\bar{u}_i(t) = \infty$  for all  $t \geq t_e$ , then the mild solution  $(u_1, u_2)$  of system (1) blows up in finite time.*

*Proof.* Let  $t_0$  be as in Lemma 2. If  $\bar{u}_i(t) = \infty$  for all  $t \geq t_e$ , we consider  $t_{i'}$  such that  $t_0, t_e < t_{i'} < \infty$ . From (2) we have that there exists  $t \geq t_{i'}$  such that

$$K_{i'}(t+t_0, t_{i'}+t_0) \geq 2^{\alpha_{i'}} K_2^{\frac{\alpha_{i'}}{\alpha_2}}(t_{i'}+t_0, 0).$$

Thus, for every  $0 \leq s \leq t_{i'}$ ,

$$K_{i'}(t+t_0, s+t_0) \geq 2^{\alpha_{i'}} K_2^{\frac{\alpha_{i'}}{\alpha_2}}(s+t_0, 0).$$

Therefore

$$\tau_{i'} \equiv \frac{K_{i'}^{\frac{1}{\alpha_{i'}}}(t+t_0, s+t_0)}{K_2^{\frac{1}{\alpha_2}}(s+t_0, 0)} \geq 2.$$

Furthermore, from (i) of Lemma 1 and (13) it follows that

$$p_2(K_2(s+t_0), 0) \leq p_2(K_2(t_0, 0), 0) \leq 1.$$

Thus, using (i) and (iii) of Lemma 1 we obtain

$$\begin{aligned} p_2 \left( K_{i'}^{\frac{\alpha_2}{\alpha_{i'}}}(t+t_0, s+t_0), y-x \right) &= \tau_{i'}^{-d} p_2 \left( K_2(s+t_0, 0), \frac{1}{\tau_{i'}}(y-x) \right) \\ &\geq \tau_{i'}^{-d} p_2(K_2(s+t_0, 0), x) p_2(K_2(s+t_0, 0), y). \end{aligned}$$

From (14), (iv) of Lemma 1 and Jensen's inequality we get

$$\begin{aligned} u_{i'}(t+t_0, x) &\geq c \int_0^{t'} h_{i'}(s+t_0) \int_{\mathbb{R}^d} p_2 \left( K_{i'}^{\frac{\alpha_2}{\alpha_{i'}}}(t+t_0, s+t_0), y-x \right) u_i^{\beta_{i'}}(s+t_0, y) dy ds \\ &\geq c \int_0^{t'} \tau_{i'}^{-d} h_{i'}(s+t_0) p_2(K_2(s+t_0, 0), x) \bar{u}_i^{\beta_{i'}}(s+t_0) ds, \end{aligned}$$

and thus  $u_{i'}(t+t_0, x) = \infty$ .  $\square$

### 3. Proof of the main theorem

*Proof.* Multiplying equations (15) by  $p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right), x \right)$ , and then integrating with respect  $x$ , we obtain

$$\begin{aligned} &\bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0 \right) \\ &\geq cc_0 p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ &\quad + \int_0^{t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \int_{\mathbb{R}^d} c p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, s+t_0 \right), y \right) \\ &\quad \cdot h_1(s+t_0) u_2^{\beta_1}(s+t_0, y) dy ds \end{aligned}$$

and

$$\begin{aligned} &\bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0 \right) \\ &\geq c_0 p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ &\quad + \int_0^{t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \int_{\mathbb{R}^d} p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, s+t_0 \right), y \right) \\ &\quad \cdot h_2(s+t_0) u_1^{\beta_2}(s+t_0, y) dy ds. \end{aligned}$$

Using (i) and (ii) of Lemma 1, we get

$$\begin{aligned} &\bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0 \right) \\ &\geq cc_0 p_2 \left( K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ &\quad + \int_0^{t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \int_{\mathbb{R}^d} c \left( \frac{K_2(s+t_0, 0)}{K_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} + t_0, s+t_0 \right)} \right)^{\frac{d}{\alpha_2}} \end{aligned}$$

$$\cdot p_2(K_2(s+t_0, 0), y) h_1(s+t_0) u_2^{\beta_1}(s+t_0, y) dy ds$$

and

$$\begin{aligned} & \bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\ & \geq c_0 p_2 \left( K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ & \quad + \int_0^{t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}}} \int_{\mathbb{R}^d} \left( \frac{K_2(s+t_0, 0)}{K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, s+t_0 \right)} \right)^{\frac{d}{\alpha_2}} \\ & \quad \cdot p_2(K_2(s+t_0, 0), y) h_2(s+t_0) u_1^{\beta_2}(s+t_0, y) dy ds. \end{aligned}$$

Applying now the Jensen's inequality, we obtain

$$\begin{aligned} & \bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\ & \geq c c_0 p_2 \left( K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ & \quad + \int_0^{t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}}} c \left( \frac{K_2(s+t_0, 0)}{K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, s+t_0 \right)} \right)^{\frac{d}{\alpha_2}} \\ & \quad \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds \end{aligned}$$

and

$$\begin{aligned} & \bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\ & \geq c_0 p_2 \left( K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, t_0 \right) + \gamma_0, 0 \right) \\ & \quad + \int_0^{t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}}} \left( \frac{K_2(s+t_0, 0)}{K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, s+t_0 \right)} \right)^{\frac{d}{\alpha_2}} \\ & \quad \cdot h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds. \end{aligned}$$

Using the fact that  $K_i \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, s+t_0 \right) \leq K_i \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right)$ ,  $i = 1, 2$ , we obtain

$$\bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)$$

$$\begin{aligned}
&\geq c c_0 p_2 \left( K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, t_0 \right) + \gamma_0, 0 \right) \\
&\quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} c \left( \frac{K_2(s+t_0, 0)}{K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_1^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right)} \right)^{\frac{d}{\alpha_2}} \\
&\quad \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
\end{aligned}$$

and

$$\begin{aligned}
&\bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\
&\geq c_0 p_2 \left( K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right) + K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, t_0 \right) + \gamma_0, 0 \right) \\
&\quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( \frac{K_2(s+t_0, 0)}{2K_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0, 0 \right)} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds.
\end{aligned}$$

Now, using (2), we have

$$\begin{aligned}
&\bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\
&\geq c c_0 p_2 \left( a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}} + \gamma_0, 0 \right) \\
&\quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} c \left( \frac{a_1^2(s+t_0)^{\rho_2}}{a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} \\
&\quad \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
\end{aligned}$$

and

$$\begin{aligned}
&\bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \geq c_0 p_2 \left( 2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + \gamma_0, 0 \right) \\
&\quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( \frac{a_1^2(s+t_0)^{\rho_2}}{2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2}} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds.
\end{aligned}$$

Using (i) of Lemma 1 we get

$$\begin{aligned}
& \bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \\
& \geq c c_0 \left( a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}} + \gamma_0 \right)^{-\frac{d}{\alpha_2}} p_2(1,0) \\
& \quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} c \left( \frac{a_1^2(s+t_0)^{\rho_2}}{a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} \\
& \quad \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
\end{aligned}$$

and

$$\begin{aligned}
& \bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) \geq c_0 \left( 2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + \gamma_0 \right)^{-\frac{d}{\alpha_2}} p_2(1,0) \\
& \quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( \frac{a_1^2(s+t_0)^{\rho_2}}{2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2}} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \\
& \geq c c_0 \left( \frac{t+t_0}{a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}} + \gamma_0} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\
& \quad + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} c \left( \frac{a_1^2(s+t_0)^{\rho_2} (t+t_0)}{a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + (a_2^1)^{\frac{\alpha_2}{\alpha_1}} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\frac{\rho_1 \alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} \\
& \quad \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
\end{aligned}$$

and

$$\begin{aligned} \bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} &\geq c_0 \left( \frac{t+t_0}{2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2} + \gamma_0} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\ &+ \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( \frac{a_1^2 (s+t_0)^{\rho_2} (t+t_0)}{2a_2^2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right)^{\rho_2}} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds. \end{aligned}$$

Let us fix  $T_0 > 0$  such that  $t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \geq t_0$ ,  $2^{\rho_2} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}} + 2^{\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}} t^{\frac{\rho_1}{\rho_1 \vee \rho_2}} \geq \gamma_0$  and  $2^{\rho_2+1} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}} \geq \gamma_0$  for all  $t > T_0$ . Then for any  $t > T_0$

$$\begin{aligned} \bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} &\geq cc_0 \left( \frac{t+t_0}{2^{\rho_2+1} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}} + 2^{1+\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}} t^{\frac{\rho_1}{\rho_1 \vee \rho_2}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\ &+ \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} c \left( \frac{a_1^2 (s+t_0)^{\rho_2} (t+t_0)}{2^{\rho_2} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}} + 2^{\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}} t^{\frac{\rho_1}{\rho_1 \vee \rho_2}}} \right)^{\frac{d}{\alpha_2}} \\ &\cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds \end{aligned}$$

and

$$\begin{aligned} \bar{u}_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} &\geq c_0 \left( \frac{t+t_0}{2^{\rho_2+2} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\ &+ \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( \frac{a_1^2 (s+t_0)^{\rho_2} (t+t_0)}{2^{\rho_2+1} a_2^2 t^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}}} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds. \end{aligned}$$

From here

$$\begin{aligned} \bar{u}_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} &\geq cc_0 \left( \frac{t+t_0}{2^{\rho_2+1} a_2^2 (t+t_0)^{\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \vee \rho_2)}} + 2^{1+\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}} (t+t_0)^{\frac{\rho_1}{\rho_1 \vee \rho_2}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} c \left( \frac{a_1^2(s+t_0)^{\rho_2}(t+t_0)}{2\rho_2 a_2^2(t+t_0) \frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \sqrt{\rho_2})} + 2 \frac{\rho_1 \alpha_2}{\alpha_1} (a_2^1)^{\frac{\alpha_2}{\alpha_1}} (t+t_0) \frac{\rho_1}{\rho_1 \sqrt{\rho_2}}} \right)^{\frac{d}{\alpha_2}} \\
 & \cdot h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq c_0 \left( \frac{t+t_0}{2\rho_2+2 a_2^2(t+t_0) \frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \sqrt{\rho_2})}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\
 & + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} \left( \frac{a_1^2(s+t_0)^{\rho_2}(t+t_0)}{2\rho_2+1 a_2^2(t+t_0) \frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \sqrt{\rho_2})}} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds.
 \end{aligned}$$

Using that  $\frac{\alpha_1 \rho_2}{\alpha_2(\rho_1 \sqrt{\rho_2})} \leq 1$  and  $\frac{\rho_1}{\rho_1 \sqrt{\rho_2}} \leq 1$ , we obtain

$$\begin{aligned}
 & \bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq cc_0 \left( \frac{1}{2\rho_2+1 a_2^2 + 2^{1+\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\
 & + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} c \left( \frac{a_1^2(s+t_0)^{\rho_2}}{2\rho_2 a_2^2 + 2 \frac{\rho_1 \alpha_2}{\alpha_1} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} h_1(s+t_0) (\bar{u}_2(s+t_0))^{\beta_1} ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq c_0 \left( \frac{1}{2\rho_2+2 a_2^2} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\
 & + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} \left( \frac{a_1^2(s+t_0)^{\rho_2}}{2\rho_2+1 a_2^2} \right)^{\frac{d}{\alpha_2}} h_2(s+t_0) (\bar{u}_1(s+t_0))^{\beta_2} ds.
 \end{aligned}$$

Using now (3), we get

$$\begin{aligned}
 & \bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq cc_0 \left( \frac{1}{2\rho_2+1 a_2^2 + 2^{1+\frac{\rho_1 \alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0) \\
 & + \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} c \left( \frac{a_1^2}{2\rho_2 a_2^2 + 2 \frac{\rho_1 \alpha_2}{\alpha_1} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} b_1^1(s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_1} (\bar{u}_2(s+t_0))^{\beta_1} ds
 \end{aligned}$$

and

$$\bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq c_0 \left( \frac{1}{2\rho_2+2 a_2^2} \right)^{\frac{d}{\alpha_2}} p_2(1,0)$$

$$+ \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} \left( \frac{a_1^2}{2\rho_2+1 a_2^2} \right)^{\frac{d}{\alpha_2}} b_1^2 (s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_2} (\bar{u}_1(s+t_0))^{\beta_2} ds.$$

Let  $\eta$  be the minimum value of

$$cc_0 \left( \frac{1}{2\rho_2+1 a_2^2 + 2^{1+\frac{\rho_1\alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} p_2(1,0), \quad c \left( \frac{a_1^2}{2\rho_2+1 a_2^2 + 2^{\frac{\rho_1\alpha_2}{\alpha_1}} (a_2^1)^{\frac{\alpha_2}{\alpha_1}}} \right)^{\frac{d}{\alpha_2}} b_1,$$

where  $b_1 := b_1^1 \wedge b_1^2$ . Since  $0 < c \leq 1$ , we have

$$\bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq \eta + \eta \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} (s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_1} (\bar{u}_2(s+t_0))^{\beta_1} ds$$

and

$$\bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \geq \eta + \eta \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} (s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_2} (\bar{u}_1(s+t_0))^{\beta_2} ds,$$

or equivalently

$$\begin{aligned} & \bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \\ & \geq \eta + \eta \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} (s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_1 - \frac{d\beta_1}{\alpha_2}} \left( (s+t_0)^{\frac{d}{\alpha_2}} \bar{u}_2(s+t_0) \right)^{\beta_1} ds \end{aligned}$$

and

$$\begin{aligned} & \bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} + t_0 \right) (t+t_0)^{\frac{d}{\alpha_2}} \\ & \geq \eta + \eta \int_0^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} (s+t_0)^{\frac{d\rho_2}{\alpha_2} + \sigma_2 - \frac{d\beta_2}{\alpha_2}} \left( (s+t_0)^{\frac{d}{\alpha_2}} \bar{u}_1(s+t_0) \right)^{\beta_2} ds. \end{aligned}$$

From here, we have

$$\bar{u}_1 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} \right) t^{\frac{d}{\alpha_2}} \geq \eta + \eta \int_{t_0+T_0}^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} s^{\frac{d\rho_2}{\alpha_2} + \sigma_1 - \frac{d\beta_1}{\alpha_2}} \left( s^{\frac{d}{\alpha_2}} \bar{u}_2(s) \right)^{\beta_1} ds$$

and

$$\bar{u}_2 \left( t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} \right) t^{\frac{d}{\alpha_2}} \geq \eta + \eta \int_{t_0+T_0}^t \frac{\alpha_1}{\alpha_2(\rho_1 \sqrt{\rho_2})} s^{\frac{d\rho_2}{\alpha_2} + \sigma_2 - \frac{d\beta_2}{\alpha_2}} \left( s^{\frac{d}{\alpha_2}} \bar{u}_1(s) \right)^{\beta_2} ds,$$

with  $t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \geq t_0 + T_0$ .

Consider the integral system

$$w_1 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right) = \eta + \eta \int_{t_0+T_0}^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( s^{\theta_1} \wedge s^{\theta_2} \right) w_2^{\beta_1}(s) ds, \quad t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \geq t_0 + T_0, \quad (16)$$

$$w_2 \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right) = \eta + \eta \int_{t_0+T_0}^t \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} \left( s^{\theta_1} \wedge s^{\theta_2} \right) w_1^{\beta_2}(s) ds, \quad t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \geq t_0 + T_0,$$

where  $\theta_i = \frac{d}{\alpha_2}(\rho_2 - \beta_i) + \sigma_i$ ,  $i = 1, 2$ . The differential expression of (17) is

$$\begin{aligned} & \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} - 1} w_1' \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right) \\ &= \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} - 1} \eta \left[ \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right)^{\theta_1} \wedge \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right)^{\theta_2} \right] w_2^{\beta_1} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right), \\ & \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} - 1} w_2' \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right) \\ &= \frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)} - 1} \eta \left[ \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right)^{\theta_1} \wedge \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right)^{\theta_2} \right] w_1^{\beta_2} \left( t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} \right), \end{aligned}$$

$t^{\frac{\alpha_1}{\alpha_2(\rho_1 \vee \rho_2)}} > t_0 + T_0$ , with initial conditions  $w_1(t_0 + T_0) = \eta = w_2(t_0 + T_0)$ , or equivalently

$$\begin{aligned} w_1'(t) &= \eta \left( t^{\theta_1} \wedge t^{\theta_2} \right) w_2^{\beta_1}(t), \quad t > t_0 + T_0, \\ w_2'(t) &= \eta \left( t^{\theta_1} \wedge t^{\theta_2} \right) w_1^{\beta_2}(t), \quad t > t_0 + T_0, \\ w_1(t_0 + T_0) &= \eta = w_2(t_0 + T_0), \end{aligned} \quad (17)$$

whose solution satisfies

$$\frac{w_1^{\beta_2+1}(t) - \eta^{\beta_2+1}}{\beta_2 + 1} = \frac{w_2^{\beta_1+1}(t) - \eta^{\beta_1+1}}{\beta_1 + 1}.$$

If  $\frac{\eta^{\beta_2+1}}{\beta_2+1} \geq \frac{\eta^{\beta_1+1}}{\beta_1+1}$ , then

$$\frac{w_1^{\beta_2+1}(t)}{\beta_2 + 1} \geq \frac{w_2^{\beta_1+1}(t)}{\beta_1 + 1}$$

or, equivalently

$$w_1(t) \geq \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{1}{\beta_2+1}} w_2^{\frac{\beta_1+1}{\beta_2+1}}(t), \quad t \geq t_0 + T_0.$$

Substituting this into the second equation of (17) we obtain

$$w_2'(t) \geq \eta \left( t^{\theta_1} \wedge t^{\theta_2} \right) \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{\beta_2}{\beta_2 + 1}} w_2^{\frac{\beta_2(\beta_1 + 1)}{\beta_2 + 1}}(t), \quad t > t_0 + T_0.$$

Let us suppose (without loss of generality) that  $\theta_1 \leq \theta_2$ , then  $t^{\theta_1} \leq t^{\theta_2}$  and thus,

$$w_2^{-\frac{\beta_2(\beta_1 + 1)}{\beta_2 + 1}}(t) w_2'(t) \geq \eta \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{\beta_2}{\beta_2 + 1}} t^{\theta_1}, \quad t > t_0 + T_0.$$

Integrating from  $t_0 + T_0$  to  $t$  yields

$$\frac{\beta_2 + 1}{1 - \beta_1 \beta_2} \left[ w_2^{\frac{1 - \beta_1 \beta_2}{\beta_2 + 1}}(t) - \eta^{\frac{1 - \beta_1 \beta_2}{\beta_2 + 1}} \right] \geq \eta \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{\beta_2}{\beta_2 + 1}} \int_{t_0 + T_0}^t s^{\theta_1} ds.$$

Thus (recalling that  $\beta_1 \beta_2 > 1$ ), we obtain

$$w_2(t) \geq \left[ \eta^{\frac{1 - \beta_1 \beta_2}{\beta_2 + 1}} - \eta \left( \frac{\beta_2 + 1}{\beta_1 + 1} \right)^{\frac{\beta_2}{\beta_2 + 1}} \int_{t_0 + T_0}^t s^{\theta_1} ds \right]^{\frac{\beta_2 + 1}{1 - \beta_1 \beta_2}}.$$

Since  $\theta_1 + 1 = \frac{d(\rho_2 - \beta_1)}{\alpha_2} + \sigma_1 + 1 \geq 0$ , it follows that  $\int_{t_0 + T_0}^t s^{\theta_1} ds \rightarrow \infty$  when  $t \rightarrow \infty$ . So there exists  $t_e > t_0 + T_0$  such that  $w_2(t) = \infty$  for  $t = t_e$ . By comparison we have

$$\bar{u}_2(t) t^{\frac{d\rho_1}{\alpha_2}} \geq w_2(t) = \infty \quad \text{for } t = t_e,$$

and Lemma 3 implies that  $(u_1, u_2)$  blows up in finite time. In the case when  $\frac{\eta^{\beta_2 + 1}}{\beta_2 + 1} \leq \frac{\eta^{\beta_1 + 1}}{\beta_1 + 1}$ , it can be shown that there exists  $t_e > t_0 + T_0$  such that  $\bar{u}_1(t_e) = \infty$ , which implies again that  $(u_1, u_2)$  blows up in finite time.  $\square$

#### 4. Other results

In this section  $\alpha_i \in (1, 2)$  and  $g_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function in the Kato class  $\mathcal{K}_d^{\alpha_i - 1}$  on  $\mathbb{R}^d$ ,  $i = 1, 2$ .

Let  $p_i^{g_i}(t, y - x)$  be the transition density of the semigroup generated by  $\Delta_{\alpha_i} + g_i(x) \cdot \nabla$ ,  $i = 1, 2$ . It is known (see Bogdan and Jakubowski [2], Theorems 1 and 2) that  $p_i^{g_i}(t, y - x)$  is continuous, and for every  $0 < T < \infty$ , there exists  $C_i = C_i(d, \alpha_i, g_i, T) > 1$  such that

$$C_i^{-1} p_i(t, y - x) \leq p_i^{g_i}(t, y - x) \leq C_i p_i(t, y - x), \quad 0 < t \leq T, \quad x, y \in \mathbb{R}^d, \quad (18)$$

and  $C_i \rightarrow 1$  as  $T \rightarrow 0$ .

The associated integral system to the nonautonomous fractional diffusion-convection-reaction system (11) is given by

$$u_i(t, x) = \int_{\mathbb{R}^d} p_i^{g_i}(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i^{g_i}(K_i(t, s), y - x) h_i(s) u_{i'}^{\beta_i}(s, y) dy ds, \tag{19}$$

$t \geq 0, x \in \mathbb{R}^d$ , where  $i \in \{1, 2\}$  and  $i' = 3 - i$ . A solution of integral system (19) is called a mild solution of (11).

The proof of Theorem 2 is a simple adaptation of the proof given in Pérez [12] for the autonomous case.

*Proof.* Let us fix  $0 < T < \infty$ . From (18) we get

$$\begin{aligned} & C_i^{-1} \left[ \int_{\mathbb{R}^d} p_i(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t, s), y - x) h_i(s) u_{i'}^{\beta_i}(s, y) dy ds \right] \\ & \leq \int_{\mathbb{R}^d} p_i^{g_i}(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i^{g_i}(K_i(t, s), y - x) h_i(s) u_{i'}^{\beta_i}(s, y) dy ds \\ & \leq C_i \left[ \int_{\mathbb{R}^d} p_i(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t, s), y - x) h_i(s) u_{i'}^{\beta_i}(s, y) dy ds \right], \end{aligned}$$

for all  $t \geq 0$  such that  $K_i(t, 0) \leq T, i = 1, 2$ .

Letting the integral system

$$v_i(t, x) = \int_{\mathbb{R}^d} p_i(K_i(t, 0), y - x) \varphi_i(y) dy + \int_0^t \int_{\mathbb{R}^d} p_i(K_i(t, s), y - x) h_i(s) v_{i'}^{\beta_i}(s, y) dy ds,$$

$t \geq 0, x \in \mathbb{R}^d, i = 1, 2$ , we have that  $(v_1, v_2)$  is a nonnegative mild solution of the nonautonomous reaction-diffusion system (1). Thus, by comparison

$$C_i^{-1} v_i(t, x) \leq u_i(t, x) \leq C_i v_i(t, x),$$

for all  $t \geq 0$  such that  $K_i(t, 0) \leq T$ . Therefore  $(u_1, u_2)$  blows up in finite time if and only if  $(v_1, v_2)$  blows up in finite time.  $\square$

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