

## ESTIMATES INVOLVING THE $\omega$ -RIEMANN—LIOUVILLE FRACTIONAL INTEGRAL OPERATORS BY MEANS OF $\eta$ -QUASICONVEXITY WITH APPLICATIONS TO MEANS

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*Abstract.* Since not every quasiconvex function is convex, it is our purpose in this present paper to extend some already established inequalities of the Hermite–Hadamard–Fejér type and its companions for convex functions to the class of  $\eta$ -quasiconvex functions. The new results obtained herein are in terms of the  $\omega$ -Riemann–Liouville fractional integral operators and they reduce to inequalities for quasiconvex functions for a particular choice of the bifunction  $\eta$ . In addition, we apply some of our results to certain special means of positive real numbers to obtain more estimates in this regard.

### 1. Introduction

In the 17th century, the french mathematicians Charles Hermite [8] and Jacques S. Hadamard [7] stipulate the following two-sided estimate of the mean value of a continuous convex function  $h : [\alpha, \beta] \rightarrow \mathbb{R}$ :

$$h\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(r) dr \leq \frac{h(\alpha) + h(\beta)}{2}. \quad (1)$$

The above inequality, known as the Hermite–Hadamard inequality, has motivated loads of papers in this direction. In 1906, the award winning mathematician Lipót Fejér established the following weighted version of (1) in the following theorem:

**THEOREM 1.** ([4]) *Suppose  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  is a nonnegative integrable function with  $g(x) = g(\alpha + \beta - x)$ . If  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  is a convex function on  $[\alpha, \beta]$ , then the following inequalities hold:*

$$h\left(\frac{\alpha + \beta}{2}\right) \int_{\alpha}^{\beta} g(r) dr \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(r)g(r) dr \leq \frac{h(\alpha) + h(\beta)}{2} \int_{\alpha}^{\beta} g(r) dr. \quad (2)$$

In 2013, Sarikaya et al. [24] obtained the following fractional version of (1):

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**THEOREM 2.** Let  $\varepsilon > 0$  and  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq \alpha < \beta$  and  $h \in L([\alpha, \beta])$ . If  $h$  is a convex function on  $[\alpha, \beta]$ , then the following inequalities hold:

$$h\left(\frac{\alpha + \beta}{2}\right) \leq \frac{\Gamma(\varepsilon + 1)}{2(\beta - \alpha)^\varepsilon} \left[ \mathbf{I}_{\alpha^+}^\varepsilon h(\beta) + \mathbf{I}_{\beta^-}^\varepsilon h(\alpha) \right] \leq \frac{h(\alpha) + h(\beta)}{2},$$

where the Riemann–Liouville fractional integrals,  $\mathbf{I}_{\alpha^+}^\varepsilon$  and  $\mathbf{I}_{\beta^-}^\varepsilon$ , are defined as thus:

$$\mathbf{I}_{\alpha^+}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_\alpha^x (x-r)^{\varepsilon-1} h(r) dr$$

and

$$\mathbf{I}_{\beta^-}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^\beta (r-x)^{\varepsilon-1} h(r) dr.$$

Here,  $\Gamma(\varepsilon)$  is the Gamma function defined by  $\Gamma(\varepsilon) = \int_0^\infty e^{-x} x^{\varepsilon-1} dx$ .

More papers in this sense can be found in [9, 17]. We now recall the definition of a function integrated with respect to another function in the fractional sense:

**DEFINITION 1.** ([13]) Let  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  be an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . The left and right-sided fractional integral of  $h$  with respect to the function  $\omega$  on  $[\alpha, \beta]$  of order  $\varepsilon > 0$  are defined respectively by:

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_\alpha^x \frac{\omega'(r)}{[\omega(x) - \omega(r)]^{1-\varepsilon}} h(r) dr, \quad x > \alpha$$

and

$$\mathfrak{J}_{\beta^-; \omega}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^\beta \frac{\omega'(r)}{[\omega(r) - \omega(x)]^{1-\varepsilon}} h(r) dr, \quad x < \beta.$$

**REMARK 1.** In view of the above definition, we make the following observations that will aid the readability of this article.

1. If  $\omega(x) = x$ , then

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon h(x) = \mathbf{I}_{\alpha^+}^\varepsilon h(x) \quad \text{and} \quad \mathfrak{J}_{\beta^-; \omega}^\varepsilon h(x) = \mathbf{I}_{\beta^-}^\varepsilon h(x).$$

2. Let  $\omega(x) = \ln x$ . Then the fractional operators become the Hadamard fractional integrals,  $\mathbf{J}_{\alpha^+}^\varepsilon$  and  $\mathbf{J}_{\beta^-}^\varepsilon$ , defined as follows:

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon h(x) = \mathbf{J}_{\alpha^+}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_\alpha^x \left(\ln \frac{x}{r}\right)^{\varepsilon-1} \frac{h(r)}{r} dr$$

and

$$\mathfrak{J}_{\beta^-; \omega}^\varepsilon h(x) = \mathbf{J}_{\beta^-}^\varepsilon h(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^\beta \left(\ln \frac{r}{x}\right)^{\varepsilon-1} \frac{h(r)}{r} dr.$$

Using the operators given in Definition 1, Budak [2] recently established the following inequalities of the Hermite–Hadamard–Fejér type:

**THEOREM 3.** ([2]) *Let  $\varepsilon > 0$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta)$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$  and  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  a nonnegative integrable function. If  $h$  is a convex function on  $[\alpha, \beta]$ , then we have the following Hermite–Hadamard–Fejér type inequality for generalized fractional integrals:*

$$\begin{aligned} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right] h\left(\frac{\alpha + \beta}{2}\right) &\leq \frac{1}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon gH(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon gH(\alpha) \right] \\ &\leq \frac{h(\alpha) + h(\beta)}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right], \end{aligned}$$

where

$$H(x) = h(x) + \tilde{h}(x) \quad \text{and} \quad \tilde{h}(x) = h(\alpha + \beta - x). \quad (3)$$

In the same paper, Budak established the following trapezoid type result:

**THEOREM 4.** ([2]) *Let  $\varepsilon > 0$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta)$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $|h'|$  is a convex function on  $[\alpha, \beta]$ , then we have the following trapezoid type inequality for generalized fractional integrals:*

$$\begin{aligned} &\left| \frac{h(\alpha) + h(\beta)}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon gH(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon gH(\alpha) \right] \right| \\ &\leq \frac{K_\omega^\varepsilon(\alpha, \alpha) + K_\omega^\varepsilon(\beta, \beta)}{2\Gamma(\varepsilon)(\beta - \alpha)} \left[ |h'(\alpha)| + |h'(\beta)| \right], \end{aligned}$$

where the function  $K_\omega^\varepsilon : [\alpha, \beta] \times [\alpha, \beta] \rightarrow \mathbb{R}$  is defined by

$$\begin{aligned} K_\omega^\varepsilon(x, y) &:= \int_\alpha^{\frac{\alpha+\beta}{2}} \left( \int_r^{\alpha+\beta-r} \frac{\omega'(s)g(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) |x - r| dr \\ &\quad + \int_{\frac{\alpha+\beta}{2}}^\beta \left( \int_{\alpha+\beta-r}^r \frac{\omega'(s)g(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) |x - r| dr. \end{aligned} \quad (4)$$

Recently, Liu et al. [15] proved the following two results:

**THEOREM 5.** *Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta)$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $|h'|$  is convex on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\begin{aligned} &\left| \frac{h(\alpha) + h(\beta)}{2} - \frac{\Gamma(\varepsilon + 1)}{2(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega)(\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega)(\omega^{-1}(\alpha)) \right] \right| \\ &\leq \frac{\beta - \alpha}{2(\varepsilon + 1)} \left( 1 - \frac{1}{2^\varepsilon} \right) \left[ |h'(\alpha)| + |h'(\beta)| \right]. \end{aligned}$$

**THEOREM 6.** *Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta)$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $|h'|$  is convex on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\begin{aligned} & \left| \frac{\Gamma(\varepsilon+1)}{2(\beta-\alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+, \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-, \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] - h \left( \frac{\alpha+\beta}{2} \right) \right| \\ & \leq \frac{|h(\beta) - h(\alpha)|}{2} + \frac{\beta - \alpha}{2(\varepsilon+1)} \left( 1 - \frac{1}{2^\varepsilon} \right) [ |h'(\alpha)| + |h'(\beta)| ]. \end{aligned}$$

Next, we recall the notion of quasiconvexity.

**DEFINITION 2.** A function  $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is called quasiconvex on the interval  $I$ , if

$$h(\tau x + (1 - \tau)y) \leq \max \{h(x), h(y)\}$$

for all  $x, y \in I$  and  $\tau \in [0, 1]$ .

Not too long ago, Gordji et al. [6] further generalized the class of quasiconvex functions in the following manner:

**DEFINITION 3.** ([6]) A function  $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is called  $\eta$ -quasiconvex on  $I$  with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , if

$$h(\tau x + (1 - \tau)y) \leq \max \{h(y), h(y) + \eta(h(x), h(y))\}$$

for all  $x, y \in I$  and  $\tau \in [0, 1]$ .

Inspired by the above mentioned articles, it is our purpose, in this paper, to extend Theorems 3–6 to the class of  $\eta$ -quasiconvex functions. It is generally known that this class of functions contains strictly the class of convex functions, and thus the inequalities mentioned in the above theorems may not hold for those functions that are quasiconvex but not convex. In view of this limitation, it is of interest to establish estimates for this family of functions involving the  $\omega$ -Riemann–Liouville fractional integral operators. Results with the Riemann–Liouville and Hadamard fractional integral operators can be deduced as special cases of our theorems by taking  $\omega(x) = x$  and  $\omega(x) = \ln x$ , respectively. Besides, we also obtained more inequalities in this direction and applied them to some special means of real numbers. For some recent results involving these generalized fractional integral operators, we invite the interested reader to see the papers [16, 10] and the references cited therein.

This paper is organized in the following fashion: in Section 2, we frame and give proofs to our main results. Some applications to certain special means are presented in Section 3, followed by a brief conclusion in the last section.

### 2. Main results

In the sequel, we will make use of the following notations: for any  $\eta$ -quasiconvex function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$ , we denote

$$\mathcal{P}_\alpha^\beta(f; \eta) := \max \{f(\alpha), f(\alpha) + \eta(f(\beta), f(\alpha))\} \tag{5}$$

and

$$\mathcal{Q}_\alpha^\beta(f; \eta) := \max \{f(\beta), f(\beta) + \eta(f(\alpha), f(\beta))\}. \tag{6}$$

Now, we state and prove a right-sided inequality of the Hermite–Hadamard–Fejér type.

**THEOREM 7.** *Let  $\varepsilon > 0$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$  and  $g \in L([\alpha, \beta])$  a nonnegative function. If  $h \in L([\alpha, \beta])$  is an  $\eta$ -quasiconvex function on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon(gH)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon(gH)(\alpha) \leq \left( \mathfrak{J}_{\alpha^+; \omega}^\varepsilon(g)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon(g)(\alpha) \right) \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right],$$

where the function  $H$  is defined by (3).

*Proof.* Using the  $\eta$ -quasiconvexity of  $h$ , one gets that for all  $\tau \in [0, 1]$ , the following inequalities hold:

$$h(\tau\beta + (1 - \tau)\alpha) \leq \mathcal{P}_\alpha^\beta(h; \eta) \tag{7}$$

and

$$h(\tau\alpha + (1 - \tau)\beta) \leq \mathcal{Q}_\alpha^\beta(h; \eta). \tag{8}$$

Adding (7) and (8) gives:

$$h(\tau\beta + (1 - \tau)\alpha) + h(\tau\alpha + (1 - \tau)\beta) \leq \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta). \tag{9}$$

Now multiplying both sides of (9) by

$$\frac{\beta - \alpha}{\Gamma(\varepsilon)} \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha)$$

and integrate the resultant inequality over  $[0, 1]$  to give:

$$\begin{aligned} & \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\beta + (1 - \tau)\alpha) \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau \\ & + \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\alpha + (1 - \tau)\beta) \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau \\ & \leq \frac{\beta - \alpha}{\Gamma(\varepsilon)} \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \int_0^1 \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau. \end{aligned} \tag{10}$$

If we substitute  $x = \tau\beta + (1 - \tau)\alpha$ , then  $d\tau = \frac{1}{\beta - \alpha} dx$ ,  $\alpha + \beta - x = \tau\alpha + (1 - \tau)\beta$ . Using this change of variable, we get:

$$\begin{aligned} & \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\beta + (1 - \tau)\alpha) \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau \\ &= \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_\alpha^\beta h(x) \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} g(x) \frac{1}{\beta - \alpha} dx \\ &= \frac{1}{\Gamma(\varepsilon)} \int_\alpha^\beta \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} h(x) g(x) dx \\ &= \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gh)(\beta), \end{aligned} \tag{11}$$

$$\begin{aligned} & \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\alpha + (1 - \tau)\beta) \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau \\ &= \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 \tilde{h}(\tau\beta + (1 - \tau)\alpha) \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau \\ &= \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (g\tilde{h})(\beta) \end{aligned} \tag{12}$$

and

$$\frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 \frac{\omega'(\tau\beta + (1 - \tau)\alpha)}{[\omega(\beta) - \omega(\tau\beta + (1 - \tau)\alpha)]^{1-\varepsilon}} g(\tau\beta + (1 - \tau)\alpha) d\tau = \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (g)(\beta). \tag{13}$$

Using (11), (12) and (13) in (10), we obtain:

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gh)(\beta) + \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (g\tilde{h})(\beta) \leq \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (g)(\beta). \tag{14}$$

Equivalently,

$$\mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gH)(\beta) \leq \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (g)(\beta). \tag{15}$$

Similarly, one gets, by multiplying (9) by

$$\frac{\beta - \alpha}{\Gamma(\varepsilon)} \frac{\omega'(\tau\alpha + (1 - \tau)\beta)}{[\omega(\tau\alpha + (1 - \tau)\beta) - \omega(\alpha)]^{1-\varepsilon}} g(\tau\alpha + (1 - \tau)\beta)$$

and integrating the resultant inequality over  $[0, 1]$ , the succeeding inequality:

$$\begin{aligned} & \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\beta + (1 - \tau)\alpha) \frac{\omega'(\tau\alpha + (1 - \tau)\beta)}{[\omega(\tau\alpha + (1 - \tau)\beta) - \omega(\alpha)]^{1-\varepsilon}} g(\tau\alpha + (1 - \tau)\beta) d\tau \\ &+ \frac{\beta - \alpha}{\Gamma(\varepsilon)} \int_0^1 h(\tau\alpha + (1 - \tau)\beta) \frac{\omega'(\tau\alpha + (1 - \tau)\beta)}{[\omega(\tau\alpha + (1 - \tau)\beta) - \omega(\alpha)]^{1-\varepsilon}} g(\tau\alpha + (1 - \tau)\beta) d\tau \\ &\leq \frac{\beta - \alpha}{\Gamma(\varepsilon)} \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \int_0^1 \frac{\omega'(\tau\alpha + (1 - \tau)\beta)}{[\omega(\tau\alpha + (1 - \tau)\beta) - \omega(\alpha)]^{1-\varepsilon}} g(\tau\alpha + (1 - \tau)\beta) d\tau. \end{aligned} \tag{16}$$

Changing variable to  $y = \tau\alpha + (1 - \tau)\beta$  and applying Definition 1, we get:

$$\mathfrak{J}_{\beta^-; \omega}^\varepsilon(g\tilde{h})(\alpha) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon(gh)(\alpha) \leq \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \mathfrak{J}_{\beta^-; \omega}^\varepsilon(g)(\alpha). \tag{17}$$

That is;

$$\mathfrak{J}_{\beta^-; \omega}^\varepsilon(gH)(\alpha) \leq \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \mathfrak{J}_{\beta^-; \omega}^\varepsilon(g)(\alpha). \tag{18}$$

The desired inequality is established by adding (15) and (18).

The next theorem contains a right-sided Hermite–Hadamard type inequality.

**THEOREM 8.** *Let  $\varepsilon \in (0, 1)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $h \in L([\alpha, \beta])$  is a positive  $\eta$ -quasiconvex function on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\begin{aligned} & \frac{\Gamma(\varepsilon + 1)}{(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon(h \circ \omega)(\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon(h \circ \omega)(\omega^{-1}(\alpha)) \right] \\ & \leq \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta). \end{aligned}$$

*Proof.* To obtain the desired inequality, we multiply both sides of (9) by  $\tau^{\varepsilon-1}$  and integrate over  $\tau$  on the interval  $[0, 1]$  to get:

$$\begin{aligned} & \int_0^1 \tau^{\varepsilon-1} h(\tau\beta + (1 - \tau)\alpha) d\tau + \int_0^1 \tau^{\varepsilon-1} h(\tau\alpha + (1 - \tau)\beta) d\tau \\ & \leq \left[ \mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta) \right] \int_0^1 \tau^{\varepsilon-1} d\tau \\ & = \left[ \frac{\mathcal{P}_\alpha^\beta(h; \eta) + \mathcal{Q}_\alpha^\beta(h; \eta)}{\varepsilon} \right]. \end{aligned} \tag{19}$$

Next, we observe that

$$\begin{aligned} & \frac{\Gamma(\varepsilon)}{(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon(h \circ \omega)(\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon(h \circ \omega)(\omega^{-1}(\alpha)) \right] \\ & = \frac{\Gamma(\varepsilon)}{(\beta - \alpha)^\varepsilon} \frac{1}{\Gamma(\varepsilon)} \left[ \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \frac{\omega'(r)}{[\beta - \omega(r)]^{1-\varepsilon}} h(\omega(r)) dr \right. \\ & \quad \left. + \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \frac{\omega'(r)}{[\omega(r) - \alpha]^{1-\varepsilon}} h(\omega(r)) dr \right] \\ & = \left[ \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \left( \frac{\beta - \alpha}{\beta - \omega(r)} \right)^{1-\varepsilon} h(\omega(r)) \frac{\omega'(r)}{\beta - \alpha} dr \right. \\ & \quad \left. + \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \left( \frac{\beta - \alpha}{\omega(r) - \alpha} \right)^{1-\varepsilon} h(\omega(r)) \frac{\omega'(r)}{\beta - \alpha} dr \right] \end{aligned} \tag{20}$$

$$\begin{aligned} &= \int_0^1 \tau^{\varepsilon-1} h(\tau\alpha + (1-\tau)\beta) d\tau + \int_0^1 t^{\varepsilon-1} h(t\beta + (1-t)\alpha) dt \\ &\quad \left[ \text{with } \tau = \frac{\beta - \omega(t)}{\beta - \alpha} \right] \quad \left[ \text{with } t = \frac{\omega(r) - \alpha}{\beta - \alpha} \right] \\ &= \int_0^1 \tau^{\varepsilon-1} h(\tau\beta + (1-\tau)\alpha) d\tau + \int_0^1 \tau^{\varepsilon-1} h(\tau\alpha + (1-\tau)\beta) d\tau. \end{aligned}$$

Substituting (20) into the left hand side of (19), one gets the desired result.

If we let  $\eta = (x - y)$ , we then obtain the following corollary:

**COROLLARY 1.** *Let  $\varepsilon \in (0, 1)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $h \in L([\alpha, \beta])$  is a positive quasiconvex function on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\begin{aligned} &\frac{\Gamma(\varepsilon + 1)}{2(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] \\ &\leq \max \{ h(\alpha), h(\beta) \}. \end{aligned}$$

The following lemma will be useful in the proof of the next result.

**LEMMA 1.** ([2]) *Let  $\varepsilon > 0$  and let the mappings  $\omega$  and  $g$  be as in Theorem 7. If  $h : [\alpha, \beta] \rightarrow \mathbb{R}$  is a differentiable mapping on  $(\alpha, \beta)$ , then the following identity for generalized fractional integrals holds:*

$$\begin{aligned} &\frac{h(\alpha) + h(\beta)}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gH)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon (gH)(\alpha) \right] \\ &= \frac{1}{2\Gamma(\varepsilon)} \int_\alpha^\beta L_{g; \omega}(r) h'(r) dr, \end{aligned}$$

where the mapping  $L_{g; \omega} : [\alpha, \beta] \rightarrow \mathbb{R}$  is defined by

$$L_{g; \omega}(r) := \int_{\alpha+\beta-r}^r \frac{\omega'(x)}{[\omega(x) - \omega(\alpha)]^{1-\varepsilon}} g(x) dx + \int_{\alpha+\beta-r}^r \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} g(x) dx.$$

**REMARK 2.** Since  $\omega$  is an increasing function on  $[\alpha, \beta]$  and  $g$  is nonnegative, it follows therefore from the definition of  $L_{g; \omega}(r)$  that:

$$L_{g; \omega}(\alpha) = - \int_\alpha^\beta \frac{\omega'(x)}{[\omega(x) - \omega(\alpha)]^{1-\varepsilon}} g(x) dx - \int_\alpha^\beta \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} g(x) dx < 0$$

and

$$L_{g; \omega} \left( \frac{\alpha + \beta}{2} \right) = 0.$$

Hence,

$$L_{g; \omega}(r) \begin{cases} \leq 0 & \text{if } \alpha \leq r \leq \frac{\alpha + \beta}{2} \\ > 0 & \text{if } \frac{\alpha + \beta}{2} < r \leq \beta. \end{cases}$$

**THEOREM 9.** Let  $\varepsilon > 0$  and let the mappings  $\omega$  and  $g$  be as in Theorem 7. If  $|h'|$  is  $\eta$ -quasiconvex on  $[\alpha, \beta]$ , then the following inequality of the trapezoid type holds:

$$\begin{aligned} & \left| \frac{h(\alpha) + h(\beta)}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gH)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon (gH)(\alpha) \right] \right| \\ & \leq \frac{\Theta_\omega^\varepsilon(\alpha) + \Theta_\omega^\varepsilon(\beta)}{2\Gamma(\varepsilon)} \mathcal{Q}_\alpha^\beta(|h'|; \eta), \end{aligned} \quad (21)$$

where the function  $\Theta_\omega^\varepsilon : [\alpha, \beta] \rightarrow \mathbb{R}$  is defined by

$$\begin{aligned} \Theta_\omega^\varepsilon(y) := & \int_\alpha^{\frac{\alpha+\beta}{2}} \left( \int_r^{\alpha+\beta-r} \frac{\omega'(s)g(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) dr \\ & + \int_{\frac{\alpha+\beta}{2}}^\beta \left( \int_{\alpha+\beta-r}^r \frac{\omega'(s)g(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) dr. \end{aligned} \quad (22)$$

*Proof.* Using the  $\eta$ -quasiconvexity of  $|h'|$  on  $[\alpha, \beta]$ , we get the following inequality:

$$|h'(r)| = \left| h' \left[ \frac{\beta-r}{\beta-\alpha} \alpha + \left( 1 - \frac{\beta-r}{\beta-\alpha} \right) \beta \right] \right| \leq \mathcal{Q}_\alpha^\beta(|h'|; \eta). \quad (23)$$

From Lemma 1 and (23), one obtains:

$$\begin{aligned} & \left| \frac{h(\alpha) + h(\beta)}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon g(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon (gH)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon (gH)(\alpha) \right] \right| \\ & \leq \frac{1}{2\Gamma(\varepsilon)} \int_\alpha^\beta |L_{g; \omega}(r)| |h'(r)| dr \leq \frac{\mathcal{Q}_\alpha^\beta(|h'|; \eta)}{2\Gamma(\varepsilon)} \int_\alpha^\beta |L_{g; \omega}(r)| dr. \end{aligned} \quad (24)$$

Now, using Remark 2 and (22), we obtain:

$$\begin{aligned} \int_\alpha^\beta |L_{g; \omega}(r)| dr &= \int_\alpha^{\frac{\alpha+\beta}{2}} -L_{g; \omega}(r) dr + \int_{\frac{\alpha+\beta}{2}}^\beta L_{g; \omega}(r) dr \\ &= \int_\alpha^{\frac{\alpha+\beta}{2}} \left[ \int_r^{\alpha+\beta-r} \frac{\omega'(x)}{[\omega(x) - \omega(\alpha)]^{1-\varepsilon}} g(x) dx \right] dr \\ & \quad + \int_{\frac{\alpha+\beta}{2}}^\beta \left[ \int_{\alpha+\beta-r}^r \frac{\omega'(x)}{[\omega(x) - \omega(\alpha)]^{1-\varepsilon}} g(x) dx \right] dr \\ & \quad + \int_\alpha^{\frac{\alpha+\beta}{2}} \left[ \int_r^{\alpha+\beta-r} \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} g(x) dx \right] dr \\ & \quad + \int_{\frac{\alpha+\beta}{2}}^\beta \left[ \int_{\alpha+\beta-r}^r \frac{\omega'(x)}{[\omega(\beta) - \omega(x)]^{1-\varepsilon}} g(x) dx \right] dr \\ &= \Theta_\omega^\varepsilon(\alpha) + \Theta_\omega^\varepsilon(\beta). \end{aligned} \quad (25)$$

The intended result is achieved by using (25) in (24).

REMARK 3. Let  $\eta(x, y) = x - y$ . From Theorem 9, we deduce the following inequalities:

1. If  $g(x) = 1$ , then (21) becomes:

$$\begin{aligned} & \left| \frac{h(\alpha) + h(\beta)}{2} - \frac{\Gamma(\varepsilon + 1)}{4[\omega(\beta) - \omega(\alpha)]^\varepsilon} \left[ \mathfrak{J}_{\alpha^+; \omega}^\varepsilon(H)(\beta) + \mathfrak{J}_{\beta^-; \omega}^\varepsilon(H)(\alpha) \right] \right| \\ & \leq \frac{\Phi_\omega^\varepsilon(\alpha) + \Phi_\omega^\varepsilon(\beta)}{4\varepsilon^{-1}[\omega(\beta) - \omega(\alpha)]^\varepsilon} \max \{ |h'(\alpha)|, |h'(\beta)| \}, \end{aligned}$$

where

$$\begin{aligned} \Phi_\omega^\varepsilon(y) := \Theta_\omega^\varepsilon(y)|_{g \equiv 1} &= \int_\alpha^{\frac{\alpha+\beta}{2}} \left( \int_r^{\alpha+\beta-r} \frac{\omega'(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) dr \\ &+ \int_{\frac{\alpha+\beta}{2}}^\beta \left( \int_{\alpha+\beta-r}^r \frac{\omega'(s)}{|\omega(y) - \omega(s)|^{1-\varepsilon}} ds \right) dr. \end{aligned}$$

2. Suppose  $\omega(x) = k(x) = x$ . Then (21) amounts to the following Riemann–Liouville fractional inequality:

$$\begin{aligned} & \left| \frac{h(\alpha) + h(\beta)}{2} \left[ \mathbf{I}_{\alpha^+}^\varepsilon g(\beta) + \mathbf{I}_{\beta^-}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathbf{I}_{\alpha^+}^\varepsilon (gH)(\beta) + \mathbf{I}_{\beta^-}^\varepsilon (gH)(\alpha) \right] \right| \\ & \leq \frac{\Theta_k^\varepsilon(\alpha) + \Theta_k^\varepsilon(\beta)}{2\Gamma(\varepsilon)} \max \{ |h'(\alpha)|, |h'(\beta)| \}, \end{aligned}$$

where

$$\Theta_k^\varepsilon(y) := \int_\alpha^{\frac{\alpha+\beta}{2}} \left( \int_r^{\alpha+\beta-r} \frac{g(s)}{|y-s|^{1-\varepsilon}} ds \right) dr + \int_{\frac{\alpha+\beta}{2}}^\beta \left( \int_{\alpha+\beta-r}^r \frac{g(s)}{|y-s|^{1-\varepsilon}} ds \right) dr.$$

3. Suppose  $\omega(x) = \ln x$ . Then (21) reduces to the following Hadamard fractional inequality:

$$\begin{aligned} & \left| \frac{h(\alpha) + h(\beta)}{2} \left[ \mathbf{J}_{\alpha^+}^\varepsilon g(\beta) + \mathbf{J}_{\beta^-}^\varepsilon g(\alpha) \right] - \frac{1}{2} \left[ \mathbf{J}_{\alpha^+}^\varepsilon (gH)(\beta) + \mathbf{J}_{\beta^-}^\varepsilon (gH)(\alpha) \right] \right| \\ & \leq \frac{\Theta_{\ln}^\varepsilon(\alpha) + \Theta_{\ln}^\varepsilon(\beta)}{2\Gamma(\varepsilon)} \max \{ |h'(\alpha)|, |h'(\beta)| \}, \end{aligned}$$

where

$$\begin{aligned} \Theta_{\ln}^\varepsilon(y) &:= \int_\alpha^{\frac{\alpha+\beta}{2}} \left( \int_r^{\alpha+\beta-r} \left[ \ln \left( \frac{y}{s} \right) \right]^{\varepsilon-1} \frac{g(s)}{s} ds \right) dr \\ &+ \int_{\frac{\alpha+\beta}{2}}^\beta \left( \int_{\alpha+\beta-r}^r \left[ \ln \left( \frac{y}{s} \right) \right]^{\varepsilon-1} \frac{g(s)}{s} ds \right) dr. \end{aligned}$$

For the rest of our theorems, the following two lemmas will be useful.

LEMMA 2. ([15]) Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $h' \in L([\alpha, \beta])$ ,  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . Then we have the succeeding equality for generalized fractional integrals:

$$\begin{aligned} & \frac{h(\alpha)+h(\beta)}{2} - \frac{\Gamma(\varepsilon+1)}{2(\beta-\alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] \\ &= \frac{1}{2(\beta-\alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \left[ (\omega(r) - \alpha)^\varepsilon - (\beta - \omega(r))^\varepsilon \right] (h' \circ \omega)(r) \omega'(r) dr. \end{aligned}$$

LEMMA 3. ([15]) Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $h' \in L([\alpha, \beta])$ ,  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . Then we have the succeeding equality for generalized fractional integrals:

$$\begin{aligned} & \frac{\Gamma(\varepsilon+1)}{2(\beta-\alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] - h \left( \frac{\alpha+\beta}{2} \right) \\ &= \frac{1}{2(\beta-\alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \left[ (\beta - \omega(r))^\varepsilon - (\omega(r) - \alpha)^\varepsilon \right] (h' \circ \omega)(r) \omega'(r) dr \\ & \quad + \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} p(h' \circ \omega)(r) \omega'(r) dr, \end{aligned}$$

where

$$p = \begin{cases} \frac{1}{2}, & \omega^{-1} \left( \frac{\alpha+\beta}{2} \right) \leq r \leq \omega^{-1}(\beta) \\ -\frac{1}{2}, & \omega^{-1}(\alpha) < r < \omega^{-1} \left( \frac{\alpha+\beta}{2} \right). \end{cases} \tag{26}$$

We are now ready to state and prove our last two theorems.

THEOREM 10. Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $|h'|$  is  $\eta$ -quasiconvex on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:

$$\begin{aligned} & \left| \frac{h(\alpha)+h(\beta)}{2} - \frac{\Gamma(\varepsilon+1)}{2(\beta-\alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] \right| \\ & \leq \frac{\beta-\alpha}{\varepsilon+1} \left( 1 - \frac{1}{2^\varepsilon} \right) \mathcal{Q}_\alpha^\beta(|h'|; \eta). \end{aligned}$$

*Proof.* We start by observing that for every  $r \in (\omega^{-1}(\alpha), \omega^{-1}(\beta))$ , we have  $\alpha < \omega(r) < \beta$ . Letting  $\tau = \frac{\beta-\omega(r)}{\beta-\alpha}$ , then  $\omega(r) = \alpha\tau + (1-\tau)\beta$ . Using Lemma 2 and the  $\eta$ -quasiconvexity of  $|h'|$ , we get

$$\left| \frac{h(\alpha)+h(\beta)}{2} - \frac{\Gamma(\varepsilon+1)}{2(\beta-\alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] \right|$$

$$\begin{aligned}
&\leq \frac{1}{2(\beta - \alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} |(\omega(r) - \alpha)^\varepsilon - (\beta - \omega(r))^\varepsilon| |(h' \circ \omega)(r)| |\omega'(r)| dr \\
&= \frac{1}{2(\beta - \alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} |(\omega(r) - \alpha)^\varepsilon - (\beta - \omega(r))^\varepsilon| |(h' \circ \omega)(r)| d\phi(r) \\
&= \frac{\beta - \alpha}{2} \int_0^1 |(1 - \tau)^\varepsilon - \tau^\varepsilon| |h'(\alpha\tau + (1 - \tau)\beta)| d\tau \\
&\leq \frac{\beta - \alpha}{2} \left( \int_0^1 |(1 - \tau)^\varepsilon - \tau^\varepsilon| d\tau \right) \mathcal{Q}_\alpha^\beta(|h'|; \eta).
\end{aligned}$$

To finish up, we need to compute  $\int_0^1 |(1 - \tau)^\varepsilon - \tau^\varepsilon| d\tau$ . To do this, we note that:

$$(1 - \tau)^\varepsilon - \tau^\varepsilon \begin{cases} > 0, & \text{if } 0 \leq \tau < \frac{1}{2} \\ = 0, & \text{if } \tau = \frac{1}{2} \\ < 0, & \text{if } \frac{1}{2} < \tau \leq 1. \end{cases} \quad (27)$$

Now, using (27), we write:

$$\begin{aligned}
\int_0^1 |(1 - \tau)^\varepsilon - \tau^\varepsilon| d\tau &= \int_0^{\frac{1}{2}} [(1 - \tau)^\varepsilon - \tau^\varepsilon] d\tau + \int_{\frac{1}{2}}^1 [\tau^\varepsilon - (1 - \tau)^\varepsilon] d\tau \\
&= \frac{1}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right) + \frac{1}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right) \\
&= \frac{2}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right).
\end{aligned} \quad (28)$$

The proof is complete by using (28).

**THEOREM 11.** *Let  $\varepsilon \in (0, 1)$  and  $h : [\alpha, \beta]$  be a differentiable function on  $(\alpha, \beta)$ . Suppose  $\omega : [\alpha, \beta] \rightarrow \mathbb{R}$  is an increasing and positive monotone function on  $(\alpha, \beta]$  having a continuous derivative  $\omega'(x)$  on  $(\alpha, \beta)$ . If  $|h'|$  is  $\eta$ -quasiconvex on  $[\alpha, \beta]$ , then we have the succeeding inequality for generalized fractional integrals:*

$$\begin{aligned}
&\left| \frac{\Gamma(\varepsilon + 1)}{2(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] - h \left( \frac{\alpha + \beta}{2} \right) \right| \\
&\leq \frac{\beta - \alpha}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right) \mathcal{Q}_\alpha^\beta(|h'|; \eta) + \frac{|h(\beta) - h(\alpha)|}{2}.
\end{aligned}$$

*Proof.* Taking the absolute value of both sides of the identity in Lemma 3 and utilizing the  $\eta$ -quasiconvexity of  $|h'|$  on  $[\alpha, \beta]$  to get:

$$\begin{aligned}
&\left| \frac{\Gamma(\varepsilon + 1)}{2(\beta - \alpha)^\varepsilon} \left[ \mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) + \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) \right] - h \left( \frac{\alpha + \beta}{2} \right) \right| \\
&= \left| \frac{1}{2(\beta - \alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} [(\beta - \omega(r))^\varepsilon - (\omega(r) - \alpha)^\varepsilon] (h' \circ \omega)(r) \omega'(r) dr \right|
\end{aligned}$$

$$\begin{aligned}
 & + \left| \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} p(h' \circ \omega)(r) \omega'(r) dr \right| \\
 \leq & \frac{1}{2(\beta - \alpha)^\varepsilon} \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} \left| (\beta - \omega(r))^\varepsilon - (\omega(r) - \alpha)^\varepsilon \right| |(h' \circ \omega)(r) \omega'(r)| dr \\
 & + \left| \int_{\omega^{-1}(\alpha)}^{\omega^{-1}(\beta)} p(h' \circ \omega)(r) \omega'(r) dr \right| \\
 \leq & \frac{\beta - \alpha}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right) \mathcal{Q}_\alpha^\beta(|h'|; \eta) + \left| p(\beta - \alpha) \int_0^1 h'(\alpha\tau + (1 - \tau)\beta) d\tau \right| \\
 = & \frac{\beta - \alpha}{\varepsilon + 1} \left( 1 - \frac{1}{2^\varepsilon} \right) \mathcal{Q}_\alpha^\beta(|h'|; \eta) + |p| |h(\beta) - h(\alpha)|,
 \end{aligned}$$

from which the desired result is deduced.

### 3. Applications

In this section, we present some applications of some of our results to the following special means of  $u, v \in \mathbb{R}$ .

1. Arithmetic mean:

$$\mathcal{A}(u, v) = \frac{u + v}{2}.$$

2. Harmonic mean:

$$\mathcal{H}(u, v) = \frac{2uv}{u + v}.$$

3. Logarithmic mean:

$$\mathcal{L}(u, v) = \frac{u - v}{\ln|u| - \ln|v|}, \quad |u| \neq |v|, \quad u, v \neq 0.$$

4. Generalized logarithmic mean:

$$\mathcal{L}_m(u, v) = \left[ \frac{v^{m+1} - u^{m+1}}{(m+1)(v-u)} \right]^{\frac{1}{m}}, \quad m \in \mathbb{N}, \quad u \neq v.$$

PROPOSITION 1. *If  $0 < x < y$  and  $m \geq 2$ , then the following inequality holds:*

$$|\mathcal{A}(x^m, y^m) - \mathcal{L}_m^m(x, y)| \leq m \frac{y-x}{4} \max \{x^{m-1}, y^{m-1}\}. \tag{29}$$

*Proof.* Let  $\eta(s, t) = s - t$ ,  $h(r) = r^m$  and  $\omega(r) = r$ . Then,  $|h'(r)| = mr^{m-1}$  is quasiconvex on  $[x, y]$ . Also,

$$\mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\beta)) = \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^\varepsilon (h \circ \omega) (\omega^{-1}(\alpha)) = \frac{1}{\Gamma(\varepsilon)} \frac{y^{m+1} - x^{m+1}}{m+1}.$$

As  $\varepsilon \rightarrow 1^-$ , the inequality in Theorem 10 amounts to (29).

PROPOSITION 2. *If  $0 < x < y$ , then the following inequality holds:*

$$|\mathcal{A}(e^x, e^y) - \mathcal{L}(e^x, e^y)| \leq \frac{y-x}{4} \max\{e^x, e^y\}. \quad (30)$$

*Proof.* Let  $\eta(s, t) = s - t$ ,  $h(r) = e^r$  and  $\omega(r) = r$ . Then,  $|h'(r)| = e^r$  is quasi-convex on  $[x, y]$ . In this case,

$$\mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^{\varepsilon}(h \circ \omega)(\omega^{-1}(\beta)) = \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^{\varepsilon}(h \circ \omega)(\omega^{-1}(\alpha)) = \frac{1}{\Gamma(\varepsilon)}(e^y - e^x).$$

As  $\varepsilon \rightarrow 1^-$ , the inequality in Theorem 10 becomes (30).

PROPOSITION 3. *If  $0 < x < y$ , then the following inequality holds:*

$$|\mathcal{H}^{-1}(x, y) - \mathcal{L}^{-1}(x, y)| \leq \frac{y-x}{4} \max\left\{\frac{1}{x^2}, \frac{1}{y^2}\right\}. \quad (31)$$

*Proof.* Suppose  $\eta(s, t) = s - t$ ,  $h(r) = \frac{1}{r}$  and  $\omega(r) = r$ . Then,  $|h'(r)| = \frac{1}{r^2}$  is quasiconvex on  $[x, y]$ . In this case,

$$\mathfrak{J}_{\omega^{-1}(\alpha)^+; \omega}^{\varepsilon}(h \circ \omega)(\omega^{-1}(\beta)) = \mathfrak{J}_{\omega^{-1}(\beta)^-; \omega}^{\varepsilon}(h \circ \omega)(\omega^{-1}(\alpha)) = \frac{1}{\Gamma(\varepsilon)}(\ln y - \ln x).$$

By letting  $\varepsilon \rightarrow 1^-$ , we get the desired inequality by applying Theorem 10.

PROPOSITION 4. *If  $0 < x < y$ , then the following inequality holds:*

$$|\mathcal{L}^{-1}(x, y) - \mathcal{A}^{-1}(x, y)| \leq \frac{y-x}{4} \max\left\{\frac{1}{x^2}, \frac{1}{y^2}\right\} + \frac{y-x}{2xy}. \quad (32)$$

*Proof.* If we let, as in the proof of Proposition 3,  $\eta(s, t) = s - t$ ,  $h(r) = \frac{1}{r}$  and  $\omega(r) = r$ , then we arrive at the intended inequality by utilizing Theorem 11 with  $\varepsilon \rightarrow 1^-$ .

#### 4. Conclusion

New estimates of the Hermite–Hadamard–Fejér type and its associates have been established. We did this by utilizing fractional integral of a function  $h$ , whose derivative in absolute value is  $\eta$ -quasiconvex, with respect to another function  $\omega$ . In addition, we applied Theorems 10 and 11 to the arithmetic, harmonic, logarithmic and generalized logarithmic means to deduce more results in this regard. We anticipate that this work will arouse interest in finding more applications in other field of the mathematical sciences and beyond. Some recent results associated with the  $\eta$ -quasi(convex) functions can be found in the papers [14, 18, 1, 3, 5, 19, 20, 21, 22, 23, 11, 12] and the references cited therein.

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