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WEAK SOLUTIONS FOR CAPUTO–PETTIS FRACTIONAL q–DIFFERENCE INCLUSIONS

SAÏD ABBAS, BASHIR AHMAD, MOUFFAK BENCHOHRA AND SOTIRIS NTOUYAS

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Abstract. This article deals with some existence of weak solutions for a class of Caputo fractional q-difference inclusions and a coupled system of Caputo fractional q-difference inclusions by using the set-valued analysis, and Mönch's fixed point theorem associated with the technique of measure of weak noncompactness. Two illustrative examples are given in the end.

1. Introduction

Considerable interest has recently been shown in the study of fractional order differential equations and inclusions [4, 18, 19, 20, 25, 26]. Motivated by aforementioned work, many researchers turned to investigate initial and boundary value problems of fractional q-difference equations; for instance, see [7, 8, 9, 10, 13, 14] and references therein.

Coupled differential and integro-differential equations appear in mathematical modeling of many biological phenomena and environmental issues. For further details on the utility of coupled systems, see [3, 16, 23], and references therein.

The technique of measure of weak noncompactness is found to be a fruitful one to obtain the existence results for a variety of differential and integral equations, for example, see [1, 2, 5, 11, 21].

In the present work, we investigate the existence of weak solutions for an initial value problem of fractional q-difference inclusions given by

$$({}^cD_q^\alpha u)(t) \in F(t,u(t)),\ t \in I := [0,T], T > 0, \tag{1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{2}$$

where ${}^cD_q^\alpha$ is the Caputo fractional q-difference operator of order $\alpha \in (0,1]$, $q \in (0,1)$, $F : I \times E \to \mathscr{P}(E)$ is a multivalued map, $\mathscr{P}(E)$ is the family of all nonempty subsets of a real (or complex) Banach space E with norm $\|\cdot\|$.

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Next we switch onto the study of a coupled system of fractional q-difference inclusions:

$$\begin{cases} ({}^cD_q^\alpha u)(t) \in F(t, v(t)) \\ ({}^cD_q^\alpha v)(t) \in G(t, u(t)) \end{cases} \quad ; t \in I, \tag{3}$$

with the initial conditions

$$(u(0), v(0)) = (u_0, v_0) \in E \times E, \tag{4}$$

where $F, G : I \times E \to \mathscr{P}(E)$ are multivalued maps.

Here we emphasize that we initiate the study of weak solutions for the problems hand.

2. Preliminaries

Let $(C(I), \|\cdot\|_\infty)$ denote the Banach space of continuous functions from I into E endowed with the norm $\|u\|_\infty := \sup_{t\in I} \|u(t)\|$ and $\mathscr{C} := C(I) \times C(I)$ be a Banach space with the norm: $\|(u,v)\|_{\mathscr{C}} = \|u\|_\infty + \|v\|_\infty$. By $(L^1(I), \|\cdot\|_1)$ we denote the space of measurable functions $v : I \to E$ which are Bochner integrable with $\|v\|_1 = \int_I \|v(t)\| dt$. Denote by $(E, w) = (E, \sigma(E, E^*))$ the Banach space E with its weak topology, where E^* denotes the dual to E.

DEFINITION 1. A multivalued map $G : I \to \mathscr{P}_{cl}(E)$ is said to be measurable if for each $\omega \in E$ the function $t \to d(\omega, G(t)) = \inf\{\|\omega - \upsilon\| : \upsilon \in G(t)\}$ is measurable.

DEFINITION 2. A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

DEFINITION 3. [22] A function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s)) ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s) ds$).

Let $P(I, E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I, E)$ be the Banach space of Bochner integrable functions $u : I \to E$. Let $P_1(I, E)$ denote the space $P_1(I, E) = \{u \in P(I, E) : \varphi(u) \in L^1(I, \mathbb{R});\ for\ every\ \varphi \in E^*\}$ normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \leqslant 1} \int_0^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[22], Theorem 3.4 and Corollary 3.41]).

PROPOSITION 1. [22] *If $u \in P_1(I,E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(I,E)$.*

DEFINITION 4. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (x_n) in E with $x_n \to x$ in (E,w) then $h(x_n) \to h(x)$ in (E,w)).

DEFINITION 5. Let $\mathscr{P}_{cl,cv}(Q) = \{Y \in \mathscr{P}(Q) : Y \text{ is closed and convex}\}$. A function $F : Q \to \mathscr{P}_{cl,cv}(Q)$ has a weakly sequentially closed graph, if for any sequence $(x_n, y_n) \in Q \times Q, y_n \in F(x_n)$ for $n \in \{1,2,\ldots\}$, with $x_n \to x$ in (E,ω), and $y_n \to y$ in (E,ω), then $y \in F(x)$.

From the Hahn-Banach theorem, we have the following result

PROPOSITION 2. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$ let us denote by $V(t) = \{v(t) : v \in V\}$; $t \in I$, and $V(I) = \{v(t) : v \in V,\ t \in I\}$.

LEMMA 1. [15] *Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t \in I} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)ds\right) \leqslant \int_I \beta(H(s))ds,$$

where $H(t) = \{u(t) : u \in H\}$; $t \in I$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

Recall that the map $\beta : \Omega_E \to [0,\infty)$ defined by

$$\beta(X) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \varepsilon B_1 + \Omega\}$$

is the De Blasi measure of weak noncompactness, where Ω_E is the bounded subset of the Banach space E and B_1 is the unit ball of E (for details, see [12]).

In the sequel, we rely on the following fixed point theorem.

THEOREM 1. [21] *Let E be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of a metrizable locally convex vector space C such that $0 \in Q$. Suppose $T : Q \to \mathscr{P}_{cl,cv}(Q)$ has weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact,} \tag{5}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

Let us now recall some basic concepts of fractional q-calculus.

DEFINITION 6. [17] The q-Gamma function satisfying the relation: $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$ is defined by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}};\ \xi \in \mathbb{R} - \{0,-1,-2,\ldots\},$$

where

$$(1-q)^{(\xi-1)} = \prod_{k=0}^{\infty}\left(\frac{1-q^{k+1}}{1-q^{k+\xi}}\right),\ [\xi]_q = \frac{1-q^{\xi}}{1-q}.$$

DEFINITION 7. [6] The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0,\infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^{\alpha} u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

DEFINITION 8. [24] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^{\alpha} u)(t) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u\right)(t);\ t \in I,$$

where $[\alpha]$ is the integer part of α.

DEFINITION 9. [24] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$\left({}^C D_q^{\alpha} u\right)(t) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u\right)(t);\ t \in I.$$

LEMMA 2. [24] *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$\left(I_q^{\alpha}\, {}^C D_q^{\alpha} u\right)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$\left(I_q^{\alpha}\, {}^C D_q^{\alpha} u\right)(t) = u(t) - u(0).$$

COROLLARY 1. *Let $F : I \times E \to \mathscr{P}(E)$ be such that $S_{F\circ u} \subset C(I)$ for any $u \in C(I)$. Then problem (1)-(2) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,\ v \in S_{F\circ u}.$$

REMARK 1. Let $g \in P_1([I,E)$. Then, for every $\varphi \in E^*$, $\varphi(I_q^{\alpha} g)(t) = (I_q^{\alpha} \varphi g)(t)$, for *a.e.* $t \in I$.

3. Caputo-Pettis fractional q-difference inclusions

Let us start by defining what we mean by a weak solution of the problem (1)-(2).

DEFINITION 10. By a weak solution of the problem (1)-(2) we mean a measurable function $u \in C(I)$ that satisfies

$$u(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s,$$

with $v \in S_{F\circ u}$, where

$$S_{F\circ u} = \{v \in L^1(I) : v(t) \in F(t,u(t)) \ , \ a.e. \ t \in I\} \tag{6}$$

is the set of selectors from $F \circ u$.

We introduce the following hypotheses:

(H_1) $F : I \times E \to \mathscr{P}_{cp,cl,cv}(E)$ has weakly sequentially closed graph, where

$$\mathscr{P}_{cp,cl,cv}(E) = \{Y \in \mathscr{P}(E) : Y \text{ is compact, closed and convex}\};$$

(H_2) For each $u \in C(I)$, there exists a function $v \in S_{F\circ u}$ which is measurable a.e. on I and Pettis integrable on I, where $F \circ u$ is defined by (6);

(H_3) There exists a function $p \in L^\infty(I, \mathbb{R}_+)$ such that for all $\varphi \in E^*$, we have $\|F(t,u)\|_{\mathscr{P}} = \sup_{v \in S_{F\circ u}} |\varphi(v)| \leqslant p(t)$; *for a.e.* $t \in I$, *and each* $u \in E$;

(H_4) For each bounded and measurable set $B \subset E$, $\beta(F(t,B)) \leqslant p(t)\beta(B)$ for each $t \in I$.

THEOREM 2. *Assume that the hypotheses* $(H_1) - (H_4)$ *hold. If*

$$L := \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} < 1, \tag{7}$$

where $p^* = \operatorname{ess\,sup}_{t\in I} p(t)$, *then the problem (1)-(2) has a weak solution on* I.

Proof. Consider the multi-valued map $N : C(I) \to \mathscr{P}_{cl}(C(I))$ defined by:

$$(Nu)(t) = \left\{ h \in C(I) : h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s; \ v \in S_{F\circ u} \right\}. \tag{8}$$

For each $u \in C(I)$, it follows by the given hypotheses there exists a Pettis integrable function $v \in S_{F\circ u}$, and for each $s \in [0,t]$, the function

$$t \mapsto (t-qs)^{\alpha-1} v(s), \ for \ a.e. \ t \in I,$$

is Pettis integrable. Thus, N is well defined. Let $R>0$ be such that

$$R > \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)},$$

and consider the set

$$Q = \Big\{ u \in C(I) : \|u\|_\infty \leqslant R \text{ and } \|u(t_2)-u(t_1)\| \\ \leqslant \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)}(t_2-t_1)^\alpha + \frac{p^*}{\Gamma_q(\alpha)} \int_0^{t_1} \left|(t_2-qs)^{\alpha-1}-(t_1-qs)^{\alpha-1}\right| d_q s,\ t_1,t_2 \in I \Big\}.$$

The set Q is closed, convex end equicontinuous. We shall show in several steps that N satisfies the assumptions of Theorem 1.

Step 1. *$N(u)$ is convex for each $u \in Q$.*
Let $h_1,h_2 \in N(u)$. Then there exist $v_1,v_2 \in S_{F\circ u}$ such that, for each $t \in I$, and for any $i=1,2$, we have

$$h_i(t) = u_0 + \int_0^t (t-s)^{\alpha-1} \frac{v_i(s)}{\Gamma_q(\alpha)} ds.$$

Let $0 \leqslant \lambda \leqslant 1$. Then, for each $t \in I$, we have

$$[\lambda h_1 + (1-\lambda)h_2](t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} (\lambda v_1(s) + (1-\lambda)v_2(s)) d_q s.$$

Since F has convex values, $S_{F\circ u}$ is convex. Hence, it follows that

$$\lambda h_1 + (1-\lambda)h_2 \in N(u).$$

Step 2. *N maps Q into itself.*
Let $h \in N(Q)$, then there exists $u \in Q$ such that $h \in N(u)$, and there exists a Pettis integrable function $v \in S_{F\circ u}$. Assume that $h(t) \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\|=1$ such that $\|h(t)\| = |\varphi(h(t))|$. Thus

$$\|h(t)\| = \varphi\left(u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} v(s) d_q s\right).$$

Hence

$$\|h(t)\| \leqslant \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |\varphi(v(s))| d_q s \leqslant \frac{p^*}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{\alpha-1} d_q s \leqslant \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} \leqslant R.$$

Next, let $t_1,t_2 \in I$ such that $t_1 < t_2$ and let $h \in N(u)$, with

$$h(t_2) - h(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|h(t_2)-h(t_1)\| = |\varphi(h(t_2)-h(t_1))|,$$

and $\|\varphi\| = 1$. Then, we have

$$\begin{aligned}\|h(t_2)-h(t_1)\| &= |\varphi(h(t_2)-h(t_1))| \\ &\leqslant \varphi\left(\int_0^{t_2}(t_2-qs)^{\alpha-1}\frac{v(s)}{\Gamma_q(\alpha)}d_qs-\int_0^{t_1}(t_1-qs)^{\alpha-1}\frac{v(s)}{\Gamma_q(\alpha)}d_qs\right).\end{aligned}$$

Thus, we get

$$\begin{aligned}\|h(t_2)-h(t_1)\| &\leqslant \int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\frac{|\varphi(v(s))|}{\Gamma_q(\alpha)}d_qs \\ &\quad+\int_0^{t_1}\left|(t_2-qs)^{\alpha-1}-(t_1-qs)^{\alpha-1}\right|\frac{|\varphi(v(s))|}{\Gamma_q(\alpha)}d_qs \\ &\leqslant \int_{t_1}^{t_2}(t_2-qs)^{\alpha-1}\frac{p(s)}{\Gamma_q(\alpha)}d_qs \\ &\quad+\int_0^{t_1}\left|(t_2-qs)^{\alpha-1}-(t_1-qs)^{\alpha-1}\right|\frac{p(s)}{\Gamma_q(\alpha)}d_qs.\end{aligned}$$

Hence, we obtain

$$\|h(t_2)-h(t_1)\| \leqslant \frac{p^*T^\alpha}{\Gamma_q(1+\alpha)}(t_2-t_1)^\alpha+\frac{p^*}{\Gamma_q(\alpha)}\int_0^{t_1}\left|(t_2-qs)^{\alpha-1}-(t_1-qs)^{\alpha-1}\right|d_qs.$$

This implies that $h \in Q$. Hence $N(Q) \subset Q$.

Step 3. *N has weakly-sequentially closed graph.*
Let (u_n, w_n) be a sequence in $Q \times Q$, with $u_n(t) \to u(t)$ in (E,ω) for each $t \in I$, $w_n(t) \to w(t)$ in (E,ω) for each $(t \in I$, and $w_n \in N(u_n)$ for $n \in \{1,2,\ldots\}$.
We show that $w \in \Omega(u)$. Since $w_n \in \Omega(u_n)$, there exists $v_n \in S_{F\circ u_n}$ such that

$$w_n(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}v_n(s)d_qs.$$

We show that there exists $v \in S_{F\circ u}$ such that, for each $t \in I$,

$$w(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}v(s)d_qs.$$

From the fact that $F(\cdot,\cdot)$ has compact values, there exists a Pettis integrable subsequence v_{n_m}; such that

$$v_{n_m}(t) \in F(t,u_n(t)) \text{ a.e. } t \in I,$$

and

$$v_{n_m}(\cdot) \to v(\cdot) \text{ in } (E,\omega) \text{ as } m \to \infty.$$

As $F(t,\cdot)$ has weakly sequentially closed graph, $v(t) \in F(t,u(t))$. Then by the Lebesgue dominated convergence theorem for the Pettis integral, we obtain

$$\varphi(w_n(t)) \to \varphi\left(u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}v_n(s)d_qs\right),$$

i.e., $w_n(t) \to (Nu)(t)$ in (E,ω) for each $t \in I$, which implies that $w \in N(u)$.

Step 4. *The Mönch condition holds*, i.e.,

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{is relatively weakly compact for every subset } V \subset Q. \quad (9)$$

Let $V \subset Q$ such that $\overline{V} = \overline{conv}(\Omega(V) \cup \{0\})$. Then, for each $t \in I$, $V(t) \subset \overline{conv}(\Omega(V(t)) \cup \{0\})$. Since V is bounded and equicontinuous, the function $t \to v(t) = \beta(V(t))$ is continuous on I. By (H_4) and the properties of β ([12]) for any $t \in I$, we have

$$\begin{aligned} v(t) &\leqslant \beta((NV)(t) \cup \{0\}) \leqslant \beta((NV)(t)) \leqslant \beta\{(Nu)(t) : u \in V\} \\ &\leqslant \beta\left\{\int_0^t (t-qs)^{\alpha-1} \frac{v(s)}{\Gamma_q(\alpha)} d_q s : v(t) \in S_{F\circ u},\ u \in V\right\} \\ &\leqslant \beta\left\{\int_0^t (t-qs)^{\alpha-1} \frac{F(s,V(s))}{\Gamma_q(\alpha)} d_q s\right\} \leqslant \int_0^t (t-qs)^{\alpha-1} \frac{\beta(V(s))}{\Gamma_q(\alpha)} d_q s \\ &\leqslant \int_0^t (t-qs)^{\alpha-1} \frac{p(s)v(s)}{\Gamma_q(\alpha)} d_q s \leqslant \frac{p^* T^\alpha}{\Gamma_q(1+\alpha)} \|v\|_\infty = L\|v\|_\infty. \end{aligned}$$

In particular,

$$\|u\|_\infty \leqslant L\|v\|_\infty.$$

By (7) it follows that $\|v\|_\infty = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$, and then V is weakly relatively compact in $C(I)$. In view of the foregoing arguments, we deduce that the conclusion of Theorem 1 applies and hence the operator N has a fixed point, which corresponds to a weak solution of the problem (1)-(2).

4. Coupled systems of Caputo-Pettis fractional q-difference inclusions

DEFINITION 11. A pair of coupled measurable functions $(u,v) \in \mathscr{C}$ is said to be a weak solution of the problem (3)-(4) if it satisfies

$$\begin{cases} u(t) = u_0 + \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s) d_q s, \\ v(t) = v_0 + \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} z(s) d_q s, \end{cases}$$

where $w \in S_{F\circ v}$, and $z \in S_{F\circ u}$.

THEOREM 3. *Assume that*

(H_{01}) $F,G : I \times E \to \mathscr{P}_{cp,cl,cv}(E)$ *have weakly sequentially closed graph;*

(H_{02}) *For all continuous functions* $u,v : I \to E$, *there exist measurable functions* $w \in S_{F\circ v}$, $z \in S_{F\circ u}$, *a.e. on* I *and* w,z *are Pettis integrable on* I;

(H_{03}) *There exist* $p,d \in L^{\infty}(I,\mathbb{R}_+)$ *such that for all* $\varphi \in E^*$, *we have*

$$\|F(t,v)\|_{\mathscr{P}} \leqslant p(t), \text{ for a.e. } t \in I, \text{ and each } v \in E,$$

$$\|G(t,u)\|_{\mathscr{P}} \leqslant d(t), \text{ for a.e. } t \in I, \text{ and each } u \in E;$$

(H_{04}) *For each bounded and measurable set* $B \subset E$ *and for each* $t \in I$, *we have*

$$\beta(F(t,B) \leqslant p(t)\beta(B), \text{ and } \beta(G(t,B) \leqslant d(t)\beta(B).$$

Then there exists at least one weak solution for the problem (3)-(4) on I *provided that*

$$\frac{p^*T^{\alpha}}{\Gamma_q(1+\alpha)} < 1, \text{ and } \frac{d^*T^{\alpha}}{\Gamma_q(1+\alpha)} < 1, \tag{10}$$

where $p^* = \operatorname{ess\,sup}_{t\in I} p(t)$, $d^* = \operatorname{ess\,sup}_{t\in I} d(t)$.

Proof. Consider the multi-valued map $N : \mathscr{C} \to \mathscr{P}_{cl}(\mathscr{C})$ defined by:

$$(N(u,v))(t) = ((N_1u)(t),(N_2v)(t)),$$

where $N_1,N_2 : C(I) \to \mathscr{P}_{cl}(C(I))$ are given by

$$(N_1u)(t) = \left\{ h \in C(I) : h(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} w(s)d_qs;\ w \in S_{F\circ v} \right\}, \tag{11}$$

and

$$(N_2v)(t) = \left\{ h \in C(I) : h(t) = v_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} z(s)d_qs;\ z \in S_{F\circ u} \right\}. \tag{12}$$

For each $(u,v) \in \mathscr{C}$, there exist Pettis integrable functions $w \in S_{F\circ v}$, $z \in S_{F\circ u}$, and for each $s \in [0,t]$, the functions

$$t \mapsto (t-qs)^{\alpha-1}w(s), \text{ and } t \mapsto (t-qs)^{\alpha-1}z(s);\ for\ a.e.\ t \in I,$$

are Pettis integrable. Thus, the multi-function N is well defined. Let $R > 0$ be such that

$$R > \max\left\{ \frac{p^*T^{\alpha}}{\Gamma_q(1+\alpha)}, \frac{d^*T^{\alpha}}{\Gamma_q(1+\alpha)} \right\}$$

and consider the set

$$\begin{aligned}\Lambda = \Bigg\{ (u,v) \in \mathscr{C} : \|(u,v)\|_{\mathscr{C}} \leqslant R \text{ and } \|u(t_2)-u(t_1)\| \leqslant \frac{p^*T^{\alpha}}{\Gamma_q(1+\alpha)}(t_2-t_1)^{\alpha} \\ + \frac{p^*}{\Gamma_q(\alpha)} \int_0^{t_1} \left|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}\right| d_qs, \text{ and } \|v(t_2)-v(t_1)\| \\ \leqslant \frac{d^*T^{\alpha}}{\Gamma_q(1+\alpha)}(t_2-t_1)^{\alpha} + \frac{d^*}{\Gamma_q(\alpha)} \int_0^{t_1} \left|(t_2-qs)^{\alpha-1} - (t_1-qs)^{\alpha-1}\right| d_qs,\ t_1,t_2 \in I \Bigg\}.\end{aligned}$$

The subset Λ of $\mathscr{C}$ is closed, convex end equicontinuous. As in the proof of Theorem 2, we can show that $N(u,v)$ is convex for each $(u,v) \in \Lambda$, $N(\Lambda) \subset \Lambda$, N has weakly-sequentially closed graph, and the Mönch condition (9) holds. Hence, the operator N satisfies all the assumptions of Theorem 1. Therefore; we conclude that N has a fixed point which is a weak solution of the problem (3)-(4).

5. Examples

Let $E = l^1 = \{u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$ be the Banach space equipped with the norm $\|u\|_E = \sum_{n=1}^{\infty} |u_n|$.

EXAMPLE 1. Consider the following problem of fractional $\frac{1}{4}-$difference inclusion

$$\begin{cases} \left({}^{C}D^{\frac{1}{2}}_{\frac{1}{4}} u_n\right)(t) \in F_n(t, u(t)); \ t \in [0,1], \\ u(0) = (1, 0, \dots, 0, \dots), \end{cases} \tag{13}$$

where

$$F_n(t, u(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E} [u_n(t) - 1, u_n(t)]; \ t \in [0,1],$$

$$u = (u_1, u_2, \dots, u_n, \dots), \ c := \frac{e^4}{4} \Gamma_{\frac{1}{4}}\left(\frac{1}{2}\right),$$

and F is closed and convex valued. Set

$$F = (F_1, F_2, \dots, F_n, \dots).$$

For each $u \in E$ and $t \in [0,1]$, we have

$$\|F(t, u(t))\|_{\mathscr{P}} \leqslant ct^2 \frac{1}{e^{t+4}}.$$

Hence, the hypothesis (H_3) is satisfied with $p^* = ce^{-4}$. We shall show that condition (7) holds with $T = 1$. Indeed,

$$L = \frac{ce^{-4}}{\Gamma_{\frac{1}{4}}(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 2 are satisfied. Hence, the problem (13) has at least one weak solution defined on $[0,1]$.

EXAMPLE 2. We consider now the following coupled system of fractional $\frac{1}{4}-$difference inclusions

$$\begin{cases} \left({}^{C}D^{\frac{1}{2}}_{\frac{1}{4}} u_n\right)(t) \in F_n(t, v(t)) \\ \left({}^{C}D^{\frac{1}{2}}_{\frac{1}{4}} v_n\right)(t) \in G_n(t, u(t)) \\ u(0) = (1, 0, \ldots, 0, \ldots),\ v(0) = (0, 1, 0, \ldots, 0, \ldots) \end{cases} ;\ t \in [0,1], \tag{14}$$

where

$$F_n(t, v(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E}[v_n(t) - 1, v_n(t)],$$

$$G_n(t, u(t)) = \frac{ct^2 e^{-4-t}}{1 + \|u(t)\|_E}[u_n(t), 1 + u_n(t)];\ t \in [0,1],$$

with $u = (u_1, u_2, \ldots, u_n, \ldots),\ v = (v_1, v_2, \ldots, v_n, \ldots)$, and $c := \frac{e^4}{4}\Gamma_{\frac{1}{4}}\left(\frac{1}{2}\right)$. Set

$$F = (F_1, F_2, \ldots, F_n, \ldots),\ G = (G_1, G_2, \ldots, G_n, \ldots).$$

It is easy to show that all conditions of Theorem 3 are satisfied. Hence, the problem (14) has at least one weak solution (u, v) defined on $[0,1]$.

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Saïd Abbas
Department of Mathematics
Tahar Moulay University of Saïda
P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria
e-mail: abbasmsaid@yahoo.fr

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group
Department of Mathematics, Faculty of Science, King Abdulaziz University
P.P. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com

Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbès
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: benchohra@yahoo.com

Sotiris Ntouyas
Department of Mathematics
University of Ioannina
Ioannina, 451 10, Greece
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group
Department of Mathematics, Faculty of Science, King Abdulaziz University
P.P. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: sntouyas@uoi.gr