

EXISTENCE AND UNIQUENESS ANALYSIS FOR FRACTIONAL ∇ -DIFFERENCE TWO-POINT BOUNDARY VALUE PROBLEMS WITH FULLY DIFFERENCE BOUNDARY CONDITIONS

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Abstract. In this article a fractional-order ∇ -difference two-point boundary value problem of order $[2, 3)$ is investigated. One of the interesting parts of this boundary value problem is its boundary conditions that includes all of the possible fractional orders $\alpha \in [0, 1)$, $\alpha = 1$, and $\alpha \in [1, 2)$. Thanks to the Green's function approach, the main problem is transferred into an appropriate functional space and then, making use of fixed point techniques such as nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems existence of at least one solution is approved for the boundary value problem under study. Next, we choose the Banach contraction principle to make a uniqueness criterion. At the end, we examine our theoretical findings with some numerical prototypes to show applicability of the solvability results in practice.

Mathematics subject classification (2020): Primary 34A08, 34F15, 39A12; Secondary 39A10, 34A12.

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