Journal of Classical Analysis
Volume 3, Number 2 (2013), 109–127
doi:10.7153/jca-03-10

EXTENSION OF SOME CLASSICAL SUMMATION THEOREMS FOR THE GENERALIZED HYPERGEOMETRIC SERIES WITH INTEGRAL PARAMETER DIFFERENCES

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Abstract. We derive extensions of the classical summation theorems of Kummer and Watson for the generalized hypergeometric series where r pairs of numeratorial and denominatorial parameters differ by positive integers. The results are obtained with the help of a generalization of Kummer's second summation theorem for the ${}_2F_1$ series given recently by Rakha and Rathie [Integral Transforms and Special Functions, **22**, 823–840 (2011)] together with generalizations of the Euler transformations for the ${}_{r+2}F_{r+1}(z)$ function. A few interesting special cases are also presented.

1. Introduction

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series [21, p. 41]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1.1} $$

where for nonnegative integer n the Pochhammer symbol (or ascending factorial) is defined by $(a)_0 = 1$ and for $n \geqslant 1$ by $(a)_n = a(a+1)\ldots(a+n-1)$. However, for all n (whether an integer or *non-integer*) we write

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.2} $$

In what follows we shall adopt the convention of writing the finite sequence of parameters $(a_1, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$ ((a_p))_k \equiv (a_1)_k \ldots (a_p)_k, $$

where an empty product ($p = 0$) is understood to be unity. When $p \leqslant q$ the above series on the right-hand side of (1.1) converges for $|z| < \infty$, but when $p = q+1$ convergence occurs when $|z| < 1$ (unless the series terminates).

In the theory of hypergeometric and generalized hypergeometric functions summation and transformation formulas play a key role. Recently, considerable progress

Mathematics subject classification (2010): Primary 33C20, Secondary 33C15.
Keywords and phrases: Generalized hypergeometric series, generalized Gauss and Kummer summation theorems, generalized Watson summation theorem.

Paper JCA-03-10

has been made in the direction of generalizing the classical summation theorems, such as those of Gauss and Kummer for the series ${}_2F_1$, and of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, and others. The summation theorems that we extend in our present investigation are given by the following:

KUMMER'S SUMMATION THEOREM I

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ 1+a-b \end{matrix};-1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}, \tag{1.3} $$

KUMMER'S SUMMATION THEOREM II

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2} \end{matrix};\frac{1}{2}\right] = \frac{\pi^{\frac{1}{2}}\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}, \tag{1.4} $$

KUMMER'S SUMMATION THEOREM III

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a \\ c \end{matrix};\frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}, \tag{1.5} $$

WATSON'S SUMMATION THEOREM

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2},\, 2c \end{matrix};1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\Gamma(c+\frac{1}{2}-\frac{1}{2}b)} \tag{1.6} $$

provided $\Re(2c-a-b) > -1$. It has been recently pointed out in [1] that the summation theorems (1.4) and (1.5) were first obtained by Kummer [7] and have been erroneously attributed to Gauss and Bailey respectively in the book by Slater [21, p. 243–245].

During 1992–1996, Lavoie *et al.* [8, 9, 10] generalized the summation theorems in (1.3)–(1.5) and also those of Watson, Dixon and Whipple for the ${}_3F_2$ series by considering their contiguous extensions, and obtained a large number of special and limiting cases (all verified with the help of *Mathematica* and *Maple*). Specifically, they obtained the explicit expressions of

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ 1+a-b+i \end{matrix};-1\right], \tag{1.7} $$

$$ {}_2F_1\left[\begin{matrix} a,\, b \\ \frac{1}{2}(a+b+i+1) \end{matrix};\frac{1}{2}\right] \tag{1.8} $$

and

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a+i \\ c \end{matrix};\frac{1}{2}\right], \tag{1.9} $$

each for $i = 0, \pm 1, \ldots, \pm 5$. For $i = 0$, these results reduce to (1.3), (1.4) and (1.5) respectively. In addition, they also obtained the explicit expression of the series

$$ {}_3F_2\left[\begin{matrix} a,\, b,\, c \\ \frac{1}{2}(a+b+i+1),\, 2c+j \end{matrix};1\right], \tag{1.10} $$

for $i, j = 0, \pm 1, \pm 2$, which when $i = j = 0$ reduces to (1.6). In [19], Rakha and Rathie generalized the above results (1.7)–(1.9) and (1.10) with $j = 0$ to the most general case for any integer i.

In this paper we provide extensions of the above-mentioned classical summation theorems to higher-order hypergeometric series when there are r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r). An early result of this type was the generalized Karlsson-Minton theorem (which extends the well-known Gauss summation theorem) given in Theorem 2, together with the special case when the series terminates (which extends the Vandermonde-Chu summation formula) derived by Miller [13]. A similar extension of the Saalschütz summation theorem has recently been given in [6]. The approach we adopt here is to make use of the contiguous form of Kummer's second summation theorem for the ${}_2F_1$ series derived in [19], together with two recently obtained Euler-type transformations for ${}_{r+2}F_{r+1}$ functions in [15, 17].

The plan of the paper is as follows. In Section 2 we collect together several theorems that will be required in the sequel. In Section 3, we derive the extension of Kummer's second summation theorem in the form

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ \frac{1}{2}(a+b+j+1), & (f_r) \end{matrix}; \frac{1}{2}\right] $$

for $j = 0, \pm 1, \pm 2, \ldots$, thereby extending earlier results in [5]. Sections 4 and 5 deal with the extension of the first and third Kummer summation theorems in the form

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ 1+a-b+j, & (f_r) \end{matrix}; -1\right] $$

and

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, 1-a+j, (f_r+m_r) \\ c, \qquad (f_r) \end{matrix}; \frac{1}{2}\right] $$

for $j = 0, \pm 1, \pm 2, \ldots$.

Finally, in Section 6 we provide the extension of Watson's summation theorem with a single pair of numeratorial and denominatorial parameters differing by a positive integer m in the form

$$ {}_4F_3\left[\begin{matrix} a, b, c, & f+m \\ \frac{1}{2}(a+b+1), 2c, & f \end{matrix}; 1\right]. $$

In addition, several interesting special cases of the above-mentioned results are also given. These results are of a general character and should help to advance the theory of the evaluation of higher-order hypergeometric series.

2. Results required

In this section we collect together several results that will be required in our present investigation. The generalization of Kummer's second summation theorem for the ${}_2F_1(\frac{1}{2})$ series and the generalized Karlsson-Minton summation formula are given by the following two theorems.

THEOREM 1. *For $j = 0, \pm 1, \pm 2, \ldots$, we have the contiguous form of Kummer's second summation theorem for the ${}_2F_1$ series given by* [19]

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+j+1) \end{matrix}; \frac{1}{2}\right] \\
&= \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}j+\frac{1}{2})}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|j|+\frac{1}{2})} \sum_{n=0}^{|j|} \binom{|j|}{n} \frac{(\mp 1)^n \Gamma(\frac{1}{2}b+\frac{1}{2}n)}{\Gamma(\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}|j|+\frac{1}{2}n)}, \qquad (2.1)
\end{aligned}
$$

where the upper or lower sign is chosen according as $j \geqslant 0$ or $j < 0$ respectively.

THEOREM 2. *Let (m_r) be a sequence of positive integers and $m := m_1 + \cdots + m_r$. The generalized Karlsson-Minton summation theorem is given by* [17, 18]

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, (f_r+m_r) \\ c, \quad (f_r) \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \frac{(-1)^k (a)_k (b)_k C_{k,r}}{(1+a+b-c)_k} \qquad (2.2)
$$

provided $\Re(c-a-b) > m$.

The coefficients $C_{k,r}$ appearing in (2.2) are defined for $0 \leqslant k \leqslant m$ by

$$
C_{k,r} = \frac{1}{\Lambda} \sum_{j=k}^{m} \sigma_j \mathbf{S}_j^{(k)}, \qquad \Lambda = (f_1)_{m_1} \ldots (f_r)_{m_r}, \qquad (2.3)
$$

with $C_{0,r} = 1$, $C_{m,r} = 1/\Lambda$. The $\mathbf{S}_j^{(k)}$ denote the Stirling numbers of the second kind and the σ_j $(0 \leqslant j \leqslant m)$ are generated by the relation

$$
(f_1+x)_{m_1} \cdots (f_r+x)_{m_r} = \sum_{j=0}^{m} \sigma_j x^j. \qquad (2.4)
$$

In [16], an alternative representation for the coefficients $C_{k,r}$ is given as the terminating hypergeometric series of unit argument

$$
C_{k,r} = \frac{(-1)^k}{k!} {}_{r+1}F_r\left[\begin{matrix} -k, (f_r+m_r) \\ (f_r) \end{matrix}; 1\right]. \qquad (2.5)
$$

When $r = 1$, with $f_1 = f$, $m_1 = m$, Vandermonde's summation theorem [21, p. 243] can be used to show that

$$
C_{k,1} = \binom{m}{k} \frac{1}{(f)_k}. \qquad (2.6)
$$

The following three theorems give transformations of the ${}_{r+2}F_{r+1}(z)$ hypergeometric functions in the special case that r pairs of numeratorial and denominatorial parameters differ by positive integers. Theorems 4 and 5 generalize the familiar Euler transformations for the Gauss ${}_2F_1(z)$ function.

THEOREM 3. *Let* (m_r) *denote a set of positive integers with* $m := m_1 + \cdots + m_r$. *Then, when* $|z| < 1$, *we have* [14]

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, (f_r + m_r) \\ c, \quad (f_r) \end{matrix}; z\right] = \sum_{k=0}^{m} \frac{(a)_k (b)_k}{(c)_k} z^k C_{k,r}\, {}_2F_1\left[\begin{matrix} a+k, b+k \\ c+k \end{matrix}; z\right], \tag{2.7} $$

where the coefficients $C_{k,r}$ *are defined in (2.3) or (2.5). The expansion (2.7) also holds when* $z = 1$ *provided* $\Re(c-a-b) > m$.

The result (2.7) can be deduced as a particular case of the more general expansion given by Luke in [11, Eq. (5.10.2(4))] combined with the fact that $C_{k,r} = 0$ for $k > m$ [16]. It is also related to the more general expansion obtained by Karlsson [3] for a hypergeometric function with integral parameter differences expressed as an r-fold sum of lower-order hypergeometric functions; see also [22] for a simpler derivation.

THEOREM 4. *Let* (m_r) *be a sequence of positive integers and* $m := m_1 + \cdots + m_r$. *Then if* $b \neq f_j$ $(1 \leqslant j \leqslant r)$, $(\lambda)_m \neq 0$, *where* $\lambda := c - b - m$, *we have the first Euler-type transformation* [15, 17]

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, (f_r + m_r) \\ c, \quad (f_r) \end{matrix}; z\right] = (1-z)^{-a}\, {}_{m+2}F_{m+1}\left[\begin{matrix} a, c-b-m, (\xi_m + 1) \\ c, \quad (\xi_m) \end{matrix}; \frac{z}{z-1}\right] \tag{2.8} $$

for $|\arg(1-z)| < \pi$. *The* (ξ_m) *are the nonvanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* m *given by*

$$ Q_m(t) = \frac{1}{(\lambda)_m} \sum_{k=0}^{m} (b)_k C_{k,r} (t)_k (\lambda - t)_{m-k} \equiv \sum_{j=0}^{m} d_j t^j, \qquad d_0 = 1, \tag{2.9} $$

where the coefficients $C_{k,r}$ *are defined in (2.3). The polynomial* $Q_m(t)$ *is normalized so that* $Q_m(0) = 1$.

THEOREM 5. *Let* (m_r) *be a sequence of positive integers and* $m := m_1 + \cdots + m_r$. *Then if* $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$, *we have the second Euler-type transformation* [15], [17, (5.8), (5.9)]

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a, b, (f_r + m_r) \\ c, \quad (f_r) \end{matrix}; z\right] = (1-z)^{c-a-b-m}\, {}_{m+2}F_{m+1}\left[\begin{matrix} c-a-m, c-b-m, (\eta_m + 1) \\ c, \quad (\eta_m) \end{matrix}; z\right] \tag{2.10} $$

for $|\arg(1-z)| < \pi$.

The (η_m) *are the nonvanishing zeros of the associated parametric polynomial* $\hat{Q}_m(t)$ *of degree* m $(\hat{d}_m \neq 0)$ *given by*

$$ \hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,r} (a)_k (b)_k (t)_k}{(c-a-m)_k (c-b-m)_k} G_{m,k}(t) \equiv \sum_{j=0}^{m} \hat{d}_j t^j, \tag{2.11} $$

where

$$ G_{m,k}(t) := {}_3F_2\left[\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix}; 1\right] $$

and the coefficients $C_{k,r}$ are defined in (2.3). For $0 \leqslant k \leqslant m$, the function $G_{m,k}(t)$ is a polynomial in t of degree $m-k$ and $\hat{Q}_m(t)$ is normalized so that $\hat{Q}_m(0) = 1$ $(\hat{d}_0 = 1)$.

We remark that in [15, 17] the transformations (2.8) and (2.10) were established in the domains $|z| < 1$, $\Re(z) < \frac{1}{2}$ and $|z| < 1$, respectively; but these domains can be extended to $|\arg(1-z)| < \pi$ by analytic continuation.

In the particular case $r = 1$, $m_1 = m = 1$, $f_1 = f$, we have the associated parametric polynomial from (2.9)

$$Q_1(t) = 1 + \frac{(b-f)t}{(c-b-1)f}$$

with the nonvanishing zero $\xi_1 = \xi$ (provided $b \neq f$, $c-b-1 \neq 0$) given by

$$\xi = \frac{(c-b-1)f}{f-b}, \tag{2.12}$$

and from (2.11)

$$\hat{Q}_1(t) = 1 - \frac{\{(c-a-b-1)f + ab\}t}{(c-a-1)(c-b-1)f}$$

with the nonvanishing zero $\eta_1 = \eta$ (provided $c-a-1, c-b-1 \neq 0$ and $\hat{d}_1 \neq 0$) given by[1]

$$\eta = \frac{(c-a-1)(c-b-1)f}{ab + (c-a-b-1)f}. \tag{2.13}$$

In the case $r = 1$, the transformation (2.8) has been obtained previously in [20] and both (2.8) and (2.10) in [12] using different methods.

The transformation (2.10) has been employed recently to obtain a generalization of Saalschütz's theorem in [6] when there are r pairs of numeratorial and denominatorial parameters differing by positive integers.

3. Extension of Kummer's second summation theorem

Our main result in this section is given by the following theorem which extends Kummer's second summation theorem (1.4) to the generalized hypergeometric series with r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r).

THEOREM 6. *Let (m_r) be a set of positive integers with $m := m_1 + \cdots + m_r$. Then, we have the summation*

$$\begin{aligned} &{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a+b+j+1), & (f_r) \end{matrix}; \frac{1}{2}\right] \\ &= \frac{2^{b-1}\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}j + \frac{1}{2})}{\Gamma(b)\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}|j| + \frac{1}{2})} \sum_{n=0}^{|j|} (\mp 1)^n \binom{|j|}{n} G_n(a,b) \end{aligned} \tag{3.1}$$

[1]If the transformation (2.8) is applied again to the hypergeometric function on the right-hand side with f replaced by ξ we find from (2.12) $\eta = (c-a-1)\xi/(\xi - a)$, which simplifies to yield (2.13).

for $j = 0, \pm 1, \pm 2, \dots$ *, where*

$$G_n(a,b) := \sum_{k=0}^{m} \frac{(a)_k C_{k,r} \Gamma(\frac{1}{2}b + \frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}(n+k-|j|))} \tag{3.2}$$

and the upper or lower sign is chosen according as $j \geqslant 0$ *or* $j < 0$*, respectively. The coefficients* $C_{k,r}$ *are defined in (2.3) or (2.5) and, when* $r = 1$*, in (2.6).*

Proof. If we set $z = \frac{1}{2}$ and $c = \frac{1}{2}(a+b+j+1)$ in (2.7), where j is an integer, we obtain

$$\begin{aligned} F \equiv {}_{r+2}F_{r+1} &\left[\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a+b+j+1), & (f_r) \end{matrix}; \frac{1}{2}\right] \\ &= \sum_{k=0}^{m} \frac{2^{-k}(a)_k (b)_k C_{k,r}}{(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})_k} \, {}_2F_1 \left[\begin{matrix} a+k, b+k \\ \frac{1}{2}(a+b+j+2k+1) \end{matrix}; \frac{1}{2}\right]. \end{aligned} \tag{3.3}$$

We now employ the contiguous form of Gauss' second summation theorem stated in (2.1) with the substitutions $a \to a+k$, $b \to b+k$. Application of the duplication formula for the gamma function

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + \tfrac{1}{2}),$$

followed by some routine algebra, then yields

$$\begin{aligned} F = &\frac{2^{b-1}\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}j + \frac{1}{2})}{\Gamma(b)\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}|j| + \frac{1}{2})} \\ &\times \sum_{k=0}^{m} (a)_k C_{k,r} \sum_{n=0}^{|j|} (\mp 1)^n \binom{|j|}{n} \frac{\Gamma(\frac{1}{2}b + \frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}(n+k-|j|))}, \end{aligned}$$

where the choice of sign is made according as $j \geqslant 0$ or $j < 0$, respectively. A straightforward reversal of the order of summation then leads to the result stated in (3.1). □

REMARK 1. The summation (3.1) is valid for any integer j and so represents a more general result than that recently given in [5], which is only applicable for $|j| \leqslant 5$.

The summation in (3.1) for $j = 0, \pm 1$ takes the forms

$$\begin{aligned} &{}_{r+2}F_{r+1} \left[\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a+b+1), & (f_r) \end{matrix}; \frac{1}{2}\right] \\ &= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} 2^k C_{k,r} \Gamma(\tfrac{1}{2}a + \tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b + \tfrac{1}{2}k), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &{}_{r+2}F_{r+1} \left[\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a+b+2), & (f_r) \end{matrix}; \frac{1}{2}\right] \\ &= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \frac{a+b}{a-b} \sum_{k=0}^{m} 2^k C_{k,r} D_k^{-}(a,b), \end{aligned} \tag{3.5}$$

and

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ \frac{1}{2}(a+b), & (f_r) \end{matrix}; \frac{1}{2}\right]
= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} 2^k C_{k,r} D_k^+(a,b) \tag{3.6}
$$

respectively, where

$$
D_k^{\pm}(a,b) := \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k+\tfrac{1}{2})\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k) \pm \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k+\tfrac{1}{2}).
$$

When $r=0$, the sequences (f_r) and (f_r+m_r) are empty, so that $m=0$, and (3.4)–(3.6) reduce to the corresponding cases of (2.1).

REMARK 2. The results for $j=-2, -1, 0$ and 1 have been obtained by Miller and Paris [14]. However, the result for $j=-1$ is given here in its corrected form.

In the case $r=1$ (with $m_1=m$ a positive integer and $f_1=f$) the coefficients $C_{k,1}$ are given by (2.6). The summations (3.4)–(3.6) then reduce to

$$
{}_3F_2\left[\begin{matrix} a, b, & f+m \\ \frac{1}{2}(a+b+1) & f \end{matrix}; \frac{1}{2}\right]
= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} \binom{m}{k} \frac{2^k\Gamma(\frac{1}{2}a+\frac{1}{2}k)\Gamma(\frac{1}{2}b+\frac{1}{2}k)}{(f)_k}, \tag{3.7}
$$

$$
{}_3F_2\left[\begin{matrix} a, b, & f+m \\ \frac{1}{2}(a+b+2), & f \end{matrix}; \frac{1}{2}\right]
= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} \binom{m}{k} \frac{2^k D_k^-(a,b)}{(f)_k}, \tag{3.8}
$$

$$
{}_3F_2\left[\begin{matrix} a, b, & f+m \\ \frac{1}{2}(a+b), & f \end{matrix}; \frac{1}{2}\right]
= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} \binom{m}{k} \frac{2^k D_k^+(a,b)}{(f)_k}. \tag{3.9}
$$

The formula (3.7) was first given by Fox [2]. The above summations, together with those corresponding to $j=\pm 2$, have been given in [5], where certain limiting cases are also discussed.

4. Extension of Kummer's first summation theorem

The extension of Kummer's first summation theorem in (1.3) to the case where there are r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r) is given by the following theorem.

THEOREM 7. *Let (m_r) be a sequence of positive integers, with $m := m_1 + \cdots + m_r$, and let p be an integer. Set $q := |m+p|$ and define*

$$
\gamma_{p,m}(b) := \frac{\Gamma(b-p)}{\Gamma(b+m)} \quad (m+p \geqslant 0), \quad 1 \quad (m+p < 0). \tag{4.1}
$$

Then

$$
\begin{aligned}
&{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ 1+a-b+p, & (f_r) \end{matrix};-1\right] \\
&= 2^{p-m-2b}\frac{\Gamma(1+a-b+p)\gamma_{p,m}(b)}{\Gamma(1+a-2b+p-m)}\sum_{n=0}^{q}(\mp 1)^n\binom{q}{n}\hat{G}_n(a,\lambda), \qquad (4.2)
\end{aligned}
$$

where $\lambda := 1+a-2b+p-m$ *and*

$$
\hat{G}_n(a,\lambda) := \sum_{k=0}^{m}\frac{(a)_k\hat{C}_k(\xi)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}(n+k-q))} \qquad (4.3)
$$

with the upper or lower sign chosen according as $m+p \geqslant 0$ *or* $m+p<0$*, respectively. The coefficients* $\hat{C}_k(\xi)$ *are defined by the terminating series (compare (2.3) and (2.5))*

$$
\hat{C}_k(\xi) = \frac{(-1)^k}{k!}\,{}_{m+1}F_m\left[\begin{matrix}-k, (\xi_m+1) \\ (\xi_m)\end{matrix};1\right] = \sum_{j=k}^{m}(-1)^j d_j \mathbf{S}_j^{(k)}, \qquad (4.4)
$$

where $\mathbf{S}_j^{(k)}$ *is the Stirling number of the second kind and the* (ξ_m) *are the nonvanishing zeros of the associated parametric polynomial (2.9) of degree* m *expressed in the form*

$$
Q_m(t) = \frac{1}{(\lambda)_m}\sum_{k=0}^{m}(b)_k\hat{C}_k(\xi)(t)_k(\lambda-t)_{m-k} \equiv \sum_{j=0}^{m} d_j t^j \qquad (d_0=1).
$$

Proof. Put $z=-1$ and the denominatorial parameter $c=1+a-b+p$, where p is an integer, in the transformation formula (2.8) to obtain

$$
\begin{aligned}
&{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, & (f_r+m_r) \\ 1+a-b+p, & (f_r) \end{matrix};-1\right] \\
&= 2^{-a}\,{}_{m+2}F_{m+1}\left[\begin{matrix} a, 1+a-2b+p-m, (\xi_m+1) \\ 1+a-b+p, \quad (\xi_m) \end{matrix};\tfrac{1}{2}\right],
\end{aligned}
$$

where (ξ_m) are the zeros of the mth degree associated parametric polynomial $Q_m(t)$ in (2.9). Application of the generalized Gauss summation theorem in (3.1) with $j=m+p$ and the parameters (f_r) replaced by (ξ_m) then yields the result stated in (4.2).

From (2.3) and (2.5) the coefficients $\hat{C}_k(\xi)$ are

$$
\hat{C}_k(\xi) = \frac{(-1)^k}{k!}\,{}_{m+1}F_m\left[\begin{matrix}-k, (\xi_m+1) \\ (\xi_m)\end{matrix};1\right] = \frac{1}{\Lambda'}\sum_{j=k}^{m}\hat{\sigma}_j\mathbf{S}_j^{(k)},
$$

with $\Lambda' = \xi_1\ldots\xi_m$ and from (2.4) the $\hat{\sigma}_j$ $(0 \leqslant j \leqslant m)$ are generated by

$$
(\xi_1+x)\ldots(\xi_r+x) = \sum_{j=0}^{m}\hat{\sigma}_j x^j. \qquad (4.5)
$$

By comparison of (4.5) with the associated parametric polynomial (2.9) in the form

$$Q_m(t) = \frac{(-1)^m}{\Lambda'} \prod_{j=1}^{m} (t - \xi_j) = \sum_{j=0}^{m} d_j t^j \qquad (d_0 = 1),$$

it is easily seen that $\hat{\sigma}_j/\Lambda' = (-1)^j d_j$ $(0 \leqslant j \leqslant m)$. Hence we obtain

$$\hat{C}_k(\xi) = \sum_{j=k}^{m} (-1)^j d_j \mathbf{S}_j^{(k)},$$

which completes the proof. □

REMARK 3. It is important to note that in the application of the summation formula (4.2) it is not *necessary* to compute the zeros (ξ_m) of the associated parametric polynomial $Q_m(t)$ in (2.9); it is sufficient to determine only the coefficients d_j $(0 \leqslant j \leqslant m)$ associated with $Q_m(t)$.

When $r = 0$ (so that $m = 0$), we immediately obtain from (4.2) the contiguous form of Kummer's first summation formula for the ${}_2F_1$ series given by the following theorem.

THEOREM 8. *Let p be an integer and $q = |p|$. Then, with $\gamma_{p,m}(b)$ as defined in (4.1) (with $m = 0$), we have*

$$\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+p \end{matrix}; -1\right] \\
&= 2^{p-2b} \frac{\Gamma(1+a-b+p)\,\gamma_{p,0}(b)}{\Gamma(1+a-2b+p)} \sum_{n=0}^{q} (\mp 1)^n \binom{q}{n} \frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{2}n)}{\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}(n-q))} \qquad (4.6)
\end{aligned}$$

for $p = 0, \pm 1, \pm 2, \ldots$, where $\lambda := 1 + a - 2b + p$ and the upper or lower sign is chosen according as $p \geqslant 0$ or $p < 0$, respectively.

REMARK 4. The summation (4.6) is valid for any integer p and so represents a more general result than that given in Lavoie *et al.* [10], which is applicable for $|p| \leqslant 5$.

When $p = 0$, (4.6) combined with use of the duplication formula for the gamma function reduces to Kummer's summation formula in (1.3). When $r = 1$ and $m_1 = m = 1$ $(f_1 = f,\ \xi_1 = \xi)$, we have from (2.12) (with $c = 1 + a - b + p$) and (2.6) that

$$\xi = \frac{(a - 2b + p)f}{f - b}, \qquad \hat{C}_k(\xi) = \frac{1}{(\xi)_k} \quad (k = 0, 1).$$

Then, for $p = 0, \pm 1$, we find respectively after some algebra the summations

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, & f+1 \\ 1+a-b, & f \end{matrix}; -1\right] \\
&= \frac{\Gamma(1+a-b)}{\Gamma(1+a)} \left\{ \left(1 - \frac{a}{2f}\right) \frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(\frac{1}{2}a - b + 1)} + \frac{a}{2f} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2})}{\Gamma(\frac{1}{2}a - b + \frac{1}{2})} \right\}, \qquad (4.7)
\end{aligned}$$

$$
{}_3F_2\left[\begin{matrix} a,\, b, & f+1 \\ 2+a-b, & f \end{matrix};-1\right]
$$

$$
=\frac{\Gamma(2+a-b)}{2(b-1)\Gamma(a)}\left\{\left(1-\frac{1+a-b}{f}\right)\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}a-b+\frac{3}{2})}-\left(1-\frac{a}{f}\right)\frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)}\right\} \tag{4.8}
$$

and

$$
{}_3F_2\left[\begin{matrix} a,\, b,\, f+1 \\ a-b, \quad f \end{matrix};-1\right]
$$

$$
=\frac{\Gamma(a-b)}{2\,\Gamma(a)}\left\{\frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b)}+\left(1-\frac{b}{f}\right)\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}a-b+\frac{1}{2})}\right\}. \tag{4.9}
$$

REMARK 5. The summations (4.7) and (4.8) have been given previously in [4, Eqs. (5.1), (5.10)].

Finally, when $r=2$, $m_1=m_2=1$, the associated parametric polynomial is, from (2.9), given by $Q_2(t)=d_0+d_1t+d_2t^2$, where

$$
d_0=1,\qquad d_1=-\frac{((\alpha+\beta)\lambda+\beta)}{f_1f_2\lambda(\lambda+1)},\qquad d_2=\frac{\alpha}{f_1f_2\lambda(\lambda+1)},
$$

with

$$
\alpha=(f_1-b)(f_2-b),\qquad \beta=f_1f_2-b(b+1),\qquad \lambda=a-2b+p-1.
$$

From (4.4), we therefore find the coefficients

$$
\hat{C}_0(\xi)=1,\qquad \hat{C}_1(\xi)=d_2-d_1,\qquad \hat{C}_2(\xi)=d_2.
$$

Then we have the summation

$$
{}_4F_3\left[\begin{matrix} a,\, b, & f_1+1,\, f_2+1 \\ 1+a-b+p, & f_1,\, f_2 \end{matrix};-1\right]
$$

$$
=2^{p-2b-2}\frac{\Gamma(1+a-b+p)\gamma_{p,2}(b)}{\Gamma(a-2b+p-1)}\sum_{n=0}^{q}(\mp1)^n\binom{q}{n}\hat{G}_n(a,\lambda) \tag{4.10}
$$

for $p=0,\pm1,\pm2,\ldots$, where $\gamma_{p,m}(b)$ and $\hat{G}_n(a,\lambda)$ are defined in (4.1) and (4.3), $q=|p+2|$, and the upper or lower sign corresponds to $p\geqslant-2$ or $p<-2$.

We note that when $r=1$, $m_1=m=2$, we can put $f_1=f$, $f_2=f+1$ in the above to obtain the evaluation of

$$
{}_3F_2\left[\begin{matrix} a,\, b, & f+2 \\ 1+a-b+p, & f \end{matrix};-1\right]\equiv{}_4F_3\left[\begin{matrix} a,\, b, & f+1,\, f+2 \\ 1+a-b+p, & f,\, f+1 \end{matrix};-1\right]
$$

given by (4.10).

5. Extension of Kummer's third summation theorem

The extension of Kummer's third summation theorem in (1.5) to the case where there are r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r) is given by the following theorem.

THEOREM 9. *Let (m_r) be a sequence of positive integers, with $m := m_1 + \cdots + m_r$, and let $\lambda := c+a-m-1$, $\lambda' := c-a-m$. Then*

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a,\ 1-a, (f_r+m_r) \\ c, \qquad (f_r) \end{matrix}; \tfrac{1}{2}\right]
$$

$$
= \frac{2^{a-1}\Gamma(c)\Gamma(1-a-m)}{\Gamma(a+c-m-1)\Gamma(1-a+m)} \sum_{n=0}^{2m} (-1)^n \binom{2m}{n} \hat{G}_n(\lambda', \lambda), \tag{5.1}
$$

where

$$
\hat{G}_n(\lambda', \lambda) := \sum_{k=0}^{m} \frac{(\lambda')_k \hat{\mathscr{C}}_k(\eta)\Gamma(\frac{1}{2}\lambda + \frac{1}{2}(n+k))}{\Gamma(\frac{1}{2}\lambda' + \frac{1}{2} + \frac{1}{2}(n+k-2m))}. \tag{5.2}
$$

The coefficients $\hat{\mathscr{C}}_k(\eta)$ are defined by the finite series (compare (2.3) and (2.5))

$$
\hat{\mathscr{C}}_k(\eta) = \frac{(-1)^k}{k!} {}_{m+1}F_m\left[\begin{matrix} -k, (\eta_m+1) \\ (\eta_m) \end{matrix}; 1\right] = \sum_{j=k}^{m} (-1)^j \hat{d}_j \mathbf{S}_j^{(k)}, \tag{5.3}
$$

where $\mathbf{S}_j^{(k)}$ is the Stirling number of the second kind and the (η_m) are the nonvanishing zeros of the associated parametric polynomial $\hat{Q}_m(t)$ of degree m in (2.11) with the parameter $b = 1-a$.

Proof. Put $z = \frac{1}{2}$ and the numeratorial parameter $b = 1-a$ in the transformation formula (2.10) to find

$$
{}_{r+2}F_{r+1}\left[\begin{matrix} a,\ 1-a, (f_r+m_r) \\ c, \qquad (f_r) \end{matrix}; \tfrac{1}{2}\right] = 2^{m+1-c}\, {}_{m+2}F_{m+1}\left[\begin{matrix} \lambda,\ \lambda', (\eta_m+1) \\ c, \qquad (\eta_m) \end{matrix}; \tfrac{1}{2}\right],
$$

where $\lambda := c+a-m-1$, $\lambda' := c-a-m$ and the (η_m) are the nonvanishing zeros of the associated parametric polynomial of degree m given in (2.11) with $b = 1-a$.

Application of the extension of the summation theorem in (3.1) with $j = 2m$ and $r \to m$, $(f_r) \to (\eta_m)$ then yields the result stated in (5.1), with the coefficients $\hat{C}_k(\eta)$ defined in a similar manner to those appearing in Theorem 7. □

REMARK 6. It is important to note that in the application of the summation formula (5.1) it is not *necessary* to compute the zeros (η_m) of the associated parametric polynomial $\hat{Q}_m(t)$ in (2.11); it is sufficient to determine only the coefficients $\hat{d}_j$ $(0 \leqslant j \leqslant m)$ of $\hat{Q}_m(t)$.

In the case $r = 1$, $m_1 = m = 1$ we have from (2.6) that $\hat{C}_k(\eta) = 1/(\eta)_k$ $(k = 0, 1)$, where from (2.13) with $b = 1-a$

$$
\eta = \frac{(c-a-1)(c+a-2)f}{a(1-a)+(c-2)f}.
$$

Then from (5.1) we find

$$ {}_3F_2\left[\begin{matrix} a,\,1-a,f+1 \\ c, \qquad f \end{matrix};\tfrac{1}{2}\right] = \frac{2^{a-1}\Gamma(c)}{\Gamma(c+a-2)a(a-1)}\sum_{n=0}^{2}(-1)^n\binom{2}{n}\left[\frac{\Gamma(\frac{1}{2}(c+a+n)-1)}{\Gamma(\frac{1}{2}(c-a+n)-1)}+\frac{c-a-1}{\eta}\frac{\Gamma(\frac{1}{2}(c+a+n)-\frac{1}{2})}{\Gamma(\frac{1}{2}(c-a+n)-\frac{1}{2})}\right]. \tag{5.4} $$

This result has been given in a different form in [4, (5.3)].

When $r=1$, $m_1=2$ we find from (2.4) that $\sigma_0=(f)_2$, $\sigma_1=2f+1$, $\sigma_2=1$ so that $C_{0,1}=1$, $C_{1,1}=2/f$ and $C_{2,1}=1/(f)_2$. With $s:=c-a-b-2$, we obtain from (2.11) the quadratic parametric polynomial (with zeros η_1 and η_2) given by

$$ \hat{Q}_2(t)=1-\frac{2st}{\lambda\lambda'}+\frac{(t)_2(s)_2}{(\lambda)_2(\lambda')_2}-\frac{2abt}{\lambda\lambda' f}\left\{1-\frac{s(1+t)}{(\lambda+1)(\lambda'+1)}\right\}+\frac{(a)_2(b)_2}{(\lambda)_2(\lambda')_2(f)_2}t(1+t), $$

Upon rearrangement this yields

$$ \hat{Q}_2(t)=1-\frac{2Bt}{\lambda\lambda'}+\frac{Ct(1+t)}{(\lambda)_2(\lambda')_2}, $$

where

$$ B:=s+\frac{ab}{f}, \qquad C:=(s)_2+\frac{2abs}{f}+\frac{(a)_2(b)_2}{(f)_2}. $$

When $a=\frac{1}{4}$, $b=1-a$, $c=1$, $f=\frac{1}{2}$, for example, we consequently find

$$ \hat{Q}_2(t)=\frac{1}{105}(105+424t+268t^2) $$

so that, from (5.3),

$$ \hat{C}_0(\eta)=1, \quad \hat{C}_1(\eta)=-\frac{52}{35}, \quad \hat{C}_2(\eta)=\frac{268}{105}. $$

Evaluation of the right-hand side of (5.1) for the above parameters yields the value (to 10dp) ${}_3F_2[\frac{1}{4},\frac{3}{4},\frac{5}{2};1,\frac{1}{2};\frac{1}{2}]=2.2113833040$, which agrees with series evaluation of the ${}_3F_2(\frac{1}{2})$ series on the left-hand side.

6. Extension of Watson's summation theorem

We first establish two lemmas that will be required in our proof of the extension of Watson's summation theorem (1.6).

LEMMA 1. *Let (m_r) denote a set of positive integers and $m:=m_1+\cdots+m_r$. Then*

$$ {}_{r+3}F_{r+2}\left[\begin{matrix} a,b,c, & (f_r+m_r) \\ \frac{1}{2}(a+b+1),2c, & (f_r) \end{matrix};1\right] = \sum_{n=0}^{\infty}\frac{2^{-4n}(a)_{2n}(b)_{2n}((f_r+m_r))_{2n}}{(\frac{1}{2}(a+b+1))_{2n}((f_r))_{2n}(c+\frac{1}{2})_n n!}\,{}_{r+2}F_{r+1}\left[\begin{matrix} a+2n,\ b+2n,(f_r+m_r)+2n \\ \frac{1}{2}(a+b+1)+2n,(f_r)+2n \end{matrix};\tfrac{1}{2}\right] \tag{6.1} $$

provided $\Re(c-\frac{1}{2}a-\frac{1}{2}b)>m-\frac{1}{2}$.

Proof. We first observe, by application of the well-known Gauss summation theorem for the ${}_2F_1$ series of unit argument, that

$$\frac{2^k(c)_k}{(2c)_k}={}_2F_1\left[\begin{matrix}-\frac{1}{2}k,\,-\frac{1}{2}k+\frac{1}{2}\\ c+\frac{1}{2}\end{matrix};1\right]=\sum_{n=0}^{[k/2]}\frac{(-\frac{1}{2}k)_n(-\frac{1}{2}k+\frac{1}{2})_n}{(c+\frac{1}{2})_n\,n!}$$
$$=\sum_{n=0}^{[k/2]}\frac{2^{-2n}k!}{n!\,(k-2n)!\,(c+\frac{1}{2})_n},$$

where square brackets denote the integer part and use has been made of the fact that

$$(k-2n)!=\frac{2^{-2n}k!}{(-\frac{1}{2}k)_n(-\frac{1}{2}k+\frac{1}{2})_n}.$$

Then

$$F\equiv{}_{r+3}F_{r+2}\left[\begin{matrix}a,b,c, & (f_r+m_r)\\ \frac{1}{2}(a+b+1),2c, & (f_r)\end{matrix};1\right]=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k((f_r+m_r))_k2^{-k}}{(\frac{1}{2}(a+b+1))_k((f_r))_kk!}\cdot\frac{2^k(c)_k}{(2c)_k}$$
$$=\sum_{k=0}^{\infty}\sum_{n=0}^{[k/2]}\frac{(a)_k(b)_k((f_r+m_r))_k}{(\frac{1}{2}(a+b+1))_k((f_r))_k}\,\frac{2^{-k-2n}}{n!\,(k-2n)!(c+\frac{1}{2})_n}.$$

Interchange of the order of summation followed by the substitution $k\to k+2n$ then produces

$$F=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_{k+2n}(b)_{k+2n}((f_r+m_r))_{k+2n}}{(\frac{1}{2}(a+b+1))_{k+2n}((f_r))_{k+2n}}\,\frac{2^{-k-4n}}{n!\,k!(c+\frac{1}{2})_n}$$
$$=\sum_{n=0}^{\infty}\frac{2^{-4n}(a)_{2n}(b)_{2n}((f_r+m_r))_{2n}}{(\frac{1}{2}(a+b+1))_{2n}((f_r))_{2n}(c+\frac{1}{2})_nn!}\sum_{k=0}^{\infty}\frac{(a+2n)_k(b+2n)_k((f_r+m_r)+2n)_k}{(\frac{1}{2}(a+b+1)+2n)_k((f_r)+2n)_k2^kk!},$$

where we have used the identity

$$(a)_{k+j}=(a)_j(a+j)_k \tag{6.2}$$

with $j=2n$. Identification of the second series over k as a ${}_{r+2}F_{r+1}(\frac{1}{2})$ series then yields the result stated in (6.1). □

LEMMA 2. *For positive integer m and integer k satisfying $0\leqslant k\leqslant m$, we have the summation*

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}k,\,\frac{1}{2}b+\frac{1}{2}k,\,\frac{1}{2}f+\frac{1}{2}m,\,\frac{1}{2}f+\frac{1}{2}m+\frac{1}{2}\\ c+\frac{1}{2},\qquad \frac{1}{2}f+\frac{1}{2}k,\,\frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}\end{matrix};1\right]$$
$$=\frac{\Gamma(c+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b-k)}{\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k)\Gamma(c+\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}k)}\sum_{j=0}^{m-k}\frac{(\frac{1}{2}a+\frac{1}{2}k)_j(\frac{1}{2}b+\frac{1}{2}k)_j\,\mathscr{C}_{j,k}^{(m)}}{j!\,(k+\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c)_j} \tag{6.3}$$

when $\Re(c-\frac{1}{2}a-\frac{1}{2}b)>m-\frac{1}{2}$, *where for non-negative integer* j

$$\mathscr{C}_{j,k}^{(m)} = {}_3F_2\left[\begin{matrix}-j,\ \frac{1}{2}f+\frac{1}{2}m,\ \frac{1}{2}f+\frac{1}{2}m+\frac{1}{2}\\ \frac{1}{2}f+\frac{1}{2}k,\ \frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}\end{matrix};1\right]. \tag{6.4}$$

Proof. From the generalized Karlsson-Minton summation formula (2.2) we find

$${}_4F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}k,\ \frac{1}{2}b+\frac{1}{2}k,\ f_1+m_1,\ f_2+m_2\\ c+\frac{1}{2},\qquad f_1,\ f_2\end{matrix};1\right]$$
$$=\frac{\Gamma(c+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b-k)}{\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k)\Gamma(c+\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}k)}\sum_{j=0}^{m_1+m_2}\frac{(-1)^j(\frac{1}{2}a+\frac{1}{2}k)_j(\frac{1}{2}b+\frac{1}{2}k)_j C_{j,2}}{(k+\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c)_j}, \tag{6.5}$$

where $k=0,1,2,\ldots$, m_1, m_2 are positive integers and it is supposed that $\Re(c-\frac{1}{2}a-\frac{1}{2}b)>m_1+m_2+k-\frac{1}{2}$ for convergence. The coefficients $C_{j,2}$ when $r=2$ are defined in (2.6) by

$$C_{j,2}=\frac{(-1)^j}{j!}\,{}_3F_2\left[\begin{matrix}-j,\ f_1+m_1,\ f_2+m_2\\ f_1,\ f_2\end{matrix};1\right]. \tag{6.6}$$

Now consider the series

$${}_4F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}k,\ \frac{1}{2}b+\frac{1}{2}k,\ \frac{1}{2}f+\frac{1}{2}m,\ \frac{1}{2}f+\frac{1}{2}m+\frac{1}{2}\\ c+\frac{1}{2},\qquad \frac{1}{2}f+\frac{1}{2}k,\ \frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}\end{matrix};1\right]\qquad(0\leqslant k\leqslant m),$$

where m is a positive integer and $\Re(c-\frac{1}{2}a-\frac{1}{2}b)>m-\frac{1}{2}$. When m and k are of the same parity, we can let $f_1=\frac{1}{2}f+\frac{1}{2}k$, $f_2=\frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}$, with $m_1=m_2=\frac{1}{2}(m-k)$ in (6.5). When m and k are of different parity, we can let $f_1=\frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}$, $m_1=\frac{1}{2}(m-k-1)$ and $f_2=\frac{1}{2}f+\frac{1}{2}k$, $m_2=\frac{1}{2}(m-k+1)$. In both cases we see that $m_1+m_2=m-k$. Substitution of these values of f_1, f_2, m_1 and m_2 into (6.5) and (6.6) then yields the result stated in (6.3) and (6.4). □

We now state the principal result of this section, which provides a generalization of Watson's summation theorem (1.6) to the case when a single pair of numeratorial and denominatorial parameters differs by a positive integer m.

THEOREM 10. *Let* m *denote a positive integer. Define the constants*

$$B_k=(c+\tfrac{1}{2}-\tfrac{1}{2}a-\tfrac{1}{2}b)_{-k}\,\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}k)\Gamma(\frac{1}{2}b+\frac{1}{2}k)\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\Gamma(c+\frac{1}{2}-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k)\Gamma(c+\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}k)}$$

and the coefficients for non-negative integer j

$$\mathscr{C}_{j,k}^{(m)} = {}_3F_2\left[\begin{matrix}-j,\ \frac{1}{2}f+\frac{1}{2}m,\ \frac{1}{2}f+\frac{1}{2}m+\frac{1}{2}\\ \frac{1}{2}f+\frac{1}{2}k,\ \frac{1}{2}f+\frac{1}{2}k+\frac{1}{2}\end{matrix};1\right]$$

with $0 \leqslant k \leqslant m$. *Then, provided* $\Re(c-\frac{1}{2}a-\frac{1}{2}b) > m-\frac{1}{2}$, *we have the summation*

$$
{}_4F_3\left[\begin{matrix} a, b, c, & f+m \\ \frac{1}{2}(a+b+1), 2c, & f \end{matrix}; 1\right]
= \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\Gamma(c+\frac{1}{2}-\frac{1}{2}b)}\Upsilon_m, \tag{6.7}
$$

where

$$
\Upsilon_m = \sum_{k=0}^{m}\binom{m}{k}\frac{2^k B_k}{(f)_k}\sum_{j=0}^{m-k}\frac{(\frac{1}{2}a+\frac{1}{2}k)_j(\frac{1}{2}b+\frac{1}{2}k)_j}{j!(k+\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}-c)_j}\mathscr{C}_{j,k}^{(m)}.
$$

When $m=0$, *we find* $\Upsilon_0 = 1$ *so that Watson's summation theorem in (1.6) is recovered.*

Proof. From (6.1) with $r=1$ $(m_1 = m,\ f_1 = f)$ we find

$$
{}_4F_3\left[\begin{matrix} a, b, c, & f+m \\ \frac{1}{2}(a+b+1), 2c, & f \end{matrix}; 1\right]
= \sum_{n=0}^{\infty}\frac{2^{-4n}(a)_{2n}(b)_{2n}(f+m)_{2n}}{(\frac{1}{2}(a+b+1))_{2n}(f)_{2n}(c+\frac{1}{2})_n n!}\,{}_3F_2\left[\begin{matrix} a+2n,\ b+2n, f+2n+m \\ \frac{1}{2}(a+b+1)+2n, f+2n \end{matrix}; \frac{1}{2}\right].
$$

From (3.7) with $a \to a+2n$, $b \to b+2n$, $f \to f+2n$ and the duplication formula for the gamma function, we have

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a+2n,\ b+2n, f+2n+m \\ \frac{1}{2}(a+b+1)+2n, f+2n \end{matrix}; \frac{1}{2}\right] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n)}{\Gamma(\frac{1}{2}a+n)\Gamma(\frac{1}{2}b+n)\Gamma(\frac{1}{2}a+\frac{1}{2}+n)\Gamma(\frac{1}{2}b+\frac{1}{2}+n)} \\
&\quad \times \sum_{k=0}^{m}\binom{m}{k}\frac{2^k\Gamma(\frac{1}{2}a+\frac{1}{2}k+n)\Gamma(\frac{1}{2}b+\frac{1}{2}k+n)}{(f+2n)_k}.
\end{aligned}
$$

Then, making use of the identities

$$
(f+2n)_k = \frac{(f)_k(f+k)_{2n}}{(f)_{2n}}, \qquad (a)_{2n} = 2^{2n}(\tfrac{1}{2}a)_n(\tfrac{1}{2}a+\tfrac{1}{2})_n, \tag{6.8}
$$

we obtain

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, b, c, & f+m \\ \frac{1}{2}(a+b+1), 2c, & f \end{matrix}; 1\right] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{2^{-4n}(a)_{2n}(b)_{2n}(f+m)_{2n}(\frac{1}{2}a+\frac{1}{2}k)_n(\frac{1}{2}b+\frac{1}{2}k)_n}{(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n(\frac{1}{2}b)_n(\frac{1}{2}b+\frac{1}{2})_n(c+\frac{1}{2})_n n!} \\
&\quad \times \sum_{k=0}^{m}\binom{m}{k}\frac{2^k\Gamma(\frac{1}{2}a+\frac{1}{2}k)\Gamma(\frac{1}{2}b+\frac{1}{2}k)}{(f)_k(f+2k)_{2n}}
\end{aligned}
$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})} \sum_{k=0}^{m} \binom{m}{k} \frac{2^k}{(f)_k} \Gamma(\tfrac{1}{2}a+\tfrac{1}{2}k)\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}k)$$
$$\times \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a+\frac{1}{2})_n(\frac{1}{2}b+\frac{1}{2})_n(f+m)_{2n}}{(c+\frac{1}{2})_n(f+k)_{2n}n!}.$$

Use of the second relation in (6.2) enables the second sum over n to be identified as a ${}_4F_3(1)$ series, which leads to the result

$${}_4F_3\left[\begin{matrix} a,\, b,\, c, & f+m \\ \frac{1}{2}(a+b+1),\, 2c, & f \end{matrix};1\right]$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}$$
$$\times \sum_{k=0}^{m} \binom{m}{k} \frac{2^k\Gamma(\frac{1}{2}a+\frac{1}{2}k)\Gamma(\frac{1}{2}b+\frac{1}{2}k)}{(f)_k} {}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}k,\, \frac{1}{2}b+\frac{1}{2}k,\, \frac{1}{2}f+\frac{1}{2}m,\, \frac{1}{2}f+\frac{1}{2}m+\frac{1}{2} \\ c+\frac{1}{2},\, \frac{1}{2}f+\frac{1}{2}k,\, \frac{1}{2}f+\frac{1}{2}k+\frac{1}{2} \end{matrix};1\right].$$

Employing the evaluation of the ${}_4F_3(1)$ series in (6.3) and (6.4) we then obtain after a little algebra the result stated in (6.7). □

The coefficients $\mathscr{C}_{j,k}^{(m)}$ in (6.4) can also be written as the finite series

$$\mathscr{C}_{j,k}^{(m)} = \sum_{n=0}^{j} \frac{(-j)_n(f+m)_{2n}}{n!\,(f+k)_{2n}}$$

from which it is readily seen that

$$\mathscr{C}_{0,k}^{(m)} = 1, \quad \mathscr{C}_{1,k}^{(m)} = \frac{(k-m)(1+2f+k+m)}{(f+k)(f+k+1)}, \ \ldots\ , \quad \mathscr{C}_{j,m}^{(m)} = 0.$$

When $m=1$, $\mathscr{C}_{1,0}^{(1)} = -2/f$ and accordingly we obtain the summation formula

$${}_4F_3\left[\begin{matrix} a,\, b,\, c, & f+1 \\ \frac{1}{2}(a+b+1),\, 2c, & f \end{matrix};1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\Gamma(c+\frac{1}{2}-\frac{1}{2}b)}\Upsilon_1, \tag{6.9}$$

with

$$\Upsilon_1 = 1 + \frac{2}{(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2})f}\left\{\frac{1}{4}ab + \frac{(\frac{1}{2}a)_{1/2}(\frac{1}{2}b)_{1/2}}{(c+\frac{1}{2}-\frac{1}{2}a)_{-1/2}(c+\frac{1}{2}-\frac{1}{2}b)_{-1/2}}\right\} \tag{6.10}$$

valid when $\Re(c-\frac{1}{2}a-\frac{1}{2}b) > \frac{1}{2}$. In (6.10), we have employed the extended definition of the Pochhammer symbol in (1.2).

REMARK 7. Two different generalizations of Watson's summation theorem analogous to that in (6.9) in the case $m=1$ have been given recently in [4] when the denominatorial parameters are respectively $\frac{1}{2}(a+b+3)$, $2c$ and $\frac{1}{2}(a+b+1)$, $2c+1$.

7. Concluding remarks

We have obtained the extensions of three classical summation theorems of Kummer for the generalized hypergeometric series when there are r pairs of numeratorial and denominatorial parameters differing by positive integers. In the case of Watson's summation theorem, however, this has been only possible for a ${}_4F_3(1)$ series with a single pair of numeratorial and denominatorial parameters differing by a positive integer. This is due to the coefficients $C_{k,r}$ in (3.4) in the case $r = 1$ possessing a simple form, for which the n-dependence is separable by the first identity in (6.8).

We wish to point out that all the formulas developed in this paper have been tested numerically with the aid of *Mathematica*. Application of these results is under investigation and will be the subject of a forthcoming paper. It is hoped that these summation formulas will be of interest and will help advance research in this important area of classical special functions.

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$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\cdots,$$

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(Received April 15, 2013)

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