

A LOGARITHMIC MEAN AND INTERSECTIONS OF OSCULATING HYPERPLANES IN R^n

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Abstract. We discuss a special case of the means defined in [1]. Let C be the curve in R^n with vector equation $\hat{\alpha}(t) = \langle t, t \log t, \dots, t(\log t)^{n-1} \rangle$. Let $0 < a_1 < \dots < a_n$ and let O_k be the osculating hyperplane to C at a_k . Then we show that O_1, \dots, O_n have a unique point of intersection, $P = (i_1, \dots, i_n) \in R^n$, and in particular, i_1 equals the mean

$$M(a_1, \dots, a_n) = (n-1)! \sum_{\substack{j=1 \\ i \neq j}}^n \frac{a_j}{\prod_{i=1}^n (\ln a_j - \ln a_i)},$$

the logarithmic mean of Neuman.

1. Introduction

For $n \geq 3$, let C be the curve in R^n with vector equation $\hat{\alpha}(t) = \langle x_1(t), \dots, x_n(t) \rangle$, let

$$W_{j,n}(t) = W(x'_1(t), \dots, x'_{j-1}(t), x'_{j+1}(t), \dots, x'_n(t)),$$

the Wronskian of $x'_1(t), \dots, x'_{j-1}(t), x'_{j+1}(t), \dots, x'_n(t)$, $j = 1, \dots, n$, let \hat{x} be the vector $\langle x_1, \dots, x_n \rangle$, and let $\hat{n}(t)$ be the vector $\langle W_{1,n}(t), -W_{2,n}(t), \dots, (-1)^{n+1}W_{n,n}(t) \rangle$. Note that with this notation we mean that $W_{1,n}(t) = W(x'_2(t), \dots, x'_n(t))$, and $W_{n,n}(t) = W(x'_1(t), \dots, x'_{n-1}(t))$. In [1] we defined the **osculating hyperplane**, O_a , to C at $t = a$ to be the hyperplane in R^n with equation

$$\hat{x} \cdot \hat{n}(a) = \hat{\alpha}(a) \cdot \hat{n}(a), \text{ assuming that } \hat{n}(a) \neq \hat{0}.$$

It is not hard to show (see [1]) that O_a has n th order contact with C at $t = a$. That is, if $C_a(t) = (\hat{\alpha}(t) - \hat{\alpha}(a)) \cdot \hat{n}(a)$, then $C_a^{(j)}(a) = 0$ for $j = 0, 1, \dots, n-1$. This generalizes the osculating plane in R^3 , which has 3rd order contact with C at a . For example, if $\hat{\alpha}(t) = \langle t, t^2, t^3, t^4 \rangle$, then $W_{1,4}(t) = W(2t, 3t^2, 4t^3) = 48t^3$, $W_{2,4}(t) = W(1, 3t^2, 4t^3) = 72t^2$, $W_{3,4}(t) = W(1, 2t, 4t^3) = 48t$, $W_{4,4}(t) = W(1, 2t, 3t^2) = 12$, and $\hat{n}(t) = \langle 48t^3, -72t^2, 48t, -12 \rangle$. The equation of the osculating hyperplane at $t = 1$ is $\langle x_1, x_2, x_3, x_4 \rangle \cdot \hat{n}(1) = \hat{\alpha}(1) \cdot \hat{n}(1)$ or $4x_1 - 6x_2 + 4x_3 - x_4 = 1$.

$$\begin{aligned} C_1(t) &= \langle t-1, t^2-1, t^3-1, t^4-1 \rangle \cdot \langle 48, -72, 48, -12 \rangle \\ &= -12(t^4 - 4t^3 + 6t^2 - 4t + 1), \end{aligned}$$

and it then follows that $C_1(1) = C'_1(1) = C''_1(1) = C'''_1(1) = 0$.

Mathematics subject classification (2010): 25E60, 26B99.

Keywords and phrases: logarithmic mean; osculating hyperplane; Wronskian.

In [1] the author proved the following general result about defining means using intersections of osculating hyperplanes to curves in R^n .

THEOREM 1. *Let C be the curve in R^n with vector equation*

$$\hat{\alpha}(t) = \langle x_1(t), \dots, x_n(t) \rangle, \quad t \in I = [a, b],$$

where each $x_k \in C^{n-1}(I)$ and is strictly monotone on I .

Let

$$W_{j,n}(t) = W(x'_1(t), \dots, x'_{j-1}(t), x'_{j+1}(t), \dots, x'_n(t))$$

be the Wronskian of $x'_1(t), \dots, x'_{j-1}(t), x'_{j+1}(t), \dots, x'_n(t)$. Assume that every subset of $\{W_{1,n}, \dots, W_{n,n}\}$ is an extended complete Chebyshev system on I . Let $a = a_1 < \dots < a_n = b$ be n given points in I , and let O_k be the osculating hyperplane to C at a_k . Then

- (1) O_1, \dots, O_n have a unique point of intersection, P , in R^n , and
- (2) If $P = (i_1, \dots, i_n)$, then $a_1 < x_k^{-1}(i_k) < a_n$ for $k = 1, 2, \dots, n$.

By Theorem 1, one can define n symmetric means in a_1, \dots, a_n as follows:

$$M_k(a_1, \dots, a_n) = x_k^{-1}(i_k), \quad k = 1, \dots, n.$$

In particular, we showed in [1] that if $x_k(t) = t^k$, $k = 1, \dots, n-2$, $x_{n-1}(t) = \log t$, and $x_n(t) = \frac{1}{t}$, then $M_n(a_1, \dots, a_n) = P(a_1, \dots, a_n)$, where P is the logarithmic mean in n variables defined by Pittenger [5]. At the end of [1] we stated that perhaps another interesting generalization of the logarithmic mean to n variables would be $M_1(a_1, \dots, a_n)$, where $x_k(t) = t(\log t)^{k-1}$, $k = 1, \dots, n$. We never pursued that, but the point of this paper is to prove that $M_1(a_1, \dots, a_n)$ equals the following logarithmic mean in n variables defined by Neuman [4]:

$$L_N(a_1, \dots, a_n) = (n-1)! \sum_{j=1}^n \frac{a_j}{\prod_{\substack{i=1 \\ i \neq j}}^n (\ln a_j - \ln a_i)}.$$

That is, we show that Neuman's logarithmic mean equals the x coordinate of the intersection of the osculating hyperplanes to the curve $\hat{\alpha}(t) = \langle t, t \log t, \dots, t(\log t)^{n-1} \rangle$. L_N was also defined in a different way (and unknowingly) by Xiao and Zhang [7]. Mustonen [3] gives a good summary of these connections and other generalizations. See also the paper by Merikoski [2], where a general approach is given for extending means in two variables to n variables. The methods used in this paper are decidedly different than those in the papers just cited. We now state our main result.

THEOREM 2. *For $n \geq 3$, let C be the curve in R^n with vector equation $\hat{\alpha}(t) = \langle t, t \log t, \dots, t(\log t)^{n-1} \rangle$. Let $0 < a_1 < \dots < a_n$ and let O_k be the osculating hyperplane to C at a_k . Then O_1, \dots, O_n have a unique point of intersection, $P = (i_1, \dots, i_n) \in$*

R^n , and

$$M_1(a_1, \dots, a_n) = (n-1)! \sum_{j=1}^n \frac{a_j}{\prod_{\substack{i=1 \\ i \neq j}}^n (\ln a_j - \ln a_i)}. \tag{1}$$

Note that for $n=2$, the x coordinate of the point of intersection of the tangent lines to the curve $\hat{\alpha}(t) = \langle t, t \log t \rangle$ is the well known logarithmic mean $L(a, b) = \frac{b-a}{\ln b - \ln a}$ in two variables. So in a certain sense the logarithmic mean $L_N(a_1, \dots, a_n)$ above is a natural generalization of the logarithmic mean in two variables since it involves intersections of osculating hyperplanes to a curve in R^n whose first two components are t and $t \log t$, and where the remaining components follow the “natural pattern” of the first two components.

REMARK 1. Using the curve from Theorem 2 and Theorem 1, one also obtains means $M_k(a_1, \dots, a_n) = x_k^{-1}(i_k), k = 2, \dots, n$. Of course those means involve the inverses of the functions $y = t(\log t)^m, m \geq 1$, which are not elementary functions.

2. Preliminary Material

If f_1, \dots, f_n are n given functions of t , then we let

$$W(f_1, \dots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f'_1(t) & f'_2(t) & \cdots & f'_n(t) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}$$

denote the Wronskian determinant. Throughout the rest of the paper, we define

$$\begin{aligned} x_k(t) &= t(\log t)^{k-1}, k = 1, \dots, n. \\ W_{k,n}(t) &= W(x'_1(t), \dots, x'_{k-1}(t), x'_{k+1}(t), \dots, x'_n(t)), k = 1, \dots, n. \end{aligned} \tag{2}$$

LEMMA 1. For $r \geq 2$

$$x_{k+1}^{(r)}(t) = k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r-j)}(t). \tag{3}$$

Proof. We use induction in r . So suppose that (3) holds for some positive integer $r \geq 2$.

$$\begin{aligned} x_{k+1}^{(r+1)}(t) &= \frac{d}{dt} x_{k+1}^{(r)}(t) = k \frac{d}{dt} \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r-j)}(t) \\ &= k \sum_{j=1}^{r-1} (-1)^{j+1} (j-1)! \binom{r-2}{j-1} \frac{d}{dt} (t^{-j} x_k^{(r-j)}(t)) \end{aligned}$$

$$\begin{aligned}
&= k \sum_{j=1}^{r-1} (-1)^{j+1} (j-1)! \binom{r-2}{j-1} (t^{-j} x_k^{(r-j+1)}(t) - j t^{-j-1} x_k^{(r-j)}(t)) \\
&= k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r+1-j)}(t) - k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} j! \binom{r-2}{j-1}}{t^{j+1}} x_k^{(r-j)}(t) \\
&= k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r+1-j)}(t) - k \sum_{l=2}^r \frac{(-1)^l (l-1)! \binom{r-2}{l-2}}{t^l} x_k^{(r+1-l)}(t) \\
&= k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r+1-j)}(t) + k \sum_{j=2}^r \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-2}}{t^j} x_k^{(r+1-j)}(t).
\end{aligned}$$

Using the identity $\binom{r-2}{j-1} + \binom{r-2}{j-2} = \binom{r-1}{j-1}$, we have

$$(j-1)! \binom{r-2}{j-1} + (j-1)! \binom{r-2}{j-2} = (j-1)! \binom{r-1}{j-1},$$

which implies that

$$\begin{aligned}
&k \sum_{j=1}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-1}}{t^j} x_k^{(r+1-j)}(t) + k \sum_{j=2}^r \frac{(-1)^{j+1} (j-1)! \binom{r-2}{j-2}}{t^j} x_k^{(r+1-j)}(t) \\
&= k \sum_{j=2}^{r-1} \frac{(-1)^{j+1} (j-1)! \binom{r-1}{j-1}}{t^j} x_k^{(r+1-j)}(t) + \frac{k}{t} x_k^{(r)}(t) + \frac{k}{t^r} (-1)^{r+1} (r-1)! x_k^{(1)}(t) \\
&= k \sum_{j=1}^r \frac{(-1)^{j+1} (j-1)! \binom{r-1}{j-1}}{t^j} x_k^{(r+1-j)}(t).
\end{aligned}$$

To start the induction, $x'_{k+1}(t) = \frac{k}{t} x_k(t) + \frac{1}{t} x_{k+1}(t)$, which implies that

$$\begin{aligned}
x''_{k+1}(t) &= \frac{k}{t} x'_k(t) - \frac{k}{t^2} x_k(t) + \frac{1}{t} x'_{k+1}(t) - \frac{1}{t^2} x_{k+1}(t) \\
&= \frac{k}{t} x'_k(t) - \frac{k}{t^2} x_k(t) + \frac{1}{t} \left(\frac{k}{t} x_k(t) + \frac{1}{t} x_{k+1}(t) \right) - \frac{1}{t^2} x_{k+1}(t) \\
&= \frac{k}{t} x'_k(t) - \frac{k}{t^2} x_k(t) + \frac{k}{t^2} x_k(t) + \frac{1}{t^2} x_{k+1}(t) - \frac{1}{t^2} x_{k+1}(t) \\
&= \frac{k}{t} x'_k(t),
\end{aligned}$$

which is (3) with $r = 2$.

NOTATION 1. Let $a_{r,j} = (-1)^{j+1} (j-1)! \binom{r-2}{j-1}$. Then Lemma 1 can be written

$$x_{k+1}^{(r)}(t) = k \sum_{j=1}^{r-1} \frac{a_{r,j}}{t^j} x_k^{(r-j)}(t). \quad (4)$$

LEMMA 2. For $k \geq 2$, $x_k^{(r)}(1) = 0$ for any $r \leq k - 2$, and for $r \geq 2$, $x_r^{(r-1)}(1) = (r - 1)!$

Proof. For the first part, we use induction in k . So assume that $x_k^{(l)}(1) = 0$ for $l \leq k - 2$. By (4), $x_{k+1}^{(r)}(1) = k \sum_{j=1}^{r-1} a_{r,j} x_k^{(r-j)}(1)$. Suppose that $r \leq k - 1$. Then $r - j \leq k - j - 1 \leq k - 2$, which implies that $x_k^{(r-j)}(1) = 0$; Thus $x_{k+1}^{(r)}(1) = 0$ whenever $r \leq (k + 1) - 2$. To start the induction, consider $x'_k(t) = (\log t)^{k-2} (k - 1 + \log t)$ so $x'_k(1) = 0$ for $k \geq 3$. $x''_{k+1}(t) = \frac{k}{t} x'_k(t)$ for $k \geq 1$ therefore $x''_k(1) = 0$ for $k \geq 4$.

For the second part, we use induction in r . So assume that $x_r^{(r-1)}(1) = (r - 1)!$. By (4),

$$x_{r+1}^{(r)}(1) = r \sum_{j=1}^{r-1} a_{r,j} x_r^{(r-j)}(1) = r \left(a_{r,1} x_r^{(r-1)}(1) + \sum_{j=2}^{r-1} a_{r,j} x_r^{(r-j)}(1) \right) = r(r - 1)!$$

since $x_r^{(r-j)}(1) = 0$ for $j \geq 2$ by the first part of Lemma 2 just proven. Thus $x_{r+1}^{(r)}(1) = r!$. To start the induction, $x_2^{(1)}(1) = 1 = 1!$

LEMMA 3.
$$\sum_{k=1}^n \frac{(-1)^{k-1}}{(n+1-k)!(k-1)!} = \frac{(-1)^{n+1}}{n!}$$

Proof. Follows immediately from the binomial expansion of $1 + x$ with $x = -1$ and we omit the details.

LEMMA 4.
$$\sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \left(\sum_{j=0}^{n-k} \frac{x^{n-1-j}}{(n-k-j)!} \right) \right] = 1$$
 and
$$\sum_{k=2}^n \left[(-1)^{k-1} \frac{1}{(k-2)!} \sum_{j=0}^{n-k} \frac{x^{n-j-2}}{(n-k-j)!} \right] = -1$$

Proof. Let $d_n = \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \sum_{j=0}^{n-k} \frac{x^{n-1-j}}{(n-k-j)!} \right]$. Using induction in n , we assume that $d_n = 1$.

$$\begin{aligned} d_{n+1} &= \sum_{k=1}^{n+1} \left[(-1)^{k-1} \frac{1}{(k-1)!} \left(\sum_{j=0}^{n+1-k} \frac{x^{n-j}}{(n+1-k-j)!} \right) \right] \\ &= (-1)^n \frac{1}{n!} x^n + \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \left(\sum_{j=0}^{n+1-k} \frac{x^{n-j}}{(n+1-k-j)!} \right) \right] \\ &= (-1)^n \frac{1}{n!} x^n + \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \left(\frac{x^n}{(n+1-k)!} + \sum_{j=1}^{n+1-k} \frac{x^{n-j}}{(n+1-k-j)!} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= (-1)^n \frac{1}{n!} x^n + x^n \sum_{k=1}^n \frac{(-1)^{k-1}}{(n+1-k)!(k-1)!} + \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \sum_{j=0}^{n-k} \frac{x^{n-1-j}}{(n-k-j)!} \right] \\
&= (-1)^n \frac{1}{n!} x^n + x^n \sum_{k=1}^n \frac{(-1)^{k-1}}{(n+1-k)!(k-1)!} + 1 \\
&= (-1)^n \frac{1}{n!} x^n + x^n \frac{(-1)^{n+1}}{n!} + 1
\end{aligned}$$

(by Lemma 3) = 1.

Since $d_1 = 1$, that completes the proof of the first part of Lemma 4. The proof of the second part of Lemma 4 is similar and we omit it. The second part of Lemma 4 also follows from the first part after some manipulations and Lemma 3.

Before proving one of our main results, we introduce the functions $v_{k,n}$ below.

LEMMA 5. For $1 \leq k \leq n$ and $n \geq 3$, let

$$v_{k,n}(t) = \frac{\prod_{r=0}^{n-1} r!}{(k-1)! t^{(n-2)(n-1)/2}} \sum_{j=0}^{n-k} \frac{(\ln t)^{n-k-j}}{(n-k-j)!}, t > 0.$$

Then

$$\begin{aligned}
v_{k+1,n+1}(t) &= \frac{n!}{k} \frac{1}{t^{n-1}} v_{k,n}(t) \\
v_{1,n+1}(t) &= \left(\prod_{r=0}^{n-1} r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} + \frac{n!}{t^{n-1}} v_{1,n}(t).
\end{aligned}$$

Proof.

$$\begin{aligned}
v_{k+1,n+1}(t) &= \frac{\prod_{r=0}^{n-1} r!}{k!} \frac{(\ln t)^{n-k-j}}{t^{n(n-1)/2}} \sum_{j=0}^{n-k} \frac{1}{(n-k-j)!} \\
&= \frac{\prod_{r=0}^{n-1} r!}{k!} \frac{(\ln t)^{n-k-j}}{t^{(n-1)(n-2)/2} t^{n-1}} \sum_{j=0}^{n-k} \frac{1}{(n-k-j)!} \\
&= \frac{n!}{k} \frac{1}{t^{n-1}} \frac{\prod_{r=0}^{n-1} r!}{(k-1)!} \frac{1}{t^{(n-2)(n-1)/2}} \sum_{j=0}^{n-k} \frac{(\ln t)^{n-k-j}}{(n-k-j)!} = \frac{n!}{k} \frac{1}{t^{n-1}} v_{k,n}(t).
\end{aligned}$$

$$v_{1,n}(t) = \left(\prod_{r=0}^{n-1} r! \right) \frac{1}{t^{(n-2)(n-1)/2}} \sum_{j=0}^{n-1} \frac{(\ln t)^{n-1-j}}{(n-1-j)!} \Rightarrow v_{1,n+1}(t)$$

$$\begin{aligned}
 &= \left(\prod_{r=0}^n r! \right) \frac{1}{t^{n(n-1)/2}} \sum_{j=0}^n \frac{(\ln t)^{n-j}}{(n-j)!} \\
 &= \left(\prod_{r=0}^n r! \right) \frac{1}{t^{n(n-1)/2}} \left(\frac{(\ln t)^n}{n!} + \sum_{j=1}^n \frac{(\ln t)^{n-j}}{(n-j)!} \right) \\
 &= \left(\prod_{r=0}^n r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} \left(\frac{1}{n!} + \sum_{j=0}^{n-1} \frac{(\ln t)^{n-1-j}}{(n-1-j)!} \right) \\
 &= \left(\prod_{r=0}^{n-1} r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} + \left(\prod_{r=0}^n r! \right) \frac{1}{t^{n(n-1)/2}} \sum_{j=0}^{n-1} \frac{(\ln t)^{n-1-j}}{(n-1-j)!} \\
 &= \left(\prod_{r=0}^{n-1} r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} + \left(\prod_{r=0}^{n-1} r! \right) \frac{n!}{t^{n-1}} \frac{1}{t^{(n-2)(n-1)/2}} \sum_{j=0}^{n-1} \frac{(\ln t)^{n-1-j}}{(n-1-j)!} \\
 &= \left(\prod_{r=0}^{n-1} r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} + \frac{n!}{t^{n-1}} v_{1,n}(t).
 \end{aligned}$$

PROPOSITION 1. Let $v_{k,n}(t)$ be the functions from Lemma 5, and define the vector functions

$$\begin{aligned}
 \hat{\alpha}(t) &= \langle x_1(t), \dots, x_n(t) \rangle \\
 \hat{v}_n(t) &= \langle v_{1,n}(t), -v_{2,n}(t), \dots, (-1)^{n-1} v_{n,n}(t) \rangle.
 \end{aligned}$$

Then

$$\begin{aligned}
 \hat{\alpha}(t) \cdot \hat{v}_n(t) &= \frac{\prod_{r=0}^{n-1} r!}{t^{n(n-3)/2}} \\
 \hat{\alpha}^{(j)}(t) \cdot \hat{v}_n(t) &= 0 \text{ for } j = 1, \dots, n-1.
 \end{aligned}$$

REMARK 2. $\hat{\alpha}$ depends on n as does \hat{v}_n , but we suppress this dependence in our notation for convenience.

Proof. Case 1: $j = 0$

$$\begin{aligned}
 \tilde{\alpha}(t) \cdot \hat{v}_n(t) &= \sum_{k=1}^n (-1)^{k-1} x_k(t) v_{k,n}(t) \\
 &= \sum_{k=1}^n (-1)^{k-1} t (\ln t)^{k-1} v_{k,n}(t) \\
 &= \left(\prod_{r=0}^{n-1} r! \right) \frac{t}{t^{(n-2)(n-1)/2}} \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \left(\sum_{j=0}^{n-k} \frac{(\ln t)^{n-1-j}}{(n-k-j)!} \right) \right] \\
 &= \frac{\prod_{r=0}^{n-1} r!}{t^{n(n-3)/2}}
 \end{aligned}$$

by Lemma 4 with $x = \ln t$.

Case 2: $j = 1$

From $x_k(t) = t(\log t)^{k-1}$ it follows $x'_k(t) = t(k-1)(\log t)^{k-2} \frac{1}{t} + (\log t)^{k-1}$ and $x'_k(t) = (\log t)^{k-2}(k-1 + \log t)$.

Note that if $k = 1$, $x'_k(t) = 1$ for all t , including $t = 1$.

$$\begin{aligned}
 \hat{\alpha}'(t) \cdot \hat{v}(t) &= \sum_{k=1}^n (-1)^{k-1} x'_k(t) v_{k,n}(t) \\
 &= \sum_{k=1}^n (-1)^{k-1} (\log t)^{k-2} (k-1 + \log t) v_{k,n}(t) \\
 &= \frac{\prod_{r=0}^{n-1} r!}{t^{(n-2)(n-1)/2}} \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} (k-1 + \log t) \left(\sum_{j=0}^{n-k} \frac{(\log t)^{n-j-2}}{(n-k-j)!} \right) \right] \\
 &= \frac{\prod_{r=0}^{n-1} r!}{t^{(n-2)(n-1)/2}} \times \left(\sum_{k=2}^n \left[(-1)^{k-1} \frac{1}{(k-2)!} \sum_{j=0}^{n-k} \frac{(\log t)^{n-j-2}}{(n-k-j)!} \right] \right. \\
 &\quad \left. + \sum_{k=1}^n \left[(-1)^{k-1} \frac{1}{(k-1)!} \sum_{j=0}^{n-k} \frac{(\log t)^{n-j-1}}{(n-k-j)!} \right] \right) = 0
 \end{aligned}$$

by Lemma 4 with $x = \log t$.

Case 3: $2 \leq j \leq n$ (note that j is fixed here)

We use induction in n . So assume that

$$\hat{\alpha}^{(l)}(t) \cdot \hat{v}_n(t) = \sum_{k=1}^n (-1)^{k-1} x_k^{(l)}(t) v_{k,n}(t) = 0$$

for all $l = 1, \dots, n-1$. We have to show that $\hat{\alpha}^{(j)}(t) \cdot \hat{v}_{n+1}(t) = 0$.

Now

$$\begin{aligned}
 \hat{\alpha}^{(j)}(t) \cdot \hat{v}_{n+1}(t) &= \sum_{k=1}^{n+1} (-1)^{k-1} x_k^{(j)}(t) v_{k,n+1}(t) \\
 &= \sum_{k=0}^n (-1)^k x_{k+1}^{(j)}(t) v_{k+1,n+1}(t) \\
 &= x_1^{(j)}(t) v_{1,n+1}(t) + \sum_{k=1}^n (-1)^k x_{k+1}^{(j)}(t) v_{k+1,n+1}(t).
 \end{aligned}$$

Since $x_1^{(j)}(t) = 0$ for $j \geq 2$, we have

$$\hat{\alpha}^{(j)}(t) \cdot \hat{v}_{n+1}(t) = \sum_{k=1}^n (-1)^k x_{k+1}^{(j)}(t) v_{k+1,n+1}(t). \quad (5)$$

By Lemma 5, for $j \geq 2$,

$$\begin{aligned} x_{k+1}^{(j)}(t)v_{k+1,n+1}(t) &= k \left(\sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} x_k^{(j-i)}(t) \right) \frac{n!}{k} \frac{1}{t^{n-1}} v_{k,n}(t) \\ &= \frac{n!}{t^{n-1}} \left(\sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} x_k^{(j-i)}(t) \right) v_{k,n}(t). \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=1}^n (-1)^k x_{k+1}^{(j)}(t)v_{k+1,n+1}(t) &= \frac{n!}{t^{n-1}} \sum_{k=1}^n (-1)^k \left(\sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} x_k^{(j-i)}(t) \right) v_{k,n}(t) \\ &= \frac{n!}{t^{n-1}} \sum_{k=1}^n (-1)^k \left(\sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} x_k^{(j-i)}(t) v_{k,n}(t) \right) \\ &= \frac{n!}{t^{n-1}} \sum_{i=1}^{j-1} \left[\frac{a_{i,j}}{t^j} \left(\sum_{k=1}^n (-1)^k x_k^{(j-i)}(t) v_{k,n}(t) \right) \right] \\ &= \frac{n!}{t^{n-1}} \sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} \left(\sum_{k=1}^n (-1)^k x_k^{(j-i)}(t) v_{k,n}(t) \right) \\ &= -\frac{n!}{t^{n-1}} \sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} \left(\sum_{k=1}^n (-1)^{k-1} x_k^{(j-i)}(t) v_{k,n}(t) \right) \\ &= -\frac{n!}{t^{n-1}} \sum_{i=1}^{j-1} \frac{a_{i,j}}{t^j} \left(\sum_{k=1}^n (-1)^{k-1} x_k^{(j-i)}(t) v_{k,n}(t) \right). \end{aligned}$$

Since $j - i \leq n - 1$ for $i \geq 1$, $\sum_{k=1}^n (-1)^{k-1} x_k^{(j-i)}(t) v_{k,n}(t) = 0$ by the inductive hypothesis.

Thus $\hat{\alpha}^{(j)}(t) \cdot \hat{v}_{n+1}(t) = 0$. To start the induction, for $n = 1$ we have $\hat{\alpha}(t) = \langle t \rangle \Rightarrow \hat{\alpha}^{(j)}(t) = 0 \Rightarrow \hat{\alpha}^{(j)}(t) \cdot \hat{v}_1(t) = 0$ for $j \geq 2$.

REMARK 3. Proposition 1 could perhaps also be proven using properties of hypergeometric functions.

3. Useful Determinants

LEMMA 6. For $n \geq 3$, $W(x_1, \dots, x_n)(t) = \frac{\prod_{r=0}^{n-1} r!}{t^{n(n-3)/2}}$, $t > 0$, where W denotes the Wronskian.

Proof. It is easy to show that the n functions $\left\{t(\log t)^{k-1}\right\}_{k=1,\dots,n}$ satisfy the following n th order Euler DE:

$$t^n \frac{d^n y}{dt^n} + \frac{n^2 - 3n}{2} t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} t \frac{dy}{dt} + a_n y = 0.$$

By Abel's identity applied to the interval $(0, \infty)$,

$$\begin{aligned} W(x_1, \dots, x_n)(t) &= C_n \exp \left(- \int \frac{n^2 - 3n}{2} t^{n-1} dt \right) \\ &= C_n \exp \left(- \frac{n^2 - 3n}{2} \int \frac{dt}{t} \right) = \frac{C_n}{t^{(n^2 - 3n)/2}}. \end{aligned}$$

We shall let $t = 1$ to obtain the precise value $C_n = \prod_{r=0}^{n-1} r!$.

$$W(x_1, \dots, x_n)(t) = \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x'_1(t) & x'_2(t) & \cdots & x'_n(t) \\ x''_1(t) & x''_2(t) & \cdots & x''_n(t) \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix} = \begin{vmatrix} t x_2(t) & \cdots & x_n(t) \\ 1 x'_2(t) & \cdots & x'_n(t) \\ 0 x''_2(t) & \cdots & x''_n(t) \\ \vdots & \cdots & \vdots \\ 0 x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix}$$

implies

$$W(x_1, \dots, x_n)(1) = \begin{vmatrix} 1 \ 0 & \cdots & 0 \\ 1 \ x'_2(1) & \cdots & x'_n(1) \\ 0 \ x''_2(1) & \cdots & x''_n(1) \\ \vdots & \cdots & \vdots \\ 0 \ x_2^{(n-1)}(1) & \cdots & x_n^{(n-1)}(1) \end{vmatrix} = \begin{vmatrix} x'_2(1) & \cdots & x'_n(1) \\ x''_2(1) & \cdots & x''_n(1) \\ \vdots & \cdots & \vdots \\ x_2^{(n-1)}(1) & \cdots & x_n^{(n-1)}(1) \end{vmatrix}.$$

The diagonal entries are $x_{r+1}^{(r)}(1)$, $r = 1, \dots, n-1$ and for row i we have $\left[x_2^{(i)}(1) \cdots x_n^{(i)}(1) \right]$.

By Lemma 2, the entries in row i , column j , $j \geq i + 2$, are each 0. That shows

that the matrix $\begin{bmatrix} x'_2(1) & \cdots & x'_n(1) \\ x''_2(1) & \cdots & x''_n(1) \\ \vdots & \cdots & \vdots \\ x_2^{(n-1)}(1) & \cdots & x_n^{(n-1)}(1) \end{bmatrix}$ is upper triangular, which implies that

$$\begin{vmatrix} x'_2(1) & \cdots & x'_n(1) \\ x''_2(1) & \cdots & x''_n(1) \\ \vdots & \cdots & \vdots \\ x_2^{(n-1)}(1) & \cdots & x_n^{(n-1)}(1) \end{vmatrix} = \prod_{r=1}^{n-1} x_{r+1}^{(r)}(1) = \prod_{r=0}^{n-1} r!$$

by Lemma 2.

REMARK 4. One could also prove Lemma 6 by instead finding a formula for the Wronskian of $\left\{(\log t)^{k-1}\right\}_{k=1}^n$ and using well known properties of Wronskians. That would be easier if one did not already have the recursion for $x_{k+1}^{(r)}(t)$. Since we use that recursion elsewhere, it was easier to then prove Lemmas 2 first.

Our next result shows that the Wronskians $W_{k,n}$ are in fact identically equal to the functions $v_{k,n}$ from Lemma 5.

PROPOSITION 2. For $1 \leq k \leq n$ and $n \geq 3$,

$$W_{k,n}(t) = \frac{\prod_{r=0}^{n-1} r!}{(k-1)! t^{(n-2)(n-1)/2}} \frac{1}{\sum_{j=0}^{n-k} \frac{(\ln t)^{n-k-j}}{(n-k-j)!}}, t > 0.$$

Proof. Consider the following system of linear equations in the unknown func-

tions $u_1(t), \dots, u_n(t)$, where $k_n(t) = \frac{\prod_{r=0}^{n-1} r!}{t^{n(n-3)/2}}$:

$$\begin{aligned} x_1(t)u_1(t) + \dots + x_n(t)u_n(t) &= k_n(t) \\ x'_1(t)u_1(t) + \dots + x'_n(t)u_n(t) &= 0 \\ \vdots & \\ x_1^{(n-1)}(t)u_1(t) + \dots + x_n^{(n-1)}(t)u_n(t) &= 0. \end{aligned} \tag{6}$$

The coefficient matrix of (6) has determinant $W(x_1, \dots, x_n)(t)$, which is nonzero by Lemma 6. By Cramer's Rule, the unique solution is given by

$$x_j(t) = \frac{\begin{vmatrix} x_1(t) & \dots & x_{j-1}(t) & k_n(t) & x_{j+1}(t) & \dots & x_n(t) \\ x'_1(t) & \dots & x'_{j-1}(t) & 0 & x'_{j+1}(t) & \dots & x'_n(t) \\ x''_1(t) & \dots & x''_{j-1}(t) & 0 & x''_{j+1}(t) & \dots & x''_n(t) \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ x_1^{(n-1)}(t) & \dots & x_{j-1}^{(n-1)}(t) & 0 & x_{j+1}^{(n-1)}(t) & \dots & x_n^{(n-1)}(t) \end{vmatrix}}{W(x_1, \dots, x_n)(t)}, \quad j = 1, \dots, n.$$

Expand about column j to obtain

$$x_j(t) = k_n(t) \frac{(-1)^{j+1} W_{j,n}(t)}{W(x_1, \dots, x_n)(t)} = (-1)^{j+1} W_{j,n}(t).$$

By Proposition 1, $x_j(t) = (-1)^{j+1} v_{j,n}(t)$ also satisfies (6). By uniqueness, $W_{j,n}(t) = v_{j,n}(t)$, $j = 1, \dots, n$.

REMARK 5. It follows immediately from Proposition 2 that the $W_{k,n}$ also satisfy the following recursion from Lemma 5 for $n \geq 2$:

$$W_{k+1,n+1}(t) = \frac{n!}{k} \frac{1}{t^{n-1}} W_{k,n}(t), k \geq 1 \quad (7)$$

$$W_{1,n+1}(t) = \left(\prod_{r=0}^{n-1} r! \right) \frac{(\ln t)^n}{t^{n(n-1)/2}} + \frac{n!}{t^{n-1}} W_{1,n}(t). \quad (8)$$

(7) can also be proven using the determinant definition of the $W_{k,n}$ along with standard properties of determinants. However, we found it difficult to prove (8) this way—hence the introduction of the $v_{k,n}$ functions.

REMARK 6. We actually use the recursions (7) and (8) in the proofs below and not the explicit formula given in Proposition 2.

LEMMA 7. For $n \geq 3$,

$$\sum_{k=1}^n \left[(-1)^{k+1} b_k^{n-1} \prod_{1 \leq i < j \leq n; i, j \neq k} (b_j - b_i) \right] = (-1)^{n-1} \prod_{1 \leq i < j \leq n} (b_j - b_i).$$

Proof. It is well known that the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{n-2} & b_2^{n-2} & \cdots & b_n^{n-2} \\ b_1^{n-1} & b_2^{n-1} & \cdots & b_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (b_j - b_i),$$

which implies that

$$\begin{vmatrix} b_1^{n-1} & b_2^{n-1} & \cdots & b_n^{n-1} \\ b_1^{n-2} & b_2^{n-2} & \cdots & b_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (b_j - b_i).$$

By expanding

$$\begin{vmatrix} b_1^{n-1} & b_2^{n-1} & \cdots & b_n^{n-1} \\ b_1^{n-2} & b_2^{n-2} & \cdots & b_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

along the first row and using induction, one has

Using (8) yields

$$((n-1)!)^{n-1} \times \begin{vmatrix} \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_1)^{n-1}}{a_1^{(n-1)(n-2)/2}} + \frac{(n-1)!W_{1,n-1}(a_1)}{a_1^{n-2}} - \frac{W_{1,n-1}(a_1)}{a_1^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_1)}{(n-1)a_1^{n-2}} \\ \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_2)^{n-1}}{a_2^{(n-1)(n-2)/2}} + \frac{(n-1)!W_{1,n-1}(a_2)}{a_2^{n-2}} - \frac{W_{1,n-1}(a_2)}{a_2^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_2)}{(n-1)a_2^{n-2}} \\ \vdots \\ \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_n)^{n-1}}{a_n^{(n-1)(n-2)/2}} + \frac{(n-1)!W_{1,n-1}(a_n)}{a_n^{n-2}} - \frac{W_{1,n-1}(a_n)}{a_n^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_n)}{(n-1)a_n^{n-2}} \end{vmatrix}.$$

By adding $(n-1)! \times$ Col. 2 to Col.1 we have

$$((n-1)!)^{n-1} \times \begin{vmatrix} \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_1)^{n-1}}{a_1^{(n-1)(n-2)/2}} - \frac{W_{1,n-1}(a_1)}{a_1^{n-2}} \cdots \frac{(-1)^k W_{k,n-1}(a_1)}{ka_1^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_1)}{(n-1)a_1^{n-2}} \\ \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_2)^{n-1}}{a_2^{(n-1)(n-2)/2}} - \frac{W_{1,n-1}(a_2)}{a_2^{n-2}} \cdots \frac{(-1)^k W_{k,n-1}(a_2)}{ka_2^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_2)}{(n-1)a_2^{n-2}} \\ \vdots \\ \left(\prod_{r=0}^{n-2} r! \right) \frac{(\ln a_n)^{n-1}}{a_n^{(n-1)(n-2)/2}} - \frac{W_{1,n-1}(a_n)}{a_n^{n-2}} \cdots \frac{(-1)^k W_{k,n-1}(a_n)}{ka_n^{n-2}} \cdots \frac{(-1)^{n+1}W_{n-1,n-1}(a_n)}{(n-1)a_n^{n-2}} \end{vmatrix}.$$

Factoring out $\prod_{r=0}^{n-2} r!$ from Col. 1, factoring out $\frac{1}{k}$ from Column $k+1, k=1, \dots, n-1$, and factoring out $\frac{1}{a_j^{n-2}}$ from row $j, j=1, \dots, n$, yields

$$\frac{((n-1)!)^{n-2} \left(\prod_{r=0}^{n-2} r! \right)}{\prod_{j=1}^n a_j^{n-2}} \times \begin{vmatrix} \frac{(\ln a_1)^{n-1}}{a_1^{(n-2)(n-3)/2}} - W_{1,n-1}(a_1) \cdots (-1)^k W_{k,n-1}(a_1) \cdots (-1)^{n+1} W_{n-1,n-1}(a_1) \\ \frac{(\ln a_2)^{n-1}}{a_2^{(n-2)(n-3)/2}} - W_{1,n-1}(a_2) \cdots (-1)^k W_{k,n-1}(a_2) \cdots (-1)^{n+1} W_{n-1,n-1}(a_2) \\ \vdots \\ \frac{(\ln a_n)^{n-1}}{a_n^{(n-2)(n-3)/2}} - W_{1,n-1}(a_n) \cdots (-1)^k W_{k,n-1}(a_n) \cdots (-1)^{n+1} W_{n-1,n-1}(a_n) \end{vmatrix}.$$

By expanding about Col. 1 we obtain

$$\frac{((n-1)!)^{n-2} \left(\prod_{r=0}^{n-2} r! \right)}{\prod_{j=1}^n a_j^{n-2}} \times \left(\begin{array}{c} \frac{(\ln a_1)^{n-1}}{a_1^{(n-2)(n-3)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_2) \cdots (-1)^{n+1} W_{n-1,n-1}(a_2) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_n) \cdots (-1)^{n+1} W_{n-1,n-1}(a_n) \end{array} \right| - \\ \frac{(\ln a_2)^{n-1}}{a_2^{(n-2)(n-3)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_1) \cdots (-1)^{n+1} W_{n-1,n-1}(a_1) \\ -W_{1,n-1}(a_3) \cdots (-1)^{n+1} W_{n-1,n-1}(a_3) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_n) \cdots (-1)^{n+1} W_{n-1,n-1}(a_n) \end{array} \right| + \cdots + \\ (-1)^{n+1} \frac{(\ln a_n)^{n-1}}{a_n^{(n-2)(n-3)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_1) \cdots (-1)^{n+1} W_{n-1,n-1}(a_1) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_{n-1}) \cdots (-1)^{n+1} W_{n-1,n-1}(a_{n-1}) \end{array} \right| \end{array} \right).$$

Factoring out -1 from each column of each determinant and using the induction hypothesis gives

$$\begin{aligned} & (-1)^{n-1} \frac{((n-1)!)^{n-2} \left(\prod_{r=0}^{n-2} r! \right)}{\prod_{j=1}^n a_j^{n-2}} \times \left(\begin{array}{c} \left(\prod_{r=0}^{n-2} r! \right)^{n-3} \frac{(\ln a_1)^{n-1}}{a_1^{(n-2)(n-3)/2}} \frac{\prod_{2 \leq i < j \leq n} (\ln a_j - \ln a_i)}{\prod_{j=2}^n a_j^{(n-2)(n-3)/2}} + \cdots + \\ (-1)^{n+1} \left(\prod_{r=0}^{n-2} r! \right)^{n-3} \frac{(\ln a_n)^{n-1}}{a_n^{(n-2)(n-3)/2}} \frac{\prod_{1 \leq i < j \leq n-1} (\ln a_j - \ln a_i)}{\prod_{j=1}^{n-1} a_j^{(n-2)(n-3)/2}} \end{array} \right) \\ &= (-1)^{n-1} \frac{\left(\prod_{r=0}^{n-1} r! \right)^{n-2}}{\prod_{j=1}^n a_j^{(n-1)(n-2)/2}} \left((\ln a_1)^{n-1} \prod_{2 \leq i < j \leq n} (\ln a_j - \ln a_i) + \cdots \right. \\ & \quad \left. + (-1)^{n+1} (\ln a_n)^{n-1} \prod_{1 \leq i < j \leq n-1} (\ln a_j - \ln a_i) \right). \end{aligned}$$

$$\frac{((n-1)!)^{n-2} \prod_{r=0}^{n-1} r!}{\prod_{j=1}^n a_j^{n-2}} \times \left(\begin{array}{c} \frac{1}{a_1^{(n-1)(n-4)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_2) \cdots (-1)^{n+1} W_{n-1,n-1}(a_2) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_n) \cdots (-1)^{n+1} W_{n-1,n-1}(a_n) \end{array} \right| - \\ \frac{1}{a_2^{(n-1)(n-4)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_1) \cdots (-1)^{n+1} W_{n-1,n-1}(a_1) \\ -W_{1,n-1}(a_3) \cdots (-1)^{n+1} W_{n-1,n-1}(a_3) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_n) \cdots (-1)^{n+1} W_{n-1,n-1}(a_n) \end{array} \right| + \cdots \\ + (-1)^{n+1} \frac{1}{a_n^{(n-1)(n-4)/2}} \left| \begin{array}{ccc} -W_{1,n-1}(a_1) \cdots (-1)^{n+1} W_{n-1,n-1}(a_1) \\ \vdots & \vdots & \vdots \\ -W_{1,n-1}(a_{n-1}) \cdots (-1)^{n+1} W_{n-1,n-1}(a_{n-1}) \end{array} \right| \end{array} \right)$$

Factoring out -1 from each column of each determinant and using Proposition 3 yields

$$\begin{aligned} & (-1)^{n-1} \frac{((n-1)!)^{n-2} \prod_{r=0}^{n-1} r!}{\prod_{j=1}^n a_j^{n-2}} \times \left(\frac{1}{a_1^{(n-1)(n-4)/2}} \left(\prod_{r=0}^{n-2} r! \right)^{n-3} \frac{\prod_{2 \leq i < j \leq n} (\ln a_j - \ln a_i)}{\prod_{j=2}^n a_j^{(n-2)(n-3)/2}} + \cdots + \right. \\ & \left. (-1)^{n+1} \frac{1}{a_n^{(n-1)(n-4)/2}} \left(\prod_{r=0}^{n-2} r! \right)^{n-3} \frac{\prod_{1 \leq i < j \leq n-1} (\ln a_j - \ln a_i)}{\prod_{j=1}^{n-1} a_j^{(n-2)(n-3)/2}} \right) \\ & = (-1)^{n-1} \frac{((n-1)!)^{n-2} \left(\prod_{r=0}^{n-1} r! \right) \left(\prod_{r=0}^{n-2} r! \right)^{n-3}}{\prod_{j=1}^n a_j^{n-2}} \times \\ & \times \left(\frac{a_1 \prod_{2 \leq i < j \leq n} (\ln a_j - \ln a_i)}{\prod_{j=1}^n a_j^{(n-2)(n-3)/2}} + \cdots + (-1)^{n+1} \frac{a_n \prod_{1 \leq i < j \leq n-1} (\ln a_j - \ln a_i)}{\prod_{j=1}^n a_j^{(n-2)(n-3)/2}} \right) \end{aligned}$$

where

$$V(a_1, \dots, a_n) = \prod_{1 \leq j < i \leq n} (a_i - a_j), V_i(a_1, \dots, a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_{i-1} & a_{i+1} & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_{i-1}^2 & a_{i+1}^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^{n-2} & a_2^{n-2} & \dots & a_{i-1}^{n-2} & a_{i+1}^{n-2} & \dots & a_n^{n-2} \end{vmatrix},$$

and $m = \sum_{k=1}^{n-1} \frac{1}{k}$. For $n = 3$, if one lets $x_1(t) = t$, $x_2(t) = t^2$, $x_3(t) = \log t$, then $M_z(a, b, c) = I_Z(a, b, c) = U_2(a, b, c)$, where U_2 is given in [6]. This probably holds for all n .

CONJECTURE. *If $x_1(t) = t$, $x_2(t) = t^2$, $\dots, x_{n-1}(t) = t^{n-1}, x_n(t) = \log t$, then $M_n(a_1, \dots, a_n) = I_Z(a_1, \dots, a_n)$.*

This conjecture is probably somewhat easier to prove than Theorem 2.

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(Received April 5, 2014)

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