

MULTIPLIERS ON SPACES OF VECTOR VALUED ENTIRE DIRICHLET SERIES

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Abstract. The spaces of entire functions represented by Dirichlet series have been studied by Hussein and Kamthan and others. In this paper we have studied a sequence space which depends upon the order of an entire function represented by vector valued Dirichlet series and obtain the dual nature of this sequence space. In the later part we obtain some coefficient multipliers for some classes of vector valued Dirichlet series.

1. Introduction

Consider a vector-valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad (1)$$

where $s = \sigma + it$, (σ and t are real variables), a_n 's belong to a complex Banach algebra E with the unit element ω and $\{\lambda_n\}$ is any increasing sequence such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$; $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Let $\sigma_c(f)$ and $\sigma_a(f)$ be the abscissa of convergence and abscissa of absolute convergence respectively of the series in (1). B. L. Srivastava [5] has proved that if the sequence $\{\lambda_n\}$ satisfies the condition

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = 0, \quad (2)$$

then

$$\sigma_c(f) = \sigma_a(f) = - \limsup_{n \rightarrow \infty} \frac{\log \|a_n\|}{\lambda_n}. \quad (3)$$

If $\sigma_c(f) = \sigma_a(f) = \infty$ then (1) represents a vector valued entire Dirichlet series. Suppose that $f(s)$ is a vector valued entire Dirichlet series. We define its maximum modulus as

$$M(\sigma) = \text{lub}_{-\infty < t < \infty} \|f(\sigma + it)\|.$$

B. L. Srivastava [5] introduced the order for entire VVDS. The entire function $f(s)$ is said to be of order ρ where

$$\rho = \limsup_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma}, \quad (0 \leq \rho \leq \infty). \quad (4)$$

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$f(s)$ is said to be of slow growth if the order is zero and of fast growth if the order is infinite. When $0 < \rho < \infty$, we define the type T of $f(s)$ as

$$T = \limsup_{\sigma \rightarrow \infty} \frac{\log M(\sigma)}{\exp(\rho\sigma)}, \quad (0 \leq T \leq \infty). \quad (5)$$

The growth properties of the function $f(s)$ represented by a VVDS (1) in terms of its coefficients and exponents were given by B. L. Srivastava [5]. If $f(s)$ given by (1) is entire and (2) and (3) are satisfied, then it is of finite order ρ if and only if

$$\rho = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log ||a_n||^{-1}}. \quad (6)$$

From the equation (6), for a given $\varepsilon > 0$ there exists positive integer n_o such that for all $n > n_o$,

$$\frac{\lambda_n \log \lambda_n}{\log ||a_n||^{-1}} < \rho + \varepsilon.$$

i.e.,

$$||a_n|| \leq \lambda_n^{-\lambda_n/M}; \text{ where } M = \rho + \varepsilon.$$

Let X denote the space of all entire functions $f(s)$ defined by VVDS (1) and satisfying

$$\limsup_{n \rightarrow \infty} \frac{\log ||a_n||}{\lambda_n \log \lambda_n} \leq \frac{-1}{\rho}. \quad (7)$$

Le Hai Khoi has worked a lot on the spaces of analytic Dirichlet series with negative exponents and obtained various results. In [3] he introduced various concepts of duality for sequence spaces. The aim of this paper is to introduce a new sequence space using the order of entire functions represented by VVDS and obtain some auxiliary condition of convergence of VVDS (1). Further we throw light on the dual nature of the sequence space of entire function represented by VVDS and derive coefficient multipliers.

As given in [3], we give some definitions regarding dual spaces.

Let A and B be two sequence spaces, we denote the sequence space of “multipliers” from A to B by (A, B) such that

$$(A, B) = \{u = (u_n); (u_n a_n) \in B, \forall (a_n) \in A\}.$$

In what follows we shall always consider E to be a complex Banach algebra with unit element ω and the sequence $\{\lambda_n\}$ satisfies the condition (2). We denote by E_ρ the sequence space

$$E_\rho = \left\{ (a_n) \subseteq E : \limsup_{n \rightarrow \infty} \frac{\log ||a_n||}{\lambda_n \log \lambda_n} \leq -\frac{1}{\rho} \right\}.$$

The Köthe dual [4] of the space E_ρ is defined as

$$E_\rho^\alpha = \left\{ (u_n) \subseteq E; \sum_{n=1}^{\infty} ||u_n a_n|| < \infty, \forall (a_n) \in E_\rho \right\}.$$

Now we introduce another sequence space E_ρ^β such that

$$E_\rho^\beta = \left\{ (u_n) \subseteq E; \sum_{n=1}^{\infty} u_n a_n \text{ converges } \forall (a_n) \in E_\rho \right\}.$$

It can be easily verified that $E_\rho^\alpha \subseteq E_\rho^\beta$. In our first result we prove that in fact the two spaces are identical and characterize them.

2. Main Results

We now prove

THEOREM 1. *For every $0 < \rho < \infty$, we have $E_\rho^\alpha = E_\rho^\beta$. Moreover, $(u_n) \in E_\rho^\beta$ if and only if*

$$\limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} < \frac{1}{\rho}. \quad (8)$$

Proof. First we suppose that (8) holds, i.e.,

$$T = \limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} < \frac{1}{\rho}.$$

Then for a given $\delta > 0$, there exists N_1 such that $\forall n \geq N_1$ we have

$$\frac{\log \|u_n\|}{\lambda_n \log \lambda_n} < T + \delta$$

or

$$\|u_n\| \leq \lambda_n^{\lambda_n(T+\delta)} \quad \forall n \geq N_1.$$

Also, for the sequence $(a_n) \in E_\rho$, there exists N_2 such that $\forall n \geq N_2$,

$$\|a_n\| \leq \lambda_n^{\lambda_n((-1/\rho)+\delta)}.$$

Therefore for all $n \geq \max\{N_1, N_2\}$, $\|a_n u_n\| \leq \|a_n\| \|u_n\| < \lambda_n^{\lambda_n(T-(1/\rho)+2\delta)}$.

Since the sequence (λ_n) satisfies condition (2), the series $\sum_{n=0}^{\infty} \lambda_n^{a \lambda_n}$ converges for $a < 0$.

We can always choose $0 < 2\delta < \frac{1}{\rho} - T$. Hence the series $\sum_{n=1}^{\infty} \|u_n a_n\|$ converges. Thus the sequence $(u_n) \in E_\rho^\alpha \subseteq E_\rho^\beta$.

Conversely, let us assume that $(u_n) \in E_\rho^\beta$ is such that

$$L = \limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} \geq \frac{1}{\rho}.$$

We first suppose that $L < \infty$; then there is a subsequence $(u_{n_k})_k$ such that

$$\lim_{k \rightarrow \infty} \frac{\log \|u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} = L.$$

We now define the sequence (a_n) as

$$a_n = \begin{cases} \omega/||u_n||, & \text{if } n = n_k, k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n \rightarrow \infty} \frac{\log ||a_n||}{\lambda_n \log \lambda_n} = \lim_{k \rightarrow \infty} \frac{\log ||a_{n_k}||}{\lambda_{n_k} \log \lambda_{n_k}} = \lim_{k \rightarrow \infty} \frac{\log 1/||u_{n_k}||}{\lambda_{n_k} \log \lambda_{n_k}} = -L \leq \frac{-1}{\rho}.$$

This implies that $(a_n) \in E_\rho$. We also have $||a_{n_k} u_{n_k}|| = 1$ for every $k = 1, 2, \dots$, hence the series $\sum_{n=1}^{\infty} ||u_n a_n||$ does not converge, contradicting the fact that $(u_n) \in E_\rho^\beta$.

If $L = \infty$, then for an arbitrarily large positive number K , we can find a subsequence $(u_{n_k})_k$ such that

$$\limsup_{n \rightarrow \infty} \frac{\log ||u_n||}{\lambda_n \log \lambda_n} = K > \frac{1}{\rho}.$$

Again proceeding as above we get a contradiction. Hence $(u_n) \in E_\rho^\beta$ implies (8) must hold. Using the first part, we conclude that $(u_n) \in E_\rho^\alpha$. Hence we get $E_\rho^\alpha = E_\rho^\beta$ and the proof of Theorem 1 is complete. $\square$

Next we prove

THEOREM 2. *The space E_ρ is perfect i.e., $E_\rho^{\alpha\alpha} = E_\rho$.*

Proof. Let the sequence $(a_n) \notin E_\rho$. Then we have

$$\limsup_{n \rightarrow \infty} \frac{\log ||a_n||}{\lambda_n \log \lambda_n} > \frac{-1}{\rho}.$$

Now proceeding as in the proof of Theorem 1, we can find a sequence $(u_n) \in E_\rho^\alpha$ such that $||u_n a_n|| = 1$ i.e. $\sum a_n u_n$ does not converge. Therefore, $(a_n) \notin E_\rho^{\alpha\alpha}$. Hence $E_\rho^{\alpha\alpha} \subseteq E_\rho$. The reverse inclusion always holds. Hence we obtain $E_\rho^{\alpha\alpha} = E_\rho$ i.e. the space E_ρ is perfect. This proves Theorem 2. $\square$

Now we obtain some results for the spaces

$$(E_\rho, l^p) = \left\{ (u_n) \subseteq E; \sum_{n=1}^{\infty} ||u_n a_n||^p < \infty \forall (a_n) \in E_\rho \right\}$$

and

$$(l^p, E_\rho) = \{ (u_n) \subseteq E; (u_n a_n) \in E_\rho \forall (a_n) \in l^p \}.$$

We prove

THEOREM 3. For the sequence space E_p defined as above, we have $\forall 0 < p \leq \infty$

$$(E_p, l^p) = E_p^\alpha.$$

Proof. Suppose that a sequence $(u_n) \notin E_p^\alpha$. Then from Theorem 1, we have

$$L = \limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} \geq \frac{1}{\rho}$$

Assume that $L < \infty$; then there is a subsequence $(u_{n_k})_k$ such that

$$\lim_{k \rightarrow \infty} \frac{\log \|u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} = L.$$

Let $0 < p < \infty$. We consider the sequence (a_n) as

$$a_n = \begin{cases} \omega / \|u_n\|, & \text{if } n = n_k, k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

By proceeding as in Theorem 1 we see that the sequence $(a_n) \in E_p$ but $(a_n u_n) \notin l^p$. When $p = \infty$ we consider a sequence

$$a_n = \begin{cases} \omega n / \|u_n\|, & \text{if } n = n_k, k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

After proceeding as in Theorem 1 we see that the sequence $(a_n) \in E_p$ but $(a_n u_n) \notin l^\infty$. Hence we conclude that for $0 < p \leq \infty$, $(u_n) \notin E_p^\alpha \Rightarrow (u_n) \notin (E_p, l^p)$. Thus $(E_p, l^p) \subseteq E_p^\alpha$, $0 < p \leq \infty$.

Conversely, assume that $(u_n) \in E_p^\alpha$. Then for a given number $T < 1/\rho$ there exists N_1 such that $\forall n \geq N_1$, $\|u_n\| \leq \lambda_n^{\lambda_n T}$. Suppose that $(a_n) \in E_p$, then for $\delta \in (0, \frac{1}{\rho} - T)$ there exists N_2 such that $\forall n \geq N_2$

$$\|a_n\| \leq \lambda_n^{\lambda_n((-1/\rho)+\delta)}.$$

Consequently, for all $n \geq N = \max\{N_1, N_2\}$, we have

$$\|a_n u_n\| \leq \lambda_n^{\lambda_n(T-(1/\rho)+\delta)}.$$

If $0 < p < \infty$, then we have

$$\sum_{n=N}^{\infty} \|a_n u_n\|^p \leq \sum_{n=N}^{\infty} \lambda_n^{\lambda_n p(T-(1/\rho)+\delta)} < \infty;$$

as $p(T - (1/\rho) + \delta) < 0$, which implies that $(a_n u_n) \in l^p$.

Now let us take $p = \infty$, then we have $\|a_n u_n\| \leq \lambda_n^{\lambda_n(T-(1/\rho)+\delta)} \leq 1$, $\forall n \geq N$, which shows that $(a_n u_n) \in l^\infty$. Thus in both cases $(u_n) \in (E_p, l^p)$ and consequently, $E_p^\alpha \subset (E_p, l^p)$, $0 < p \leq \infty$. This completes the proof of Theorem 3. $\square$

In the next result we obtain the sequence space of multipliers from l^p to E_p .

THEOREM 4. For the space E_ρ we have

$$(l^p, E_\rho) = E_\rho, \quad 0 < p \leq \infty.$$

Proof. Let $(u_n) \in (l^p, E_\rho)$, $0 < p \leq \infty$. Suppose that $(u_n) \notin E_\rho$, this implies that

$$\limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} = T > \frac{1}{\rho},$$

Assume that $T < \infty$; then there exists a subsequence $(u_{n_k})_k$ such that

$$\lim_{k \rightarrow \infty} \frac{\log \|u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} = T.$$

Then for any $\delta > 0$ such that $T - \delta > \frac{1}{\rho}$, we have

$$\frac{\log \|u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} > T - \delta, \quad \forall k \geq k_0(\delta),$$

$$\|u_{n_k}\|^{-1} \leq \lambda_{n_k}^{-(T-\delta)\lambda_{n_k}}, \quad \forall k \geq k_0(\delta).$$

Define a new sequence (b_n) such that

$$b_n = \begin{cases} \omega \lambda_n^{(T-\delta)\lambda_n} / \|u_n\|, & \text{if } n = n_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{n=1}^{\infty} \|b_n\|^p = \sum_{k=1}^{\infty} \|b_{n_k}\|^p = \sum_{k=1}^{\infty} \|u_{n_k}\|^{-p} \lambda_{n_k}^{p(T-\delta)\lambda_{n_k}} \leq O(1) + \sum_{k=k_0}^{\infty} \lambda_{n_k}^{-(p\lambda_{n_k})} < \infty$$

which shows that $(b_n) \in l^p$. Now consider

$$\limsup_{n \rightarrow \infty} \frac{\log \|b_n u_n\|}{\lambda_n \log \lambda_n} = \lim_{k \rightarrow \infty} \frac{\log \|b_{n_k} u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} = (T - \delta) > 1/\rho.$$

In the second case i.e., $p = \infty$, we define a sequence (c_n) such that

$$c_n = \begin{cases} \omega \lambda_n^{(T-\delta)\lambda_n} / \|u_n\|, & \text{if } n = n_k, k = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

We can see that $\|c_n\| \leq 1$, $\forall n \geq 1$, which shows that $(c_n) \in l^\infty$. Then we have

$$\limsup_{n \rightarrow \infty} \frac{\log \|c_n u_n\|}{\lambda_n \log \lambda_n} = \limsup_{k \rightarrow \infty} \frac{\log \|c_{n_k} u_{n_k}\|}{\lambda_{n_k} \log \lambda_{n_k}} = (T - \delta) > 1/\rho.$$

Hence we see that in both cases, we get sequences $(b_n u_n)$ and $(c_n u_n)$ which do not belong to E_p even though $(b_n) \in l^p$ and $(c_n) \in l^\infty$ respectively. This is a contradiction. Thus $(l^p, E_p) \subseteq E_p, 0 < p \leq \infty$.

To prove the converse, assume that $(u_n) \in E_p$, Then we have

$$\limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} \leq \frac{1}{M}.$$

Let (d_n) be an arbitrary sequence such that $(d_n) \in l^p, 0 < p \leq \infty$. In both cases, there exists a constant P such that $\|d_n\| \leq P, \forall n \geq 1$. Hence we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log \|d_n u_n\|}{\lambda_n \log \lambda_n} &\leq \limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} + \limsup_{n \rightarrow \infty} \frac{\log \|d_n\|}{\lambda_n \log \lambda_n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log \|u_n\|}{\lambda_n \log \lambda_n} \leq \frac{1}{\rho}, \end{aligned}$$

which shows that $(d_n u_n) \in E_p$. Thus $E_p \subset (l^p, E_p), \forall 0 < p \leq \infty$. Hence the result follows. $\square$

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