

SOME SUFFICIENT CONDITIONS FOR THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract. Making use of the method of subordination chains, we obtain some sufficient conditions for the univalence of an integral operator. In particular, as special cases, our results imply certain known univalence criteria. A refinement to a quasiconformal extension criterion of the main result, is also obtained.

1. Introduction

Denote by \mathcal{U}_r ($0 < r \leq 1$) the disk of radius r centered at 0, i.e. $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$ and let $\mathcal{U} = \mathcal{U}_1$ be the unit disk.

Let \mathcal{A} denote the class of analytic functions in \mathcal{U} which satisfy the usual normalization

$$f(0) = f'(0) - 1 = 0.$$

One of the most important univalence criterion for functions in the class \mathcal{A} was obtained by Becker in 1972 [3]. His result was derived by means of Loewner chains and Loewner differential equation. During the time many extensions of Becker's criterion have been given, among them being the results due to Ahlfors [1], Lewandowski [14], Pascu [19], [20], Ruscheweyh [25], Ovesea [16, 17], Ovesa et. all [18] and Kanas and Srivastava [13].

In the present paper we use the method of subordination chains to obtain some sufficient conditions for the univalence of an integral operator. Our results generalize certain criteria obtained by Pascu [20], Danikas and Ruscheweyh [7], Moldoveanu [15], Deniz and Orhan [8], Răducanu et. all [24]. Also, we obtain a refinement to a quasiconformal extension criterion of the main result.

2. Loewner chains and quasiconformal extensions

Before proving our main theorem we need a brief summary of Loewner chains and Becker's method of constructing quasiconformal extensions by means of Loewner chains and generalized Loewner differential equation.

A function $L(z, t) : \mathcal{U} \times [0, \infty) \rightarrow \mathbb{C}$ is said to be a *subordination chain* or a *Loewner chain* if:

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- (i) $L(z, t)$ is analytic and univalent in \mathcal{U} for all $t \geq 0$.
- (ii) $L(z, t) \prec L(z, s)$ for all $0 \leq t \leq s < \infty$, where the symbol “ \prec ” stands for subordination.

The following result, due to Pommerenke, is often used to prove univalence criteria.

THEOREM 2.1. ([22], [23]) *Let $L(z, t) = a_1(t)z + \dots$ be an analytic function in \mathcal{U}_r ($0 < r \leq 1$) for all $t \geq 0$. Suppose that:*

- (i) $L(z, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$, locally uniform with respect to $z \in \mathcal{U}_r$.
- (ii) $a_1(t)$ is a complex valued continuous function on $[0, \infty)$ such that $a_1(t) \neq 0$, $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ and

$$\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$$

is a normal family of functions in \mathcal{U}_r .

- (iii) There exists an analytic function $p : \mathcal{U} \times [0, \infty) \rightarrow \mathbb{C}$ satisfying $\Re p(z, t) > 0$ for all $(z, t) \in \mathcal{U} \times [0, \infty)$ and

$$z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \quad z \in \mathcal{U}, \quad a.e \ t \geq 0. \quad (2.1)$$

Then, for each $t \geq 0$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk \mathcal{U} , i.e $L(z, t)$ is a subordination chain.

Let k be constant in $[0, 1)$. Recall that a homeomorphism f of $G \subset \mathbb{C}$ is said to be k -quasiconformal if $\partial_z f$ and $\partial_{\bar{z}} f$, in the distributional sense, are locally integrable on G and fulfill $|\partial_{\bar{z}} f| \leq k |\partial_z f|$ almost everywhere in G .

An important problem in the theory of univalent functions is to find functions that have quasiconformal extensions to \mathbb{C} .

A method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [3], [4] and also [5]).

THEOREM 2.2. *Suppose that $L(z, t)$ is a subordination chain. Consider*

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, \quad z \in \mathcal{U}, \quad t \geq 0$$

where $p(z, t)$ is defined by (2.1). If

$$|w(z, t)| \leq k, \quad 0 \leq k < 1$$

for all $z \in \mathcal{U}$ and $t \geq 0$, then $L(z, t)$ admits a continuous extension to $\bar{\mathcal{U}}$ for each $t \geq 0$ and the function $F(z, \bar{z})$ defined by

$$F(z, \bar{z}) = \begin{cases} L(z, 0) & , \text{if } |z| < 1 \\ L\left(\frac{z}{|z|}, \log |z|\right) & , \text{if } |z| \geq 1. \end{cases}$$

is a k -quasiconformal extension of $L(z, 0)$ to \mathbb{C} .

Examples of quasiconformal extension criteria can be found in [1], [2], [6], [12], [21] and more recently in [9], [10], [11].

3. Univalence criteria

In this section, making use of Theorem 2.1, we obtain certain sufficient conditions for the univalence of an integral operator.

THEOREM 3.1. *Let $f, g, \phi \in \mathcal{A}$, $g(z) \neq 0$, $\phi(z) \neq 0$ in \mathcal{U} . Let also $m \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re \gamma > 0$. If*

$$\left| \frac{(1 - |z|^{(m+1)\gamma})}{\gamma} \left[\alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right] - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.1}$$

holds $z \in \mathcal{U}$, then the function $\mathcal{F}_{\alpha, \beta, \gamma}$ defined by

$$\mathcal{F}_{\alpha, \beta, \gamma}(z) = \left[\gamma \int_0^z u^{\gamma-1} (f'(u))^\alpha \left(\frac{g(u)}{\phi(u)} \right)^\beta du \right]^{1/\gamma} \quad z \in \mathcal{U}, \tag{3.2}$$

where the principal branch is intended, is analytic and univalent in \mathcal{U} .

Proof. Let a be a positive real number. We are going to prove that there exists $r \in (0, 1]$ such that the function $L : \mathcal{U}_r \times [0, \infty) \rightarrow \mathbb{C}$, defined by

$$L(z, t) = \left\{ \gamma \int_0^{e^{-at}z} u^{\gamma-1} (f'(u))^\alpha \left(\frac{g(u)}{\phi(u)} \right)^\beta du + (e^{mat\gamma} - e^{-at\gamma}) z^\gamma (f'(e^{-at}z))^\alpha \left(\frac{g(e^{-at}z)}{\phi(e^{-at}z)} \right)^\beta \right\}^{1/\gamma} \tag{3.3}$$

is analytic in \mathcal{U}_r for all $t \in [0, \infty)$ and satisfies the conditions of Theorem 2.1. Since $f, g, \phi \in \mathcal{A}$, there exists a disk \mathcal{U}_{r_1} , $0 < r_1 \leq 1$ in which the function

$$h(z) = (f'(z))^\alpha \left(\frac{g(z)}{\phi(z)} \right)^\beta$$

is analytic. The powers are considered with their principal branches. The function $h(z)$ is analytic and does not vanish in \mathcal{U}_{r_1} .

Consider the function

$$h_1(z, t) = \gamma \int_0^{e^{-at}z} u^{\gamma-1} h(u) du, \quad z \in \mathcal{U}_{r_1}, \quad t \geq 0.$$

We can write

$$h_1(z, t) = z^\gamma h_2(z, t)$$

where $h_2(z, t)$ is analytic in \mathcal{U}_{r_1} for all $t \geq 0$. It follows that the function

$$h_3(z, t) = h_2(z, t) + (e^{mat\gamma} - e^{-at\gamma})h(e^{-at}z)$$

is also analytic in \mathcal{U}_{r_1} and

$$h_3(0, t) = e^{mat\gamma}.$$

Since $h_3(0, 0) = 1$, $h_3(0, t) \neq 0$ for $t \geq 0$ and $\lim_{t \rightarrow \infty} |h_3(0, t)| = \infty$ there is a disk \mathcal{U}_{r_2} , $0 < r_2 \leq r_1$, in which $h_3(z, t) \neq 0$. Therefore we can choose a uniform and analytic branch of $[h_3(z, t)]^{1/\gamma}$ in \mathcal{U}_{r_2} which will be denoted by $h_4(z, t)$. Now, the function defined by (3.3) can be rewritten as

$$L(z, t) = zh_4(z, t) = a_1(t)z + \dots, \quad z \in \mathcal{U}_{r_2} \text{ and } t \geq 0 \quad (3.4)$$

where $a_1(t) = e^{mat}$. Moreover $L(z, t)$ is analytic in \mathcal{U}_{r_2} for all $t \geq 0$.

Let $r_3 \in (0, r_2]$ and let $K = \{z \in \mathbb{C} : |z| \leq r_3\}$. Since the function $L(z, t)$ is analytic in \mathcal{U}_{r_2} , there exists $M > 0$ such that

$$\left| \frac{L(z, t)}{a_1(t)} \right| \leq M \text{ for } z \in K \text{ and } t \geq 0.$$

Thus, $\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$ forms a normal family in \mathcal{U}_{r_2} .

From (3.4) we obtain that $\left\{ \frac{\partial L(z, t)}{\partial t} \right\}$ is analytic in \mathcal{U}_{r_2} . It follows that $\left| \frac{\partial L(z, t)}{\partial t} \right|$ is bounded on $[0, T]$ for any fixed $T > 0$ and $z \in \mathcal{U}_{r_2}$. Therefore, the function $L(z, t)$ is locally absolutely continuous on $[0, \infty)$, locally uniform with respect to \mathcal{U}_{r_2} .

For $0 < r \leq r_2$ and $t \geq 0$, consider the function $p : \mathcal{U}_r \times [0, \infty) \rightarrow \mathbb{C}$ defined by

$$p(z, t) = z \frac{\partial L(z, t)}{\partial z} \Big/ \frac{\partial L(z, t)}{\partial t}.$$

In order to prove that the function $p(z, t)$ is analytic and has positive real part in \mathcal{U} , we will show that the function

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}$$

is analytic in \mathcal{U} and

$$|w(z, t)| < 1, \quad \text{for all } z \in \mathcal{U} \text{ and } t \geq 0. \quad (3.5)$$

Lengthy but elementary calculation gives

$$w(z, t) = \frac{(1+a)\mathcal{G}(z, t) + 1 - ma}{(1-a)\mathcal{G}(z, t) + 1 + ma}, \tag{3.6}$$

where

$$\mathcal{G}(z, t) = \frac{1}{\gamma} \left[\alpha \frac{e^{-at} z f''(e^{-at} z)}{f'(e^{-at} z)} + \beta \left(\frac{e^{-at} z g'(e^{-at} z)}{g(e^{-at} z)} - \frac{e^{-at} z \phi'(e^{-at} z)}{\phi(e^{-at} z)} \right) \right] \left(1 - e^{-(m+1)at\gamma} \right) \tag{3.7}$$

for $z \in \mathcal{U}$ and $t \geq 0$.

Inequality (3.5) is therefore, equivalent to

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad z \in \mathcal{U}, \quad t \geq 0. \tag{3.8}$$

For $t = 0$ the last inequality holds. Define

$$\mathcal{H}(z, t) = \mathcal{G}(z, t) - \frac{m-1}{2}, \quad z \in \mathcal{U}, \quad t \geq 0. \tag{3.9}$$

Since $|e^{-at} z| \leq |e^{-at}| = e^{-at} < 1$ for all $z \in \bar{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $t > 0$, we have that $\mathcal{H}(z, t)$ is analytic in $\bar{\mathcal{U}}$ for every $t > 0$. Making use of the maximum modulus principle, we obtain that, for each arbitrary fixed $t > 0$, there exists $\theta(t) \in \mathbb{R}$ such that

$$|\mathcal{H}(z, t)| < \max_{|z|=1} |\mathcal{H}(z, t)| = |\mathcal{H}(e^{i\theta}, t)| \text{ for all } z \in \mathcal{U}.$$

Let $u = e^{-at} e^{i\theta}$. Then $|u| = e^{-at}$ and $e^{-(m+1)at} = (e^{-at})^{(m+1)} = |u|^{m+1}$. Therefore

$$|\mathcal{H}(e^{i\theta}, t)| = \left| \frac{(1 - |u|^{(m+1)\gamma})}{\gamma} \left[\alpha \frac{u f''(u)}{f'(u)} + \beta \left(\frac{u g'(u)}{g(u)} - \frac{u \phi'(u)}{\phi(u)} \right) \right] - \frac{m-1}{2} \right|.$$

Inequality (3.1) from hypothesis implies

$$|\mathcal{H}(e^{i\theta}, t)| \leq \frac{m+1}{2}. \tag{3.10}$$

From (3.10) it follows that inequality (3.8) is satisfied for all $z \in \mathcal{U}$ and $t \geq 0$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(z, t)$ has an analytic and univalent extension to the whole unit disk \mathcal{U} , for all $t \geq 0$. If $t = 0$ we have $L(z, 0) = \mathcal{F}_{\alpha, \beta, \gamma}(z)$ and therefore, our integral operator $\mathcal{F}_{\alpha, \beta, \gamma}$ is analytic and univalent in \mathcal{U} . \square

Making use of Theorem 3.1, we derive another univalence criterion for the integral operator $\mathcal{F}_{\alpha, \beta, \gamma}$.

THEOREM 3.2. *Let $f, g, \phi \in \mathcal{A}$, $g(z) \neq 0$, $\phi(z) \neq 0$. Let also $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re \gamma > 0$ and $m \in \mathbb{R}_+$, $m \geq 1$. If*

$$\frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma} \left| \alpha \frac{z f''(z)}{f'(z)} + \beta \left(\frac{z g'(z)}{g(z)} - \frac{z \phi'(z)}{\phi(z)} \right) \right| \leq 1$$

holds for $z \in \mathcal{U}$ then, the function $\mathcal{F}_{\alpha, \beta, \gamma}$ defined by (3.2) is analytic and univalent in \mathcal{U} .

Proof. It can be proved (see [20]) that for $z \in \mathcal{U} \setminus \{0\}$, $\Re \gamma > 0$ and $m \in \mathbb{R}_+$

$$\left| \frac{1 - |z|^{(m+1)\gamma}}{\gamma} \right| \leq \frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma}.$$

For $m \geq 1$, we have

$$\begin{aligned} & \left| \frac{1 - |z|^{(m+1)\gamma}}{\gamma} \left[\alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right] - \frac{m-1}{2} \right| \\ & \leq \left| \frac{1 - |z|^{(m+1)\gamma}}{\gamma} \left[\alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right] \right| + \frac{m-1}{2} \\ & \leq \frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma} \left| \alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right| + \frac{m-1}{2} \\ & \leq 1 + \frac{m-1}{2} = \frac{m+1}{2}. \end{aligned}$$

Since inequality (3.1) is satisfied, making use of Theorem 3.1, we can conclude that the function $\mathcal{F}_{\alpha, \beta, \gamma}$ is analytic and univalent in \mathcal{U} . \square

EXAMPLE 3.1. Let α, β, γ be three complex numbers such that $\Re \gamma > 0$ and $\Re \gamma \geq |\alpha| + |\beta|$. Then, the function

$$\mathcal{F}_{\alpha, \beta, \gamma}(z) = z \left[{}_2F_1 \left(\gamma, -(\alpha + \beta); 1 + \gamma; -\frac{z}{2} \right) \right]^{1/\gamma}$$

is univalent in \mathcal{U} . The symbol ${}_2F_1(a, b; c; z)$ denotes the well known hypergeometric function.

Proof. Set $f(z) = z + \frac{z^2}{4}$, $g(z) = z + \frac{z^2}{2}$, $z \in \mathcal{U}$ and $\phi(z) = z$, $z \in \mathcal{U}$ in Theorem 3.2. Making use of triangle inequality, we have

$$\begin{aligned} & \left| \frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma} \left| \alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right| \right| \\ & = \frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma} \left| \alpha \frac{z}{z+2} + \beta \left(\frac{2z+2}{z+2} - 1 \right) \right| \\ & \leq \frac{1 - |z|^{(m+1)\Re \gamma}}{\Re \gamma} \frac{|z|}{2 - |z|} (|\alpha| + |\beta|) < \frac{1}{\Re \gamma} (|\alpha| + |\beta|) \leq 1. \end{aligned}$$

The last inequality follows from $1 - |z|^{(m+1)\Re\gamma} < 1$, $\frac{|z|}{2 - |z|} < 1$, $z \in \mathcal{U}$ and $\Re\gamma \geq |\alpha| + |\beta|$. Since all the conditions of Theorem 3.2 are satisfied, we obtain that the function

$$\mathcal{F}_{\alpha,\beta,\gamma}(z) = \left[\gamma \int_0^z u^{\gamma-1} \left(1 + \frac{u}{2}\right)^\alpha \left(1 + \frac{u}{2}\right)^\beta du \right]^{1/\gamma}$$

is univalent in \mathcal{U} . With the substitution $u = tz$ the function $\mathcal{F}_{\alpha,\beta,\gamma}(z)$ becomes

$$\mathcal{F}_{\alpha,\beta,\gamma}(z) = z \left[\gamma \int_0^1 t^{\gamma-1} \left(1 + t\frac{z}{2}\right)^{\alpha+\beta} dt \right]^{1/\gamma} = z \left[{}_2F_1(\gamma, -(\alpha + \beta); 1 + \gamma; -\frac{z}{2}) \right]^{1/\gamma}.$$

With this, the proof is complete.

Certain particular cases of Theorem 3.1 and Theorem 3.2 respectively, are listed below. \square

If in Theorem 3.1 we consider $\alpha = \beta$, $g(z) = z$ and $\phi = f$, we obtain the following univalence condition.

COROLLARY 3.1. *Let $f \in \mathcal{A}$ and $m \in \mathbb{R}_+$. If*

$$\left| \alpha \frac{1 - |z|^{(m+1)\gamma}}{\gamma} \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] - \frac{m-1}{2} \right| \leq \frac{m+1}{2}$$

holds $z \in \mathcal{U}$ then the function $\mathcal{F}_{\alpha,\gamma}(z)$ defined by

$$\mathcal{F}_{\alpha,\gamma}(z) = \left[\gamma \int_0^z u^{\gamma-1} \left(\frac{uf'(u)}{f(u)} \right)^\alpha du \right]^{1/\gamma} \tag{3.11}$$

is analytic and univalent in \mathcal{U} .

If we take $\alpha = \gamma = 1$ then, the integral operator $\mathcal{F}_{\alpha,\gamma}(z)$ defined by (3.11) reduces to the integral operator considered by Danikas and Ruscheweyh in [7].

An improvement of Becker's univalence criterion (see [3]) which was obtained by Pascu can be derived from Theorem 3.2 for $\alpha = 1$, $g = \phi$ and $m = 1$.

COROLLARY 3.2. ([20]) *Let $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$, $\Re\gamma > 0$. If*

$$\frac{1 - |z|^{2\Re\gamma}}{\Re\gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in \mathcal{U}$$

then the integral operator

$$\mathcal{F}_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} f'(u) du \right]^{1/\gamma}$$

is analytic and univalent in \mathcal{U} .

4. Quasiconformal extension criterion

In this section we will refine the univalence condition given in Theorem 3.1 to a quasiconformal extension criterion.

THEOREM 4.1. *Let $f, g, \phi \in \mathcal{A}$, $g(z) \neq 0$, $\phi(z) \neq 0$. Let also $m \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re \gamma > 0$ and $k \in [0, 1)$. If*

$$\left| \frac{(1 - |z|^{(m+1)\gamma})}{\gamma} \left[\alpha \frac{zf''(z)}{f'(z)} + \beta \left(\frac{zg'(z)}{g(z)} - \frac{z\phi'(z)}{\phi(z)} \right) \right] - \frac{m-1}{2} \right| \leq k \frac{m+1}{2} \quad (4.1)$$

is true for $z \in \mathcal{U}$ then, the function $\mathcal{F}_{\alpha, \beta, \gamma}$ given by (3.2) has a quasiconformal extension to \mathbb{C} .

Proof. In the proof of Theorem 3.1 has been proved that the function $L(z, t)$ given by (3.3) is a subordination chain in \mathcal{U} . Applying Theorem 2.2 to the function $w(z, t)$ given by (3.6), we obtain that the assumption

$$\left| \frac{(1+a)\mathcal{G}(z, t) + 1 - ma}{(1-a)\mathcal{G}(z, t) + 1 + ma} \right| < l, \quad z \in \mathcal{U}, \quad t \geq 0 \text{ and } l \in [0, 1) \quad (4.2)$$

where $\mathcal{G}(z, t)$ is defined by (3.7), implies l -quasiconformal extensibility of $\mathcal{F}_{\alpha, \beta, \gamma}$.

Lengthy but elementary calculation shows that the last inequality (4.2) is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} \right| \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)}. \quad (4.3)$$

It is easy to check that, under the assumption (4.1) we have

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| \leq k \frac{m+1}{2}. \quad (4.4)$$

Consider the two disks Δ and Δ' defined by (4.3) and (4.4) respectively, where $\mathcal{G}(z, t)$ is replaced by a complex variable ζ . Our theorem will be proved if we find the smallest $l \in [0, 1)$ for which Δ' is contained in Δ . This will be so if and only if the distance apart of the centers plus the smallest radius is equal, at most, to the largest radius. So, we are required to prove that

$$\left| \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{m-1}{2} \right| + k \frac{m+1}{2} \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)}$$

or equivalently

$$\frac{(1-l^2)|1-a^2|}{2[2a(1+l^2) + (1-l^2)(1+a^2)]} \leq \frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \quad (4.5)$$

with the condition

$$\frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \geq 0. \quad (4.6)$$

We will solve inequalities (4.5) and (4.6) for $1 - a^2 > 0$. In a similar way they can be solved for $1 - a^2 < 0$.

The solutions of the quadratic equation obtained from (4.5), where instead of inequality sign we put equal, are:

$$L_1 = \frac{(1-a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}, \quad L_2 = -\frac{(1+a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}.$$

Therefore, the solution of inequality (4.5) is $l \leq L_2$ and $L_1 \leq l$. Since $L_2 < 0$ it remains $L_1 \leq l$.

After similar calculations, from inequality (4.6), we get $l \leq \mathcal{L}_2$ and $\mathcal{L}_1 \leq l$, where

$$\mathcal{L}_1 = \frac{-2a + \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}, \quad \mathcal{L}_2 = \frac{-2a - \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}.$$

Since $\mathcal{L}_2 < 0$ it follows $\mathcal{L}_1 \leq l$.

It can be checked, eventually by using Mathematica program, that $\mathcal{L}_1 \leq L_1$ and thus $L_1 \leq l < 1$. If $a = 1$, both inequalities (4.5) and (4.6) reduce to $k \leq l$.

Consequently, we proved that the assumption (4.1) implies the existence of an l -quasiconformal extension of $\mathcal{F}_{\alpha, \beta, \gamma}$ to \mathbb{C} , which is given by

$$l = \begin{cases} \frac{(1-a)^2 + k|1-a^2|}{|1-a^2| + k(1-a)^2}, & a \in (0, \infty) \setminus \{1\} \\ k, & a = 1. \end{cases}$$

Therefore $L_1 \leq l < 1$ and the proof is complete. \square

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