

## SOME NEW RESULTS RELATED TO A CLASS OF GENERALIZED HURWITZ ZETA FUNCTION

MIN-JIE LUO AND R. K. RAINA

*Abstract.* In this paper, we establish some new results associated with a class of functions (related to Hurwitz-Lerch zeta function) defined and introduced by Raina and Chhajer in [Acta Math. Univ. Comenianae, 73 (2004), 89-100]. Among the results obtained are the series representation, generating function relationship and their corresponding multidimensional extensions. Some reduced cases of our results are also discussed.

### 1. Introduction

In this paper, we consider some properties of the function ([7, p. 90, Eq. (1.7)])

$$\Theta_{\mu}^{\lambda}(x, \alpha, a, b) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-at-bt^{-\lambda}} (1-xe^{-t})^{-\mu} dt \quad (1.1)$$

( $\lambda > 0, \mu \geq 1, \Re(a) > 0, \Re(b) > 0$ ; when  $\Re(b) = 0$ ,  
 then either  $|x| \leq 1$  ( $x \neq 1$ ),  $\Re(\alpha) > 0$ , or  $x = 1, \Re(\alpha) > \mu$ )

and its multidimensional analogue given by ([7, p. 99, Eq. (5.2)])

$$\Theta_{(p_i)}^{(\lambda, \mu_i)}(x_1, \dots, x_n; \alpha; a, b) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-at-bt^{-\lambda}} \prod_{i=1}^n (1-x_i e^{-p_i t})^{-\mu_i} dt. \quad (1.2)$$

( $\lambda > 0, \mu_i \geq 0, \Re(p_i) > 0$  ( $i = 1, \dots, n$ ),  $\Re(a) > 0, \Re(b) > 0$ ;  
 when  $b = 0$ , then either,  $\max_{1 \leq i \leq n} (|x_i|) < 1$  ( $x_i \neq 1$ ),  $\Re(\alpha) > 0$ ,  
 or  $x_i = 1$  ( $i = 1, \dots, n$ ),  $\Re(\alpha) > \max_{1 \leq i \leq n} (\mu_i)$ )

Other results about the function (1.1) can be found in [13]. It should also be pointed out that the function (1.1) is a special case of a significantly more general class of Hurwitz-Lerch zeta type function defined by (see [10, p. 1487, Definition 1])

$$\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(p_1, \dots, p_p; \sigma_1, \dots, \sigma_q)}(x, s, a; b, \lambda) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at-bt^{-\lambda}}$$

*Mathematics subject classification* (2010): 33C65, 33C90.

*Keywords and phrases:* Hurwitz-Lerch zeta function, series representation,  $H$ -function, generating function, multidimensional analogues.

$${}_p\Psi_q^* \left[ \begin{matrix} (\lambda_1, \rho_1), \dots, (\lambda_p, \rho_p); \\ xe^{-t} \end{matrix} \right] dt, \tag{1.3}$$

$$(\min\{\Re(a), \Re(s)\} > 0; \Re(b) \geq 0; \lambda \geq 0),$$

where  ${}_p\Psi_q^*$  denotes the Fox-Wright function (see [10, p. 1486, Eq. (1.12)]).

When we set  $b = 0$  in (1.1), it becomes

$$\Theta_\mu^\lambda(x, \alpha, a, 0) = \Phi_\mu^*(x, \alpha, a) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-at} (1 - xe^{-t})^{-\mu} dt \tag{1.4}$$

$$(\Re(a) > 0; \Re(\alpha) > 0, \text{ when } |x| \leq 1 (x \neq 1); \Re(\alpha) > \mu \text{ when } x = 1).$$

The series representation of  $\Phi_\mu^*(x, \alpha, a)$  is given by

$$\Phi_\mu^*(x, \alpha, a) = \sum_{n=0}^\infty \frac{(\mu)_n}{(a+n)^\alpha} \frac{x^n}{n!}, \tag{1.5}$$

which was studied by Goyal and Laddha [3, p. 100, Eq. (1.5)]. When we set  $\mu = 1$  in (1.5), it reduces to the classical Hurwitz-Lerch function defined by (see, for example, [11, p. 194])

$$\Phi(x, \alpha, a) = \sum_{n=0}^\infty \frac{x^n}{(a+n)^\alpha}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \alpha \in \mathbb{C} \text{ when } |x| < 1; \Re(\alpha) > 1 \text{ when } |x| = 1).$$

It is worth mentioning that by using the Riemann-Liouville fractional derivative operator  $D_x^\mu$  defined by (see, for example, [14, p. 286])

$$D_x^\mu \{f(x)\} = \begin{cases} \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} f(t) dt & (\Re(\mu) < 0) \\ \frac{d^m}{dx^m} \{D_x^{\mu-m} \{f(x)\}\} & (m-1 \leq \Re(\mu) < m (m \in \mathbb{N})), \end{cases}$$

we have

$$\Phi_\mu^*(x, \alpha, a) = \frac{1}{\Gamma(\mu)} D_x^{\mu-1} \{x^{\mu-1} \Phi(x, \alpha, a)\} \quad (\Re(\mu) > 0),$$

which (as already remarked by Lin and Srivastava [5, p. 730, Eq. (25)]) shows that  $\Phi_\mu^*(x, \alpha, a)$  is essentially a Riemann-Liouville fractional derivative of the classical Hurwitz-Lerch function  $\Phi(z, \alpha, a)$ .

It was found in [7, p. 91, Eq. (2.1)] that the function  $\Theta_\mu^\lambda(x, \alpha, a, b)$  defined by (1.1) has the series representation given by

$$\Theta_\mu^\lambda(x, \alpha, a, b) = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^\infty \frac{\Gamma(\alpha - \lambda m) (-b)^m}{m!} \Phi_\mu^*(x, \alpha - \lambda m, a) \tag{1.6}$$

$$(\lambda > 0, \mu \geq 1, \Re(a) > 0, \Re(b) \geq 0, \Re(\alpha) \neq v\lambda (v \in \mathbb{N}), |x| \leq 1).$$

The above result (1.6) was derived by using the series expansion of the function  $e^{-bt^{-\lambda}}$  occurring in the integrand of (1.1), employing a change in the order of integration and summation, and then applying the known integral representation given in [3] (see also [7, p. 91]). We observe the following:

The expression  $\alpha - \lambda m$  of  $\Phi_\mu^*(x, \alpha - \lambda m, a)$  in series (1.6) must be constrained by the condition that  $\Re(\alpha - \lambda m) > 0$ , for  $m \in \mathbb{N}_0$ , or equivalently, by the condition that

$$\Re(\alpha) > \lambda m, \text{ for } m \in \mathbb{N}_0. \tag{1.7}$$

However, (1.7) cannot be satisfied for a bounded complex parameter  $\alpha$  and a positive number  $\lambda$  when  $m$  is unbounded.

In our present investigation of the function  $\Theta_\mu^\lambda(x, \alpha, a, b)$  and its multidimensional analogue  $\Theta_{(p_i)}^{(\lambda, \mu_i)}(x_1, \dots, x_n; \alpha; a, b)$ , we focus on some further properties, especially, the generating function relationship of these functions. The main results obtained in this paper depend largely on the theory of the  $H$ -function.

### 2. Main Results

In this section, we first present the series representation and the contour integral representation of  $\Theta_\mu^\lambda(x, \alpha, a, b)$  by using the Fox's  $H$ -function. We use then these representations to derive a new generating relation involving the function  $\Theta_\mu^\lambda(x, \alpha, a, b)$ . In our present investigations, we need the following results.

The integral

$$I(\alpha, a, b; \rho) = \int_0^\infty t^{\alpha-1} e^{-at-bt^{-\rho}} dt \quad (\alpha, a, b, \rho > 0), \tag{2.8}$$

which is called the reaction rate integral (see [6]) can easily be evaluated by using the methods of  $H$ -function theory, i.e.,

$$I(\alpha, a, b; \rho) = \frac{1}{\rho a^\alpha} H_{0,2}^{2,0} \left[ ab^{\frac{1}{\rho}} \left| \begin{matrix} \phantom{(\alpha, 1)}, \left(0, \frac{1}{\rho}\right) \end{matrix} \right. \right]. \tag{2.9}$$

$$\left( \Re(\alpha) > 0, \rho > 0, \Re(a) > 0, \Re(b) > 0, \left| \arg\left(ab^{\frac{1}{\rho}}\right) \right| < \frac{1}{2}\pi(1 + 1/\rho) \right)$$

According to a standard notation, the Fox's  $H$ -function is defined as a Mellin-Barnes type contour integral as follows

$$\begin{aligned} H_{P,Q}^{M,N}(z) &= H_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_P, A_P) \\ (b_Q, B_Q) \end{matrix} \right. \right] = H_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_1, A_1), \dots, (a_P, A_P) \\ (b_1, B_1), \dots, (b_Q, B_Q) \end{matrix} \right. \right] \\ &:= \frac{1}{2\pi i} \int_L \mathcal{H}(s) z^{-s} ds, \end{aligned}$$

where

$$\mathcal{H}(s) = \frac{\prod_{j=1}^M \Gamma(b_j + B_j s) \prod_{j=1}^N \Gamma(1 - a_j - A_j s)}{\prod_{j=M+1}^Q \Gamma(1 - b_j - B_j s) \prod_{j=N+1}^P \Gamma(a_j + A_j s)}.$$

More detailed information about this celebrated function may be found in [4], [6], [14] and [12].

The following Theorem 2.1 gives a series representation of (1.1). We omit its proof since a more general result can be found in [10, p. 1488, Theorem 1].

**THEOREM 2.1** *If  $\lambda > 0$ ,  $\mu \geq 1$ ,  $\Re(a) > 0$ ,  $\Re(b) > 0$ ,  $\left| \arg \left( ab^{\frac{1}{\lambda}} \right) \right| < \frac{1}{2} \pi \alpha^*$  ( $\alpha^* = 1 + \frac{1}{\lambda}$ ), then the series representation*

$$\Theta_{\mu}^{\lambda}(x, \alpha, a, b) = \frac{1}{\lambda \Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^{\alpha}} H_{0,2}^{2,0} \left[ (a+n)b^{\frac{1}{\lambda}} \left| \begin{matrix} \phantom{\alpha, 1}, (0, \frac{1}{\lambda}) \\ (\alpha, 1), (0, \frac{1}{\lambda}) \end{matrix} \right. \right] \frac{x^n}{n!} \quad (2.10)$$

is absolutely convergent in  $|x| < 1$  for  $\Re(\alpha) > 0$ , and is absolutely convergent in  $|x| \leq 1$  for  $\Re(\alpha) > \mu$ .

**REMARK 2.2.** In order to investigate the case when  $b \rightarrow 0$ , we need an explicit power series expansion for the function  $H_{0,2}^{2,0}$ . Kilbas and Saigo [4] gave explicit power series expansion of the  $H$ -function under different conditions. Hence, by using [4, p. 6, Theorem 1.3], we get

$$H_{0,2}^{2,0} \left[ z \left| \begin{matrix} \phantom{\alpha, 1}, (0, \frac{1}{\lambda}) \\ (\alpha, 1), (0, \frac{1}{\lambda}) \end{matrix} \right. \right] = z^{\alpha} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} z^l \Gamma \left( -\frac{\alpha}{\lambda} - \frac{l}{\lambda} \right) + \lambda \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} z^{\lambda l} \Gamma(\alpha - l\lambda).$$

By putting  $\lambda = 1$  and letting  $b \rightarrow 0$ , we infer that

$$\lim_{b \rightarrow 0} H_{0,2}^{2,0} \left[ (a+n)b \left| \begin{matrix} \phantom{\alpha, 1}, (0, 1) \\ (\alpha, 1), (0, 1) \end{matrix} \right. \right] = \Gamma(\alpha).$$

The series (2.10) then reduces to the series (1.5).

**COROLLARY 2.3** *Under the conditions stated in Theorem 2.1, we have the following relation:*

$$\sum_{r=0}^{\infty} (x+y)^r \Theta_{r+1}^{\lambda}(-xy, \alpha, a+r, b) = \sum_{n=0}^{\infty} (-xy)^n \Theta_{n+1}^{\lambda}(x+y, \alpha, a+n, b). \quad (2.11)$$

*Proof.* Substituting (2.10) in the left-hand side of Eq. (2.11) and then changing the order of sums, which are absolutely convergent, we have

$$\begin{aligned} & \sum_{r=0}^{\infty} (x+y)^r \Theta_{r+1}^{\lambda}(-xy, \alpha, a+r, b) \\ &= \sum_{r=0}^{\infty} (x+y)^r \frac{1}{\lambda \Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(r+1)_n}{(a+n+r)^{\alpha}} H_{0,2}^{2,0} \left[ (a+n+r)b^{\frac{1}{\lambda}} \left| \frac{\phantom{xy}}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \frac{(-xy)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-xy)^n \frac{1}{\lambda \Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(n+1)_r}{(a+n+r)^{\alpha}} H_{0,2}^{2,0} \left[ (a+n+r)b^{\frac{1}{\lambda}} \left| \frac{\phantom{xy}}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \frac{(x+y)^r}{r!} \\ &= \sum_{n=0}^{\infty} (-xy)^n \Theta_{n+1}^{\lambda}(x+y, \alpha, a+n, b), \end{aligned}$$

where, we have used the elementary identity that  $(r+1)_n/n! = (n+1)_r/r!$ .

REMARK 2.4. The result (2.11) implies that we can change a sum involving a generalized Hurwitz-Lerch zeta function (2.10) whose argument is a product to another generalized Hurwitz-Lerch zeta function whose argument can be written as a sum. This property is similar to the *particular sum rule* of a three-parameter Mittag-Leffler function, namely, (see [9, THEOREM 3.1])

$$\begin{aligned} \sum_{r=0}^{\infty} (x+y)^r E_{2\alpha, \alpha r + \beta}^{r+1}(-xy) &= \sum_{k=0}^{\infty} (-xy)^k E_{\alpha, 2\alpha k + \beta}^{k+1}(x+y) \\ &(\Re(\alpha) > 0, \Re(\beta) > 0), \end{aligned}$$

where the function  $E_{\mu, \nu}^{\rho}(\cdot)$  is defined by ([9, Eq. (1)])

$$\begin{aligned} E_{\mu, \nu}^{\rho}(z) &= \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\mu k + \nu)} \frac{z^k}{k!} \\ &(\Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, z \in \mathbb{C}). \end{aligned}$$

With the help of the Ramanujan’s Master Theorem (see [1]) and (2.10), we can find a new contour integral representation of  $\Theta_{\mu}^{\lambda}(x, \alpha, a, b)$  which is given in the following theorem.

THEOREM 2.5 *The contour integral representation of  $\Theta_{\mu}^{\lambda}(x, \alpha, a, b)$  is given by*

$$\begin{aligned} \Theta_{\mu}^{\lambda}(-x, \alpha, a, b) &= \frac{1}{2\pi i \lambda \Gamma(\alpha) \Gamma(\mu)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s) \Gamma(\mu - s)}{(a-s)^{\alpha}} \\ &\cdot H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\phantom{xy}}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] x^{-s} ds. \end{aligned} \tag{2.12}$$

$$(\lambda > 0, \mu \geq 1, \Re(\alpha) > 0, \Re(a) > 0, \Re(b) > 0)$$

*Proof.* The Mellin transform of the function (2.10) can directly be obtained by using the Ramanujan's Master Theorem. Then, by taking the inverse Mellin transform, we at once get the contour integral representation (2.12).

REMARK 2.6. A more general result is given by Srivastava [10, p. 1489, Eq. (2.2)].

THEOREM 2.7 Under the condition stated in Theorem 2.1, we have the following generating relation:

$$\sum_{m=0}^{\infty} (\alpha)_m \Theta_{\mu}^{\lambda}(-x, \alpha + m, a, b) \frac{y^m}{m!} = \mathcal{D}_{\text{exp}} \left\{ \Theta_{\mu}^{\lambda}(-x, \alpha, a, b) \right\}, \quad (2.13)$$

where the differential operator  $\mathcal{D}_{\text{exp}}$  is defined by

$$\mathcal{D}_{\text{exp}} = e^{-y \frac{d}{da}} = \sum_{m=0}^{\infty} \frac{(-y)^m}{m!} \frac{d^m}{da^m}. \quad (2.14)$$

*Proof.* By using the contour integral representation of  $\Theta_{\mu}^{\lambda}(-x, \alpha + m, a, b)$ , and interchanging the order of integration and summation (which is permissible under the conditions stated), we have

$$\begin{aligned} & \sum_{m=0}^{\infty} (\alpha)_m \Theta_{\mu}^{\lambda}(-x, \alpha + m, a, b) \frac{y^m}{m!} \\ &= \frac{1}{2\pi i \lambda \Gamma(\mu)} \sum_{m=0}^{\infty} \frac{(\alpha)_m}{\Gamma(\alpha + m)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s) \Gamma(\mu - s)}{(a - s)^{\alpha + m}} \\ & \quad \cdot H_{0,2}^{2,0} \left[ (a - s) b^{\frac{1}{\lambda}} \left| \begin{array}{c} \text{---} \\ (\alpha + m, 1), (0, \frac{1}{\lambda}) \end{array} \right. \right] x^{-s} \frac{y^m}{m!} ds. \end{aligned} \quad (2.15)$$

Applying the known derivative formula due to Lawrynowich [6, p. 13, Eq. (1.70)] given by

$$\begin{aligned} & \frac{d^r}{dz^r} \left\{ z^{-\left(\gamma \frac{b_1}{B_1}\right)} H_{P,Q}^{M,N} \left[ z^{\gamma} \left| \begin{array}{c} (a_1, A_1), \dots, (a_P, A_P) \\ (b_1, B_1), \dots, (b_Q, B_Q) \end{array} \right. \right] \right\} \\ &= \left( -\frac{\gamma}{B_1} \right)^r z^{-\left(r + \gamma \frac{b_1}{B_1}\right)} H_{P,Q}^{M,N} \left[ z^{\gamma} \left| \begin{array}{c} (a_1, A_1), \dots, (a_P, A_P) \\ (r + b_1, B_1), \dots, (b_Q, B_Q) \end{array} \right. \right], \end{aligned} \quad (2.16)$$

which upon putting  $b_1 = \alpha$ ,  $\gamma = 1$ ,  $B_1 = 1$ , readily yields

$$\begin{aligned} (-1)^m z^{m+\alpha} \frac{d^m}{dz^m} \left\{ z^{-\alpha} H_{0,2}^{2,0} \left[ z \left| \frac{\quad}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \right\} &= H_{0,2}^{2,0} \left[ z \left| \frac{\quad}{(m+\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \\ \Rightarrow (-1)^m (a-s)^{m+\alpha} b^{\frac{\alpha}{\lambda}} \frac{d^m}{da^m} \left\{ \left[ (a-s)b^{\frac{1}{\lambda}} \right]^{-\alpha} H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\quad}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \right\} \\ &= H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\quad}{(m+\alpha, 1), (0, \frac{1}{\lambda})} \right. \right]. \quad (2.17) \end{aligned}$$

From (2.15) and (2.17), we obtain that

$$\begin{aligned} &\sum_{m=0}^{\infty} (\alpha)_m \Theta_{\mu}^{\lambda}(-x, \alpha + m, a, b) \frac{y^m}{m!} \\ &= \frac{1}{2\pi i \lambda \Gamma(\mu)} \sum_{m=0}^{\infty} \frac{(\alpha)_m}{\Gamma(\alpha + m)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s) \Gamma(\mu - s)}{(a-s)^{\alpha+m}} (-1)^m \\ &\quad \cdot (a-s)^{m+\alpha} b^{\frac{\alpha}{\lambda}} \frac{y^m}{m!} \frac{d^m}{da^m} \left\{ \left[ (a-s)b^{\frac{1}{\lambda}} \right]^{-\alpha} H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\quad}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \right\} x^{-s} ds \\ &= \frac{1}{2\pi i \lambda \Gamma(\mu) \Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \Gamma(s) \Gamma(\mu - s) \sum_{m=0}^{\infty} (-1)^m \frac{(y \frac{d}{da})^m}{m!} \\ &\quad \cdot \left\{ (a-s)^{-\alpha} H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\quad}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \right\} x^{-s} ds \\ &= \frac{1}{2\pi i \lambda \Gamma(\mu) \Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \Gamma(s) \Gamma(\mu - s) e^{-y \frac{d}{da}} \\ &\quad \cdot \left\{ (a-s)^{-\alpha} H_{0,2}^{2,0} \left[ (a-s)b^{\frac{1}{\lambda}} \left| \frac{\quad}{(\alpha, 1), (0, \frac{1}{\lambda})} \right. \right] \right\} x^{-s} ds, \end{aligned}$$

and denoting the differential operator  $e^{y \frac{d}{da}}$  by  $\mathcal{D}_{\text{exp}}$ , and making use of the contour integral representation (2.12), we finally get

$$\sum_{m=0}^{\infty} (\alpha)_m \Theta_{\mu}^{\lambda}(-x, \alpha + m, a, b) \frac{y^m}{m!} = \mathcal{D}_{\text{exp}} \left\{ \Theta_{\mu}^{\lambda}(-x, \alpha, a, b) \right\}.$$

This completes the proof of (2.13).

### 3. Multidimensional Analogues

In this section, we give the multidimensional analogues of the results obtained in Section 2.

**THEOREM 3.1** *If  $\lambda > 0$ ,  $\mu_i \geq 1$ ,  $\Re(p_i) > 0$  ( $i = 1, \dots, n$ ),  $\Re(a) > 0$ ,  $\Re(b) > 0$ ,  $\Omega(\mathbf{p}; \mathbf{k}) = \sum_{i=1}^n p_i k_i$  and  $\left| \arg \left( a + \Omega(\mathbf{p}; 1) b^{\frac{1}{\lambda}} \right) \right| < \frac{1}{2} \pi \alpha^*$  ( $\alpha^* = 1 + \frac{1}{\lambda}$ ), then the series representation given by*

$$\Theta_{(p_i)}^{(\lambda, \mu_i)}(x_1, \dots, x_n; \alpha; a, b) = \frac{1}{\lambda \Gamma(\alpha)} \sum_{k_1, \dots, k_n=0}^{\infty} \frac{H_{0,2}^{2,0} \left[ (a + \Omega(\mathbf{p}; \mathbf{k})) b^{\frac{1}{\lambda}} \middle| \begin{matrix} \text{---} \\ (\alpha, 1), (0, \frac{1}{\lambda}) \end{matrix} \right]}{(a + \Omega(\mathbf{p}; \mathbf{k}))^\alpha} \prod_{i=1}^n \left\{ \frac{(\mu_i)_{k_i} x_i^{k_i}}{k_i!} \right\} \quad (3.18)$$

is absolutely convergent in  $\max_{0 \leq i \leq n} \{|x_i|\} < 1$  for  $\Re(\alpha) > 0$ , and (3.18) is absolutely convergent in  $\max_{0 \leq i \leq n} \{|x_i|\} \leq 1$  for  $\Re(\alpha) > \max_{i \leq i \leq n} \{\mu_i\}$ .

**REMARK 3.2.** When we set  $\lambda = 1$  and let  $b \rightarrow 0$ , then the series (3.18) reduces to [8, p. 50, Eq. (1.5)], namely,

$$\begin{aligned} \Theta_{(p_n)}^{(\mu_n)}(s, a; x_1, \dots, x_n) &= \Theta_{(p_1, \dots, p_n)}^{(\mu_1, \dots, \mu_n)}(s, a; x_1, \dots, x_n) \\ &= \sum_{k_1, \dots, k_n=0}^{\infty} (a + \Omega(\mathbf{p}; \mathbf{k}))^{-s} \prod_{i=1}^n \left\{ \frac{(\mu_i)_{k_i} x_i^{k_i}}{k_i!} \right\}, \end{aligned}$$

where  $\Omega(\mathbf{p}; \mathbf{k}) = \sum_{i=1}^n p_i k_i$ ,  $\Re(a) > 0$ ,  $\mu_i \geq 1$  (either  $|x_i| < 1$ ,  $x_i \neq 1$ ; or  $|x_i| = 1$ ,  $\Re(s) > n$ ,  $i = 1, \dots, n$ ).

The multidimensional analogue of the integral representation (2.12) can be obtained by applying *Method of Bracket* to evaluate the multidimensional Mellin transform. For details about this method, one may refer to [1] and [2].

Because the method is easily applicable, we merely give the result here and the proof is omitted.

**THEOREM 3.3**

$$\Theta_{(p_i)}^{(\lambda, \mu_i)}(-x_1, \dots, -x_n; \alpha; a, b) = \frac{1}{(2\pi i)^n \lambda \Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \dots \int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^n B(u_j, \mu_j - u_j)}{(a - \Omega(\mathbf{p}; \mathbf{u}))^\alpha} \cdot H_{0,2}^{2,0} \left[ (a - \Omega(\mathbf{p}; \mathbf{u})) b^{\frac{1}{\lambda}} \middle| \begin{matrix} \text{---} \\ (\alpha, 1), (0, \frac{1}{\lambda}) \end{matrix} \right] x_1^{-u_1} \dots x_n^{-u_n} du_1 \dots du_n.$$

$$(\lambda > 0, \mu_i \geq 1, \Re(p_i) > 0 \ (i = 1, \dots, n), \Re(\alpha) > 0, \Omega(\mathbf{p}; \mathbf{u}) = \sum_{i=1}^n p_i u_i)$$

The following theorem is a multidimensional analogue of Theorem 2.5.

**THEOREM 3.4** *Under the conditions stated in Theorem 3.1, we have the multidimensional generating relation:*

$$\begin{aligned} \sum_{k_1, \dots, k_n=0}^{\infty} (\alpha)_{\sum_{i=1}^n k_i} \Theta_{(p_i)}^{(\lambda, \mu_i)} \left( -x_1, \dots, -x_n; \alpha + \sum_{i=1}^n k_i, a, b \right) \prod_{i=1}^n \frac{y_i^{k_i}}{k_i!} \\ = \mathcal{D}_{\text{exp};n} \left\{ \Theta_{(p_i)}^{(\lambda, \mu_i)} (-x_1, \dots, -x_n; \alpha, a, b) \right\}, \end{aligned} \quad (3.19)$$

where

$$\mathcal{D}_{\text{exp};n} = \exp \left( - \sum_{i=1}^n y_i \frac{d}{da} \right).$$

**REMARK 3.5.** For  $n = 1$  and  $p_1 = 1$ , (3.19) reduces to (2.13).

Indeed, in terms of the Lauricella’s hypergeometric function  $F_D^{(r)}$  defined by

$$\begin{aligned} F_D^{(n)} [a, b_1, \dots, b_n; c; x_1, \dots, x_n] = \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(a)_{k_1+\dots+k_n} (b_1)_{k_1} \dots (b_n)_{k_n}}{(c)_{k_1+\dots+k_n}} \frac{x_1^{k_1} \dots x_n^{k_n}}{k_1! \dots k_n!}, \\ (\max\{|x_1|, \dots, |x_n|\} < 1) \end{aligned}$$

we can find a further extension of (3.19). This result is contained in the following theorem.

**THEOREM 3.6** *Under the conditions stated in Theorem 3.1, we have the multidimensional generating relation:*

$$\begin{aligned} \sum_{k_1, \dots, k_n=0}^{\infty} (\beta)_{\sum_{i=1}^n k_i} \Theta_{(p_i)}^{(\lambda, \mu_i)} \left( -x_1, \dots, -x_n; \alpha + \sum_{i=1}^n k_i, a, b \right) \prod_{i=1}^n (b_i)_{k_i} \frac{y_i^{k_i}}{k_i!} \\ = \mathcal{D}_{F_D^{(n)}} \left\{ \Theta_{(p_i)}^{(\lambda, \mu_i)} (-x_1, \dots, -x_n; \alpha, a, b) \right\}, \end{aligned}$$

where

$$\mathcal{D}_{F_D^{(n)}} = F_D^{(n)} \left[ \beta, b_1, \dots, b_n; \alpha; -y_1 \frac{d}{da}, \dots, -y_n \frac{d}{da} \right]$$

and  $\max\{|y_1|, \dots, |y_n|\} < 1$ .

*Acknowledgement.* The authors express their sincerest thanks to the referee for some valuable suggestions.

## REFERENCES

- [1] T. AMDEBERHAN, O. ESPINOSA, I. GONZALEZ, M. HARRISON, V. H. MOLL AND A. STRAUB, *Ramanujan's Master Theorem*, Ramanujan J. **29** (2012), 103–120.
- [2] I. GONZALEZ AND V. MOLL, *Definite integrals by the method of brackets. Part I*. Adv. Appl. Math., **45** (2010), 50–73.
- [3] S. P. GOYAL AND R. K. LADDHA, *On the generalized Riemann zeta function and the generalized Lambert transform*, Ganita Sandesh, **11** (1997), 99–108.
- [4] A. A. KILBAS AND M. SAIGO, *H-Transforms: Theory and Applications*, Chapman & Hall/CRC, Boca Raton/London/New York/Washington, D.C., 2004.
- [5] S.-D. LIN AND H. M. SRIVASTAVA, *Some Families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput. **154** (2004), 725–733.
- [6] A. M. MATHAI, R. K. SAXENA, AND H. J. HAUBOLD, *The H-function: Theory and applications*, Springer, New York, 2010.
- [7] R. K. RAINA AND P. K. CHHAJED, *Certain results involving a class of functions associated with the Hurwitz Zeta function*, Acta Math. Univ. Comenianae, **73** (2004), 89–100.
- [8] R. K. RAINA AND T. S. NAHAR, *A note on certain class of functions related to Hurwitz zeta function and Lambert transform*, Tamkang J. Math. **31** (2000), 49–56.
- [9] A. L. SOUBHIA, R. FIGUEIREDO CAMARGO, E. CAPELAS DE OLIVEIRA AND J. JR. VAZ, *Theorems for series in three-parameter Mittag-Leffler functions*. Frac. Cal. Appl. Anal. **13** (2010), 9–20.
- [10] H. M. SRIVASTAVA, *A new family of the  $\lambda$ -generalized Hurwitz-Lerch zeta functions with applications*, Appl. Math. Inform. Sci. **8** (2014), 1485–1500.
- [11] H. M. SRIVASTAVA AND J. CHOI, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
- [12] H. M. SRIVASTAVA, K. C. GUPTA, AND S. P. GOYAL, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.
- [13] H. M. SRIVASTAVA, M.-J. LUO AND R. K. RAINA, *New results involving a class of generalized Hurwitz-Lerch zeta functions and their applications*, Turkish J. Anal. Number Theory **1** (2013), 26–35.
- [14] H. M. SRIVASTAVA AND H. L. MANOCHA, *A Treatise on Generating Functions* (569 pp.). A Halsted Press Book (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

(Received October 10, 2014)

Min-Jie Luo  
 Department of Applied Mathematics  
 Donghua University  
 Shanghai 201620, People's Republic of China  
 e-mail: mathwinnie@live.com

R. K. Raina  
 M.P. University of Agriculture and Technology  
 Present address:  
 10/11 Ganpati Vihar, Opposite Sector 5  
 Udaipur-313002, Rajasthan, India  
 e-mail: rkraina.7@hotmail.com