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RATIONAL APPROXIMATION IN $L_1(\Gamma)$ METRIC ON CURVES IN THE COMPLEX PLANE

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Abstract. In this paper, the approximation for the class of functions $L_1(\Gamma)$ is investigated by means of rational functions of the form $R_n(z) = \sum_{k=-n}^{n} a_k(z-b)^k$. This class is difficult of access and little studied. The functions from $L_1(\Gamma)$ satisfying natural condition of Lipschitz on the curve Γ, namely, $\|f(z(s+h)) - f(z(s))\|_{L_1(\Gamma)} \leqslant const|h|^\alpha$ are considered. The corresponding approximation theorem is proved.

1. Introduction

In this paper, we will consider a most difficult of access and little studied case of both polynomial and rational approximation in problems, namely, an approximation problem $L_1(\Gamma)$ in metrics where Γ is a curve in a complex plane.

Recall that $f \in L_p(\Gamma)$ $(p \geqslant 1)$ if

$$\|f\|_{L_p(\Gamma)} = \left(\int_\Gamma |f(z)|^p |dz| \right)^{1/p} < +\infty.$$

We study an approximation problem for a class of functions defined only on the boundary Γ of domain G. As is known, polynomial approximation in this case, generally speaking, is impossible. Therefore, in this case the generalized polynomials or rational functions of the form

$$R_n(z) = \sum_{k=-n}^{n} a_k(z-b)^k, \qquad (*)$$

where b is some point strictly inside the considered curve Γ, will be used as an n approximation unit (without loss of generality, we assume $b = 0$). The polynomial approximation of functions of the class $E_1(G)$ (V. I. Smirnov's class) belonging to the Hölder class on Γ and determined by the mapping functions φ and ψ that map the exterior of Γ conformally and univalently onto the exterior of a unit circle γ_0 and vice versa, normalized

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0; \quad z = \psi(w) = \varphi^{-1}(w)$$

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was treated by J. I. Mamedkhanov and A. A. Nersesyan [1].

We consider the most natural and most general analogue of Hölder class on the curves in a complex plane. Namely, $f \in H_1^{\alpha}(\Gamma)$ $(0 < \alpha \leqslant 1)$ if

$$\|f(z(s+h)) - f(z(s))\|_{L_1(\Gamma)} \leqslant const|h|^{\alpha},$$

where $z = z(s)$ $(0 \leqslant s \leqslant l$, l is the length of the curve) is an equation of the curve Γ in angular pozitions.

We also consider the most general of the available classes of curves, i.e. the class S_{θ} defined as follows:

$\Gamma \in S_{\theta}$ if there exists a constant $C(\Gamma) \geqslant 1$ such that $\theta(\delta) \leqslant C(\Gamma)\delta$, where $\theta(\delta) = \sup\limits_{t\in\Gamma} \theta_t(\delta)$, $\theta_t(\delta) = mes\Gamma_{\delta}(t)$ (Lebesgue measure); $\Gamma_{\delta}(t) = \{\tau \in \Gamma : |t - \tau| \leqslant \delta\}$ $(0 \leqslant \delta \leqslant d)$, d is a diameter of the curve Γ $\left(d = \sup\limits_{t,\tau\in\Gamma} |t - \tau|\right)$.

Now, denote by S_{θ}^* the class of curves $\Gamma \in S_{\theta}$ on which the analogue of Jackson's theorem is valid, i.e. $\Gamma \in S_{\theta}^*$ and $f \in E_1(G)$ $(\Gamma \in \partial G)$, $f \in H_1^{\alpha}(\Gamma)$ $(0 < \alpha \leqslant 1)$ then

$$\rho_n^{(1)}(f,\Gamma) = \inf_{P_n}\|f - P_n\|_{L_1} \leqslant \frac{const}{n^{\alpha}}.$$

2. Main result

The main result of this paper is the following

THEOREM 1. *Let $f \in H_1^{\alpha}(\Gamma)$ $(0 < \alpha \leqslant 1)$ and $\Gamma \in S_{\theta}^*$. Then for each natural n there exists a rational function R_n of the form $(*)$ such that*[1]

$$\|f - P_n\|_{L_1(\Gamma} \preceq n^{-\alpha}.$$

Proof. Let Γ be a closed rectifiable Jordan curve and the boundary of domain G (G^+ and G^- are the interior and the exterior of Γ, respectively). If the singular integral

$$Sf = (Sf)_{\Gamma}(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{f(z)}{z-t}dz, \;\; t \in \Gamma$$

exists almost everywhere on Γ, then, by [2], the Cauchy type integral

$$\Phi(\xi) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{f(z)}{z-t}dz, \;\; \xi \overline{\in} \Gamma$$

has certain conditional values equal to

$$\Phi^{\pm}(z) = \pm\frac{1}{2}f(z) + \frac{1}{2}Sf \tag{1}$$

[1]The relations $A \asymp B$ and $A \preceq B$ $(A > 0,\ B > 0)$ each time are determined with respect to some fixed collection of parameters and correspond to the inequalities $C_1B \leqslant A \leqslant C_2B$ and $A \leqslant C_3B$, where $C_k > 0$ $(k = 1,2,3)$ are the constants independent of the mentioned collection of parameters.

almost everywhere on Γ. Hence

$$f(z) = \Phi^+(z) - \Phi^-(z).$$

Further, we need the following statements.

LEMMA 2.1. *Let $\Gamma \in S^*_\theta$ be a closed curve and let $f \in L_1(\Gamma)$ be such a function that $Sf \in L_1(\Gamma)$. Then we have*[2]

$$(S^2 f)(t) = f(t)$$

almost everywhere on Γ.

LEMMA 2.2. [3] *If Φ^+ is a holomorphic function in G^+ and $\Phi^+ \in L_1(\Gamma)$, then for representability of this function by the Cauchy integral in G^+ it is necessary and sufficient that the equality*

$$(S\Phi^+)(t) = \Phi^+(t)$$

holds almost everywhere on Γ.

LEMMA 2.3. [3] *Let $\Phi^-(z)$ be holomorphic everywhere in G^- except maybe for infinity, and $\Phi^- \in L_1(\Gamma)$. Then for representability of this function by the Cauchy integral with the principal part $g(z)$ at infinity, it is necessary and sufficient that the equality*

$$(S\Phi^-)(t) = -\Phi^-(t) + 2g(t)$$

holds almost everywhere on Γ.

LEMMA 2.4. *Let $f \in H_1^\alpha(\Gamma)$ and $\Gamma \in S_\theta$, then*

$$\Phi^\pm \in E_1(G^\pm).$$

It is easy to show that $\Phi^+ \in L_1(\Gamma)$.

Furthermore, from (1) we have

$$\Phi^+(t) - \Phi^-(t) = (Sf)(t).$$

It follows that $Sf \in L_1(\Gamma)$.

Thus, we can assert that the condition $f \in H_1^\alpha(\Gamma)$ implies the condition $Sf \in L_1(\Gamma)$. By lemma 2.1 it allows us to assert that $(S^2 f)(t) = f(t)$ almost everywhere on Γ.

Now we show that $S\Phi^+ = \Phi^+$, $S\Phi^- = \Phi^-$ almost everywhere on Γ. Indeed, we have

$$S\Phi^+ = S\left[\frac{1}{2}Sf + \frac{1}{2}f\right] = \frac{1}{2}S^2 f + \frac{1}{2}Sf$$

since, by lemma 2.1, $f = \frac{1}{2}S^2 f$ almost everywhere on Γ. However, by virtue of (1),

$$\Phi^+ = \frac{1}{2}Sf + \frac{1}{2}f.$$

[2] This lemma is proved in the paper [3] for more narrow class of curves Γ.

It follows that $S\Phi^+ = \Phi^+$ almost everywhere on Γ. So, by lemma 2.2, the function Φ^+ is representable in terms of Cauchy integral, and this implies $\Phi^+ \in E_1(G^+)$ [2].

Similarly, we can show that $\Phi^- \in E_1(G^-)$ to complete the proof of lemma 2.4.

Now we prove that if $f \in H_1^\alpha(\Gamma)$ and $\Gamma \in S_\theta^*$, then for every n there exists a polynomial P_n of degree n such that

$$\|\Phi^+ - P_n\|_{L_1(\Gamma)} \preceq n^{-\alpha}. \tag{2}$$

Indeed, by lemma 2.1 we can assert that $\Phi^+ \in E_1(G)$. It is also known that $f \in H_1^\alpha(\Gamma)$ implies $\Phi^+ \in H_1^\alpha(\Gamma)$. In view of the fact that $\Gamma \in S_\theta^*$, this allows us to assert that the relation (2) is valid.

Also, by lemma 2.4, $f \in H_1^\alpha(\Gamma)$ implies $\Phi^- \in E_1(G^-)$. Now we show that for any positive integer n there exists $P_n\left(\frac{1}{z}\right) = P_n^-$ of degree n such that

$$\|\Phi^+ - P_n^-\|_{L_1(\Gamma)} \preceq n^{-\alpha}. \tag{3}$$

Indeed, map the plane (z) onto the plane (ϑ) by means of the function $\vartheta = \frac{1}{z}$. Obviously, under this mapping the contour Γ passes to some contour Γ_1 while the interior and the exterior of Γ (G^+ and G^-) are mapped onto the exterior and the interior of Γ_1 (G_1^- and G_1^+), respectively.

It is easy to show that $\Gamma_1 \in S_\theta^*$.[3]

Further, the function f is transformed into the function $f_1(\vartheta)$ $\left(f_1(\vartheta) = f\left(\frac{1}{\vartheta}\right)\right)$ in the plane (ϑ). Therefore, the function $\Phi(z)$ in the plane (ϑ) takes the following form for $z \in G^-$

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f_1(\xi)}{\xi - \vartheta}d\xi = \Phi_1(\vartheta).$$

Obviously, the function $\Phi_1^+(\vartheta)$ corresponds to the function $\Phi^-(t)$ in the plane (z). It is easy to see that if $f \in H_1^\alpha(\Gamma)$, then $f_1 \in H_1^\alpha(\Gamma_1)$, and it follows, in the same way as above, that $\Phi_1^\pm(\vartheta) \in L_1(\Gamma_1)$.

Taking into account that $Sf_1 = \Phi_1^+ + \Phi_1^-$, we get $Sf_1 \in L_1(\Gamma_1)$. Thus, we have $f_1 \in L_1(\Gamma_1)$ and $Sf_1 \in L_1(\Gamma_1)$.

Hence, by lemma 2.1 it follows that

$$(S_{\Gamma_1}^2 f_1)(\vartheta) = f_1(\vartheta)$$

almost everywhere on Γ_1. Hence we find:

$$(S_\Gamma \Phi_1^+)(\vartheta) = \Phi_1^+(\vartheta)$$

almost everywhere on Γ_1.

By lemma 2.2 this allows us to assert that the function Φ_1^+ is representable by Cauchy integral in its boundary values. As a conclusion we get $\Phi_1^+ \in E_1(G_1^+)$.

[3] Class of curves S_θ^* – this is a class of curves of S_θ, for which an analogue of Jackson's theorem is valid.

Now, similar to (2), we get that for any positive integer n there exists a polynomial $Q_n(\vartheta)$ of degree $\leqslant n$ such that

$$\|\Phi_1^+ - Q_n\|_{L_1(\Gamma_1)} \preceq n^{-\alpha}.$$

Substituting $\vartheta = \frac{1}{z}$ into the left-hand side of this relation and using relation

$$\|\Phi_1^+ - Q_n\|_{L_1(\Gamma_1)} \asymp \|\Phi_1^- - Q_n^-\|_{L_1(\Gamma)},$$

we obtain

$$\|\Phi^- - Q_n^-\|_{L_1(\Gamma)} \preceq n^{-\alpha},$$

where $Q_n^-(\frac{1}{z}) = Q_n(\vartheta)$.

And finally, in order to get the main result of the theorem, we consider the equality

$$f(z) = \Phi^+(z) - \Phi^-(z)$$

and apply relations (2) and (3) to the functions Φ^+ and Φ^- to complete the proof of the theorem. $\square$

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