

## CLOSURE OF THE LINEAR SPAN OF AN EXPONENTIAL SYSTEM IN A WEIGHTED BANACH SPACE

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*Abstract.* For a certain class of sequences with repeated terms,

$$\{\lambda_n, \mu_n\}_{n=1}^{\infty} := \underbrace{\{\lambda_1, \lambda_1, \dots, \lambda_1\}}_{\mu_1 \text{ times}}, \underbrace{\{\lambda_2, \lambda_2, \dots, \lambda_2\}}_{\mu_2 \text{ times}}, \dots, \underbrace{\{\lambda_k, \lambda_k, \dots, \lambda_k\}}_{\mu_k \text{ times}}, \dots,$$

we prove that every function belonging to the closed span of the exponential system

$$\{x^k e^{\lambda_n x} : n \in \mathbb{N}, k = 0, 1, 2, \dots, \mu_n - 1\},$$

in some weighted Banach spaces on the real line, extends analytically as an entire function by admitting a series representation of the form

$$\sum_{n=1}^{\infty} \left( \sum_{k=0}^{\mu_n-1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall z \in \mathbb{C}.$$

### 1. Introduction and the main result

Let  $w(x)$  be a non-negative real valued continuous function defined on the real line  $\mathbb{R}$ , not identically equal to zero, such that for some positive constants  $a$  and  $\beta$  we have

$$0 \leq w(x) \leq \begin{cases} ax^2, & x \geq 0, \\ \beta|x|, & x < 0. \end{cases} \quad (1)$$

For any  $p \geq 1$  we denote by  $L_w^p$  the Banach space of complex-valued measurable functions  $f$  defined on  $\mathbb{R}$  such that

$$\int_{-\infty}^{\infty} |f(x)e^{-w(x)}|^p dx < \infty,$$

equipped with the norm

$$\|f\|_{L_w^p} := \left( \int_{-\infty}^{\infty} |f(x)e^{-w(x)}|^p dx \right)^{\frac{1}{p}}.$$

Similarly we denote by  $C_w$  the Banach space of complex-valued continuous functions  $f$  defined on  $\mathbb{R}$ , satisfying the condition

$$\lim_{|t| \rightarrow \infty} f(t)e^{-w(t)} = 0,$$

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*Mathematics subject classification* (2010): 30B50, 30B60, 46E15, 46E20.

*Keywords and phrases:* Completeness, closure, minimality, Taylor-Dirichlet series.

equipped with the norm

$$\|f\|_{C_w} := \sup\{|f(t)e^{-w(t)}| : t \in \mathbb{R}\}.$$

Our main goal in this article is to describe the closure of the linear span of some exponential system in the above weighted Banach spaces. The system in consideration is

$$E_\Lambda := \{x^k e^{\lambda_n x} : n \in \mathbb{N}, k = 0, 1, 2, \dots, \mu_n - 1\},$$

where  $\{\lambda_n\}_{n=1}^\infty$  is a sequence of distinct complex numbers diverging to infinity and  $\{\mu_n\}_{n=1}^\infty$  is a sequence of positive integers so that

(A)  $\sup_{n \in \mathbb{N}} |\arg \lambda_n| < \pi/4,$

(B) there is a constant  $\kappa > 1$  so that  $\frac{|\lambda_{n+1}|}{|\lambda_n|} > \kappa$  for all  $n \in \mathbb{N},$

(C) there are positive constants  $\alpha$  and  $c,$  with  $0 < \alpha < 1$  such that  $\mu_n \leq c|\lambda_n|^\alpha$  for all  $n \in \mathbb{N}.$

The set with multiple terms

$$\{\lambda_n, \mu_n\}_{n=1}^\infty := \underbrace{\{\lambda_1, \lambda_1, \dots, \lambda_1\}}_{\mu_1 \text{ times}}, \underbrace{\{\lambda_2, \lambda_2, \dots, \lambda_2\}}_{\mu_2 \text{ times}}, \dots, \underbrace{\{\lambda_k, \lambda_k, \dots, \lambda_k\}}_{\mu_k \text{ times}}, \dots\}$$

is called a multiplicity-sequence  $\Lambda,$  and we denote by  $U$  the class of multiplicity-sequences whose terms satisfy conditions (A), (B), (C) as above.

EXAMPLE 1.1. Let  $\{e^{i\theta_n}\}_{n=1}^\infty$  be a sequence of points on the arc  $\{z : |z| = 1, |\arg z| \leq \pi/6\}.$  Then the multiplicity-sequence

$$\{5^n e^{i\theta_n}, 4^n\}_{n=1}^\infty$$

belongs to the class  $U.$

We now state a property of  $\Lambda \in U.$  Consider the constants  $\kappa$  and  $\alpha$  in conditions (B) and (C). Observe that from (B) one gets  $|\lambda_n| \geq \kappa^{n-1} |\lambda_1|.$  Therefore we have

$$\sum_{n=1}^\infty \frac{\mu_n}{|\lambda_n|} \leq \sum_{n=1}^\infty \frac{c|\lambda_n|^\alpha}{|\lambda_n|} \leq \sum_{n=1}^\infty \frac{c|\lambda_1|^{\alpha-1}}{(\kappa^{1-\alpha})^{n-1}} < \infty \tag{2}$$

since  $\kappa > 1$  and  $0 < \alpha < 1.$

Next, for any  $\Lambda$  belonging to the class  $U,$  we denote by  $\text{span}(E_\Lambda)$  the set of all finite linear combinations of the system. We say that a function  $f : \mathbb{R} \mapsto \mathbb{C}$  belongs to the closed span of  $E_\Lambda$  in  $L_w^p$  if for every  $\varepsilon > 0$  there is an exponential polynomial of the form

$$g_m(x) = \sum_{n=1}^m \left( \sum_{k=0}^{\mu_n-1} c_{m,n,k} x^k \right) e^{\lambda_n x}, \quad c_{m,n,k} \in \mathbb{C},$$

so that

$$\|f - g_m\|_{L_w^p} < \varepsilon.$$

Similarly for  $C_w.$

We now state our main result.

**THEOREM 1.1.** *Let  $\Lambda$  be a multiplicity-sequence belonging to the class  $U$ . Let  $f$  be a function which belongs to the closed span of  $E_\Lambda$  in  $C_w$  (or in  $L_w^p$  for some  $p \geq 1$ ). Then there is an entire function  $g(z)$  which admits a Taylor-Dirichlet series representation*

$$g(z) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\mu_n-1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall z \in \mathbb{C},$$

so that  $f(x) = g(x)$  for all  $x \in \mathbb{R}$  (or almost everywhere on the real line).

We note that similar results but with more general weights and with  $\Lambda$  satisfying the condition

$$\sum_{n=1}^{\infty} \frac{\mu_n}{|\lambda_n|} = \infty$$

were deduced in [5, 7] whose work was motivated by articles [1, 2, 3, 4] on a version of Bernstein’s weighted polynomial approximation problem on the real line. In these papers the weight  $w$  is always a convex function. Observe that in our case the function  $w$  in (1) does not have to be a convex function. We also point out that an interesting paper which deals with complete and incomplete exponential systems in weighted Banach spaces and does not assume convexity of  $w$  is [6].

In Sections 2 and 3 we state and prove several lemmas, needed for the proof of Theorem 1.1 which is given in Section 4. The various lemmas and Theorem 1.1 are then applied in Section 5 in order to derive a result concerning the Hilbert space  $L_w^2$ .

### 2. An infinite product

Let  $\Lambda$  belong to the class  $U$  and consider the positive constant  $\beta$  in (1). Then construct the infinite product

$$F(z) = \prod_{n=1}^{\infty} \left( \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\lambda_n + 4\beta}} \right)^{\mu_n}. \tag{3}$$

Due to the convergence of the series (2), this infinite product converges uniformly on every compact subset of the complex plane which does not contain the points  $\{-\overline{\lambda_n} - 4\beta\}_{n=1}^{\infty}$ . Thus it is a meromorphic function having its poles at these points and its zeros at the points  $\{\lambda_n\}_{n=1}^{\infty}$ , both poles and zeros with respective multiplicity  $\mu_n$ .

Consider then the infinite union of closed circles  $B := \bigcup_{n=1}^{\infty} \partial B_n$  where  $B_n = \{z : |z - \lambda_n| \leq 1\}$ . Then the following result holds.

**LEMMA 2.1.** *There is a positive constant  $M$  such that*

$$|F(z)| \leq M \quad \forall z : \Re z \geq -2\beta. \tag{4}$$

Moreover, for every  $\varepsilon > 0$  there is a positive constant  $m_\varepsilon$  so that

$$|F(z)| \geq m_\varepsilon e^{-\varepsilon|z|}, \quad \forall z \in B. \tag{5}$$

*Proof.* First we prove (4). It is easy to see that

$$\left| \frac{\lambda_n - z}{\overline{\lambda_n} + 4\beta + z} \right| \leq 1 \quad \forall z : \Re z \geq -2\beta.$$

Also

$$\prod_{n=1}^{\infty} \left| \frac{\overline{\lambda_n} + 4\beta}{\lambda_n} \right|^{\mu_n} < \infty$$

because the series  $\sum_{n=1}^{\infty} \mu_n / |\lambda_n|$  converges. Combining these two results gives (4).

Next we prove (5). First, for every  $z \in \mathbb{C}$  we write the sequence  $\Lambda$  as  $\Lambda = \Lambda_1 \cup \Lambda_2$  where

$$\begin{aligned} \Lambda_1 &= \{ \lambda_n : |\lambda_n| \geq 6|z| \}, \\ \Lambda_2 &= \{ \lambda_n : |\lambda_n| < 6|z| \}. \end{aligned}$$

Then, we write

$$F(z) = \prod_{\lambda_n \in \Lambda_1} \left( \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda_n} + 4\beta}} \right)^{\mu_n} \prod_{\lambda_n \in \Lambda_2} \left( \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda_n} + 4\beta}} \right)^{\mu_n}. \tag{6}$$

Our goal is to find a lower bound for each one of these products.

Consider the first product with  $\lambda_n \in \Lambda_1$ . Then  $|1 - \frac{z}{\lambda_n}| \geq \frac{5}{6}$ , thus we get

$$\left| \frac{\frac{z}{\overline{\lambda_n} + 4\beta} + \frac{z}{\lambda_n}}{1 - \frac{z}{\lambda_n}} \right| \leq \frac{6}{5} \frac{|z|}{|\lambda_n|} \left| \frac{\lambda_n + \overline{\lambda_n} + 4\beta}{\overline{\lambda_n} + 4\beta} \right| \leq \frac{12}{5} \cdot \frac{|z|}{|\lambda_n|}. \tag{7}$$

Since  $|\lambda_n| \geq 6|z|$  the above is less than  $2/5$  and we can now apply the inequality  $|\log(1+w)| \leq 3|w|/2$  which holds when  $|w| < 1/2$ . We get

$$\left| \log \left( \frac{1 + \frac{z}{\overline{\lambda_n} + 4\beta}}{1 - \frac{z}{\lambda_n}} \right) \right| = \left| \log \left( 1 + \frac{\frac{z}{\overline{\lambda_n} + 4\beta} + \frac{z}{\lambda_n}}{1 - \frac{z}{\lambda_n}} \right) \right| \leq \frac{3}{2} \left| \frac{\frac{z}{\overline{\lambda_n} + 4\beta} + \frac{z}{\lambda_n}}{1 - \frac{z}{\lambda_n}} \right|.$$

Combining this with (7) gives

$$\left| \log \left( \frac{1 + \frac{z}{\overline{\lambda_n} + 4\beta}}{1 - \frac{z}{\lambda_n}} \right) \right| \leq \frac{18}{5} \frac{|z|}{|\lambda_n|}.$$

From the inequality  $|\log|w|| \leq |\log w|$ , we now get

$$\frac{1}{|z|} \left| \sum_{\lambda_n \in \Lambda_1} \mu_n \log \left| \frac{1 + \frac{z}{\overline{\lambda_n} + 4\beta}}{1 - \frac{z}{\lambda_n}} \right| \right| \leq \frac{1}{|z|} \sum_{\lambda_n \in \Lambda_1} \mu_n \left| \log \left( \frac{1 + \frac{z}{\overline{\lambda_n} + 4\beta}}{1 - \frac{z}{\lambda_n}} \right) \right| \leq \sum_{\lambda_n \in \Lambda_1} \frac{18\mu_n}{5|\lambda_n|}.$$

Recall however that  $\sum_{n=1}^{\infty} \mu_n / |\lambda_n|$  converges. Hence as  $|z| \rightarrow \infty$  the sum  $\sum_{|\lambda_n| > 6|z|} \frac{18\mu_n}{5|\lambda_n|}$  tends to zero.

Therefore, for every  $\varepsilon > 0$  there is  $r_\varepsilon > 0$  so that

$$\left| \sum_{\lambda_n \in \Lambda_1} \log \left| \frac{1 + \frac{z}{\lambda_n + 4\beta}}{1 - \frac{z}{\lambda_n}} \right|^{\mu_n} \right| = \left| \log \prod_{\lambda_n \in \Lambda_1} \left| \frac{1 + \frac{z}{\lambda_n + 4\beta}}{1 - \frac{z}{\lambda_n}} \right|^{\mu_n} \right| \leq \varepsilon |z|, \quad \forall z : |z| > r_\varepsilon.$$

Hence, for every  $\varepsilon > 0$  there is a positive constant  $M_{\varepsilon,1}$  so that

$$\prod_{\lambda_n \in \Lambda_1} \left| \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\lambda_n + 4\beta}} \right|^{\mu_n} \geq M_{\varepsilon,1} e^{-\varepsilon |z|}, \quad \forall z \in \mathbb{C}. \tag{8}$$

Consider now the second product with  $\lambda_n \in \Lambda_2$ . Let  $z \in B$ , thus there is a unique  $\lambda_k$  so that  $|z - \lambda_k| = 1$ . We easily get

$$|\lambda_{k+1} - \lambda_k| \geq (\kappa - 1)|\lambda_k| \quad \text{and} \quad |\lambda_{k-1} - \lambda_k| \geq \frac{(\kappa - 1)}{\kappa} |\lambda_k|. \tag{9}$$

Thus

$$|\lambda_n - \lambda_k| \geq \frac{(\kappa - 1)}{\kappa} |\lambda_k| \quad \forall n \neq k.$$

Hence we get

$$|\lambda_n - z| \geq \frac{(\kappa - 1)}{\kappa} |\lambda_k| - 1 \quad \forall n \in \mathbb{N}.$$

Since  $|z - \lambda_k| = 1$  then  $|\lambda_k| \geq |z| - 1 > |z|/2$  for all  $z$  such that  $|z| > 2$ . Thus for some  $0 < \tau < 1$  we have

$$|\lambda_n - z| \geq \frac{\tau(\kappa - 1)}{|\kappa|} |z| \quad \forall n \in \mathbb{N}.$$

Then for all  $\lambda_n \in \Lambda_2$  we get

$$\left| \frac{\lambda_n - z}{\lambda_n + 4\beta + z} \right|^{\mu_n} \geq \left( \frac{\tau(\kappa - 1)}{\kappa} |z| \right)^{\mu_n} = \left( \frac{\tau(\kappa - 1)}{7\kappa} \right)^{\mu_n}.$$

Since  $\mu_n \leq c|\lambda_n|^\alpha$  (condition (C) for  $\Lambda$ ) and  $|\lambda_n| \leq 6|z|$  then  $\mu_n \leq 6c|z|^\alpha$ . Therefore

$$\left| \frac{\lambda_n - z}{\lambda_n + 4\beta + z} \right|^{\mu_n} \geq \left( \frac{\tau(\kappa - 1)}{7\kappa} \right)^{6c|z|^\alpha} \tag{10}$$

Next we find an upper bound for the number of  $\lambda_n \in \Lambda_2$ . If  $\kappa^n |\lambda_1| \geq 6|z|$  then  $|\lambda_{n+1}| \geq 6|z|$  as well. Since  $\lambda_n \in \Lambda_2$  then

$$n \leq \frac{\log \frac{6|z|}{|\lambda_1|}}{\log \kappa}.$$

Combining this with (10) gives

$$\prod_{\lambda_n \in \Lambda_2} \left| \frac{\lambda_n - z}{\lambda_n + 4\beta + z} \right|^{\mu_n} \geq \prod_{\lambda_n \in \Lambda_2} \left( \frac{\tau(\kappa - 1)}{7\kappa} \right)^{6c|z|^\alpha} \geq \left( \left( \frac{\tau(\kappa - 1)}{7\kappa} \right)^{6c|z|^\alpha} \right)^{\frac{\log \frac{6|z|}{|\lambda_1|}}{\log \kappa}}.$$

Next, we deduce that for every  $\varepsilon > 0$  there is  $R_\varepsilon > 0$  so that

$$6c|z|^\alpha \frac{\log \frac{6|z|}{|\lambda_1|}}{\log \kappa} \log \frac{7\kappa}{\tau(\kappa-1)} \leq \varepsilon|z|, \quad \forall z: |z| > R_\varepsilon.$$

Therefore, for every  $\varepsilon > 0$  there is a positive constant  $M_{\varepsilon,2}$  such that

$$\prod_{\lambda_n \in \Lambda_2} \left| \frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\lambda_n + 4\beta}} \right|^{\mu_n} \geq M_{\varepsilon,2} e^{-\varepsilon|z|}, \quad \forall z: z \in B.$$

Combining this with (8) gives (5).  $\square$

### 3. Some crucial lemmas

Throughout this section we let  $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^\infty$  belong to the class  $U$ ,  $F$  is the meromorphic function of the previous section, and  $a, \beta$  are the positive constants as in (1)

LEMMA 3.1. *Consider the meromorphic function*

$$f(z) = F(z)e^{\frac{z^2}{5a}}. \tag{11}$$

Then there exists a continuous function  $h$  defined on  $\mathbb{R}$  such that

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(t)e^{zt} dt, \quad \forall z \in \mathbb{C}_+ := \{z: \Re z > 0\}. \tag{12}$$

Moreover, for some  $\delta > 0$  we have

$$|h(t)| \leq \begin{cases} \delta \exp(-\frac{5at^2}{4}), & t \geq \frac{-4\beta}{5a}, \\ \delta \exp(\frac{4\beta^2}{5a} + 2\beta t), & t < \frac{-4\beta}{5a}. \end{cases} \tag{13}$$

*Proof.* First observe that  $f$  is analytic in the open half-plane  $\mathbb{C}_{-4\beta} := \{z: \Re z > -4\beta\}$ , and vanishes exactly on the multiplicity sequence  $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^\infty$ . We note that the zeros and poles are symmetric with respect to the line  $\Re z = -2\beta$ . Consider also the closed right half-plane  $\overline{\mathbb{C}}_{-2\beta} := \{z: \Re z \geq -2\beta\}$ . It then follows from (4) that

$$|f(z)| \leq M e^{\frac{z^2}{5a}} \cdot e^{-\frac{z^2}{5a}}, \quad \forall z \in \overline{\mathbb{C}}_{-2\beta}. \tag{14}$$

Due to this estimate, it then follows by contour integration that for any fixed  $t \in \mathbb{R}$  the value of the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x+iy)e^{-t(x+iy)} dy$$

is the same for all  $x \geq -2\beta$ . We call this  $h(t)$ , in other words we have

$$h(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x+iy)e^{-t(x+iy)} dy, \quad \forall x \geq -2\beta. \tag{15}$$

Fix now  $x \geq -2\beta$ . Then  $e^{ix}h(t)$  is the Fourier Transform of  $f(x+iy)$ . Our goal is to prove that for any fixed  $x > 0$ ,  $f(x+iy)$  is equal to the Inverse Fourier Transform of  $e^{ix}h(t)$ . In other words, (12) holds.

For this, it suffices to show that for every fixed  $x > 0$ , one has  $\int_{-\infty}^{\infty} |e^{ix}h(t)| dt < \infty$ . Thus we need to obtain an upper bound for  $|h(t)|$ . From (14) and (15) we get

$$|h(t)| \leq M \frac{e^{\frac{x^2}{5a}-tx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{5a}} dy \quad \forall x \geq -2\beta. \tag{16}$$

Let  $\delta = \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{5}} dy$ . We then deduce that

$$\inf_{x \geq -2\beta} \left\{ \frac{x^2}{5a} - tx \right\} = -\frac{5at^2}{4} \quad \text{if } t \geq \frac{-4\beta}{5a}$$

and

$$\inf_{x \geq -2\beta} \left\{ \frac{x^2}{5a} - tx \right\} = \frac{4\beta^2}{5a} + 2\beta t \quad \text{if } t < \frac{-4\beta}{5a}.$$

Combining these with (16) gives (13). From the latter relation we see that for every fixed  $x > 0$  one has  $\int_{-\infty}^{\infty} |e^{ix}h(t)| dt < \infty$ .  $\square$

LEMMA 3.2. *The exponential system  $E_{\Lambda}$  is not complete in the various weighted Banach spaces.*

*Proof.* Let  $h$  be the continuous function of the previous lemma. For  $f \in L_w^p$  let

$$V(f) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)f(t) dt.$$

Then from (13) and (1), we deduce by the Hölder inequality that  $V$  defines a bounded linear functional on  $L_w^p$ , with norm  $\|V\|_p$ . By differentiating  $f$  in (12) and since  $f$  vanishes on  $\Lambda$ , gives

$$0 = \int_{-\infty}^{\infty} h(t)t^k e^{\lambda nt} dt \quad \forall n \in \mathbb{N}, \quad k = 0, 1, \dots, \mu_n - 1.$$

Thus  $V(t^k e^{\lambda nt}) = 0$  for all elements of the exponential system. Suppose now that this system is complete in  $L_w^p$ . Then if  $f \in L_w^p$  and not equal to zero almost everywhere, for an arbitrary  $\varepsilon > 0$  there is an exponential polynomial  $P$  in span of the system such that

$$\|f - P\|_{L_w^p} < \varepsilon.$$

Therefore,

$$|V(f) - V(P)| < \varepsilon \|V\|_p.$$

But  $V(P) = 0$ , hence  $\|V(f)\|_{L_w^p} < \varepsilon \|V\|_p$ . Since the choice of  $\varepsilon$  is arbitrary, this means that  $f = 0$  a.e, a contradiction. Hence the system is not complete in  $L_w^p$ . With similar arguments we prove the result for the space  $C_w$ .  $\square$

By the notation  $f^{(l)}(z_0)$  we mean the  $l$  derivative of  $f$  at  $z_0$ .

LEMMA 3.3. *There exist analytic functions  $\{f_{n,k}(z) : k = 0, 1, \dots, \mu_n - 1\}_{n=1}^\infty$  in the half-plane  $\mathbb{C}_{-4\beta}$ , so that*

$$f_{n,k}^{(l)}(\lambda_j) = \begin{cases} 1 & j = n, l = k, \\ 0, & j = n, l \in \{0, 1, \dots, \mu_n - 1\} \setminus \{k\}, \\ 0, & j \neq n, l \in \{0, 1, \dots, \mu_j - 1\}. \end{cases} \quad (17)$$

Furthermore, there are positive constants  $M$  and  $\xi$ , independent of  $n$  and  $k$ , so that for every fixed  $n \in \mathbb{N}$  one has

$$|f_{n,k}(z)| \leq M e^{-\xi(\Re \lambda_n)^2} e^{\frac{z^2}{5a} - \frac{v^2}{5a}}, \quad \forall z : |z - \lambda_n| \geq 1 \quad \forall k = 0, 1, \dots, \mu_n - 1. \quad (18)$$

*Proof.* Consider the meromorphic function  $f$  of Lemma 3.1. Since  $1/f(z)$  has a pole of order  $\mu_n$  at the point  $\lambda_n$ , we write down its Laurent series representation

$$\frac{1}{f(z)} = \sum_{j=1}^{\mu_n} \frac{A_{n,j}}{(z - \lambda_n)^j} + g_n(z)$$

which is valid in the open punctured disk  $\{z : 0 < |z - \lambda_n| < \rho_n\}$  where  $\rho_n > 2$  as shown in (9). We note that

$$A_{n,j} = \frac{1}{2\pi i} \int_{\partial B_n} \frac{(z - \lambda_n)^{j-1}}{f(z)} dz, \quad B_n = \{z : |z - \lambda_n| \leq 1\}$$

and  $g_n(z)$  is the regular part.

It follows from (5) that  $|f(z)| \geq m_\varepsilon e^{-\varepsilon|z|} e^{\frac{z^2}{5a} - \frac{v^2}{5a}}$  for all  $z$  on the circle  $\partial B_n$ . Since  $\sup_{n \in \mathbb{N}} |\arg \lambda_n| < \pi/4$  then we deduce that there are positive constants  $M'$  and  $\xi'$  so that

$$|A_{n,j}| \leq M' e^{-\xi'(\Re \lambda_n)^2}. \quad (19)$$

We now construct the functions that satisfy (17). Fix some positive integer  $n$  and some  $k \in \{0, 1, 2, \dots, \mu_n - 1\}$  and define

$$f_{n,k}(z) := \frac{f(z)}{k!} \sum_{l=1}^{\mu_n - k} \frac{A_{n,k+l}}{(z - \lambda_n)^l}. \quad (20)$$

First suppose that  $k = 0$ , thus

$$f_{n,0}(z) = f(z) \sum_{l=1}^{\mu_n} \frac{A_{n,l}}{(z - \lambda_n)^l}.$$

Then we get  $f_{n,0}^{(l)}(\lambda_j) = 0$  for  $j \neq n$  and  $l = 0, 1, \dots, \mu_j - 1$ . Since  $f_{n,0}(z)$  is continuous at  $z = \lambda_n$ , then

$$f_{n,0}(z) = f(z) \left[ \frac{1}{f(z)} - g_n(z) \right] = 1 - f(z)g_n(z) \quad \forall z \in B_n.$$

Hence,  $f_{n,0}(\lambda_n) = 1$  and  $f_{n,0}^{(l)}(\lambda_n) = 0$  for  $l \in \{1, \dots, \mu_n - 1\}$ . Thus,  $f_{n,0}(z)$  satisfies (17).

Next, suppose that  $k \in \{1, 2, \dots, \mu_n - 1\}$ . Since  $f_{n,k}(z)$  is continuous at  $z = \lambda_n$ , we rewrite  $f_{n,k}(z)$  for all  $z$  in  $B_n$  as

$$\begin{aligned} f_{n,k}(z) &= \frac{f(z)(z - \lambda_n)^k}{k!} \sum_{l=k+1}^{\mu_n} \frac{A_{n,l}}{(z - \lambda_n)^l} \\ &= \frac{f(z)(z - \lambda_n)^k}{k!} \left[ \frac{1}{f(z)} - g_n(z) - \sum_{j=1}^k \frac{A_{n,j}}{(z - \lambda_n)^j} \right] \\ &= \frac{(z - \lambda_n)^k}{k!} - \frac{f(z)(z - \lambda_n)^k g_n(z)}{k!} - \frac{f(z)}{k!} \sum_{j=1}^k A_{n,j}(z - \lambda_n)^{k-j}. \end{aligned} \tag{21}$$

From (20) we get  $f_{n,k}^{(l)}(\lambda_j) = 0$  for  $j \neq n$  and  $l = 0, 1, \dots, \mu_j - 1$ . From (21) we get  $f_{n,k}^{(k)}(\lambda_n) = 1$  and  $f_{n,k}^{(l)}(\lambda_n) = 0$  for  $l \in \{0, 1, \dots, \mu_n - 1\} \setminus \{k\}$ . Thus,  $f_{n,k}(z)$  satisfies (17) for  $k \neq 0$ .

Finally by combining (14), (19), (20) and the fact that  $\mu_n \leq c|\lambda_n|^\alpha$  (Condition (C)), shows that there exist positive constants  $M$  and  $\xi$  so that the upper bound (18) holds outside the open disk  $|z - \lambda_n| < 1$ .  $\square$

LEMMA 3.4. *For each positive integer  $n$  and each integer  $k \in \{0, 1, 2, \dots, \mu_n - 1\}$ , there exist non-trivial bounded linear functionals  $V_{n,k}^p$  on  $L_w^p$  for  $p \in [1, \infty)$  and  $V_{n,k}$  on  $C_w$ , so that*

$$V_{n,k}^p(t^l e^{\lambda_j t}) = \begin{cases} 1, & j = n, l = k, \\ 0, & j = n, l \in \{0, 1, \dots, \mu_n - 1\} \setminus \{k\}, \\ 0, & j \neq n, l \in \{0, 1, \dots, \mu_j - 1\}. \end{cases} \tag{22}$$

Furthermore, there are positive constants  $M$  and  $\xi$ , independent of  $n$  and  $k$ , so that

$$\|V_{n,k}^p\| \leq M \exp\{-\xi(\Re \lambda_n)^2\} \tag{23}$$

where  $\|\cdot\|$  stands for the norms. Similarly for the functionals  $V_{n,k}$ .

*Proof.* Due to the upper bound (18), by contour integration one deduces that the value of the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_{n,k}(x+iy)e^{-(x+iy)t} dy, \quad x \geq -2\beta,$$

does not depend on  $x$ , thus it is a function of  $t$  only. We denote this function by  $h_{n,k}(t)$ . Repeating the steps as in Lemma 3.1 and using the bound (18) gives

$$|h_{n,k}(t)| \leq \begin{cases} M \exp\{-\xi(\Re\lambda_n)^2\} \exp\{-\frac{5at^2}{4}\}, & t \geq \frac{-4\beta}{5a}, \\ M \exp\{-\xi(\Re\lambda_n)^2\} \exp\{\frac{4\beta^2}{5a} + 2\beta t\}, & t < \frac{-4\beta}{5a}. \end{cases} \quad (24)$$

This relation implies that for every fixed  $x > 0$  we have  $e^{xz}h_{n,k}(t) \in L^1(\mathbb{R})$ . As a result

$$f_{n,k}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_{n,k}(t)e^{tz} dt, \quad \forall z \in \mathbb{C}_+.$$

Differentiating with respect to  $z$  and applying (17) gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_{n,k}(t)t^l e^{\lambda_j t} dt = \begin{cases} 1, & j = n, l = k, \\ 0, & j = n, l \in \{0, 1, \dots, \mu_j - 1\} \setminus \{k\}, \\ 0, & j \neq n, l \in \{0, 1, \dots, \mu_j - 1\}. \end{cases} \quad (25)$$

Next, for  $f \in L^p_w$  define

$$V_{n,k}^p(f) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_{n,k}(t)f(t) dt.$$

We then write

$$V_{n,k}^p(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (h_{n,k}(t)e^{w(t)}) (f(t)e^{-w(t)}) dt.$$

Due to relations (24) and (1), applying the Hölder inequality shows that the  $V_{n,k}^p$  define bounded linear functionals on  $L^p_w$ . Then relations (23) and (22) are derived from (24) and (25) respectively.  $\square$

The next result shows that every element of the exponential system  $E_\Lambda$  lies outside the closure of the span of the rest of the elements. In other words the system is minimal.

LEMMA 3.5. *The exponential system  $E_\Lambda$  is minimal in the various weighted Banach spaces.*

*Proof.* Suppose that an element  $p_{n,k}(t) := t^k e^{\lambda_n t}$  of the system belongs to the closed span of  $E_\Lambda \setminus \{p_{n,k}(t)\}$  in  $L^p_w$  for some  $p \geq 1$ . Then for an arbitrary  $\varepsilon > 0$  there is an exponential polynomial  $P$  in span of  $E_\Lambda \setminus \{p_{n,k}(t)\}$  so that

$$\|p_{n,k} - P\|_{L^p_w} < \varepsilon.$$

Consider the functional  $V_{n,k}^p$  of Lemma 3.4. Then

$$|V_{n,k}^p(p_{n,k}) - V_{n,k}^p(P)| < \varepsilon \|V_{n,k}^p\|.$$

By (22) we have  $V_{n,k}^p(p_{n,k}) = 1$  and  $V_{n,k}^p(P) = 0$ . We then get

$$1 < \varepsilon \|V_{n,k}^p\|.$$

The arbitrary choice of  $\varepsilon$  leads to a contradiction. Similarly for the space  $C_w$ .  $\square$

LEMMA 3.6. Consider the linear functionals  $V_{n,k}^p$ . Let  $f \in L_w^p$  and associate to  $f$  the series

$$g(z) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\mu_n-1} V_{n,k}^p(f) z^k \right) e^{\lambda_n z}.$$

The series is an entire function.

*Proof.* Consider the sequence  $\{T_M(z)\}_{M=1}^{\infty}$  of exponential polynomials of the form

$$T_M(z) := \sum_{n=1}^M \left( \sum_{k=0}^{\mu_n-1} V_{n,k}^p(f) z^k \right) e^{\lambda_n z}.$$

Due to the upper bound (23) and since  $\Re \lambda_n > |\Im \lambda_n|$ , this sequence converges uniformly on compact subsets of the complex plane, thus its limit is an entire function.  $\square$

#### 4. Proof of Theorem 1.1

We prove the result for the  $L_w^p$  spaces. The proof for  $C_w$  is similar. Hence, consider the space  $L_w^p$  for some  $p \geq 1$  and suppose that  $f \in \overline{\text{span}}(E_\Lambda)$  in  $L_w^p$ . We will show that  $f(x) = g(x)$  almost everywhere on  $\mathbb{R}$  where  $g$  is the Taylor-Dirichlet series as in Lemma 3.6.

By assumption, there exists a sequence  $\{P_l(x)\}_{l=1}^{\infty}$  in  $\text{span}(E_\Lambda)$ , where

$$P_l(x) = \sum_{n=1}^{r(l)} \left( \sum_{k=0}^{\mu_n-1} a_{n,k,l} x^k \right) e^{\lambda_n x}$$

such that  $\|f - P_l\|_{L_w^p} \rightarrow 0$  as  $l \rightarrow \infty$ . Choose an arbitrary closed interval  $[c, d]$ . It then follows that  $\|f - P_l\|_{L_w^p[c,d]} \rightarrow 0$  as  $l \rightarrow \infty$  where

$$\|f\|_{L_w^p[c,d]} := \left( \int_c^d |f(x) e^{-w(x)}|^p dx \right)^{\frac{1}{p}}.$$

Then by the Minkowski inequality one gets

$$\|f - g\|_{L_w^p[c,d]} \leq \|f - P_l\|_{L_w^p[c,d]} + \|P_l - g\|_{L_w^p[c,d]}.$$

We now claim that  $\|P_l - g\|_{L_w^p[c,d]} \rightarrow 0$  as  $l \rightarrow \infty$ . This implies that the right hand-side of the above inequality converges to zero as  $l \rightarrow \infty$ . Since the left hand-side does not depend on  $l$ , then  $\|f - g\|_{L_w^p[c,d]} = 0$ , hence  $f = g$  almost everywhere on  $[c, d]$ . The arbitrary choice of this interval implies that  $f = g$  almost everywhere on  $\mathbb{R}$  and our proof is finished.

So let us justify our claim. Again by the Minkowski inequality we get

$$\|P_l - g\|_{L_w^p[c,d]} \leq I + II$$

where

$$I := \sum_{n=1}^{r(l)} \sum_{k=0}^{\mu_n-1} |a_{n,k,l} - V_{n,k}^p(f)| \left( \int_c^d (|x|^k e^{x\Re\lambda_n} e^{-w(x)})^p dx \right)^{\frac{1}{p}}$$

and

$$II := \sum_{n=r(l)+1}^{\infty} \sum_{k=0}^{\mu_n-1} |V_{n,k}^p(f)| \left( \int_c^d (|x|^k e^{x\Re\lambda_n} e^{-w(x)})^p dx \right)^{\frac{1}{p}}.$$

We show below that  $I$  and  $II$  converge to zero as  $l \rightarrow \infty$ , thus  $\|P_l - g\|_{L_w^p[c,d]} \rightarrow 0$  as  $l \rightarrow \infty$  as well.

First consider the integral

$$\left( \int_c^d (|x|^k e^{x\Re\lambda_n} e^{-w(x)})^p dx \right)^{\frac{1}{p}}.$$

Let  $T = \max\{1, |c|, |d|\}$  and  $N = \max_{x \in [c,d]} e^{-w(x)}$ . Then

$$\left( \int_c^d (|x|^k e^{x\Re\lambda_n} e^{-w(x)})^p dx \right)^{\frac{1}{p}} \leq NT^k e^{d\Re\lambda_n} (d-c)^{1/p}.$$

It follows from (23) that

$$|V_{n,k}^p(f) - a_{n,k,l}| = |V_{n,k}^p(f) - V_{n,k}^p(P_l)| \leq M e^{-\xi(\Re\lambda_n)^2} \|f - P_l\|_{L_w^p}.$$

Similarly we get

$$|V_{n,k}^p(f)| \leq M e^{-\xi(\Re\lambda_n)^2} \|f\|_{L_w^p}.$$

Thus,

$$I \leq NM(d-c)^{1/p} \|f - P_l\|_{L_w^p} \sum_{n=1}^{r(l)} \mu_n T^{\mu_n} e^{-\xi(\Re\lambda_n)^2} e^{d\Re\lambda_n} \tag{26}$$

and

$$II \leq NM(d-c)^{1/p} \|f\|_{L_w^p} \sum_{n=r(l)+1}^{\infty} \mu_n T^{\mu_n} e^{-\xi(\Re\lambda_n)^2} e^{d\Re\lambda_n}. \tag{27}$$

Next, observe that for every  $\varepsilon > 0$  there is a positive constant  $m_\varepsilon$  so that

$$\mu_n T^{\mu_n} \leq m_\varepsilon e^{\varepsilon\Re\lambda_n}.$$

Therefore

$$\sum_{n=1}^{\infty} \mu_n T^{\mu_n} e^{-\xi(\Re\lambda_n)^2} e^{d\Re\lambda_n} \leq m_\varepsilon \sum_{n=1}^{\infty} e^{-\xi(\Re\lambda_n)^2} e^{(d+\varepsilon)\Re\lambda_n}.$$

Obviously the series on the right hand-side of this inequality converges. Thus there is a positive constant  $C$  so that

$$\sum_{n=1}^{r(l)} \mu_n T^{\mu_n} e^{-\xi(\Re\lambda_n)^2} e^{d\Re\lambda_n} < C \quad \forall l \in \mathbb{N},$$

and the tail

$$\sum_{n=r(l)+1}^{\infty} \mu_n T^{\mu_n} e^{-\xi(\Re\lambda_n)^2} e^{d\Re\lambda_n} \rightarrow 0 \quad l \rightarrow \infty.$$

These results together with the assumption  $\|f - P_l\|_{L^p_w} \rightarrow 0$  as  $l \rightarrow \infty$  shows that the right hand-sides in (26) and (27) tend to zero as  $l \rightarrow \infty$ . Hence both  $I$  and  $II$  tend to zero as  $l \rightarrow \infty$ . Our proof is now complete.

### 5. The Hilbert space $L^2_w$

Our second result, Theorem 5.1, concerns the weighted  $L^2_w$  space which is a Hilbert space when endowed with the inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(t) \overline{g(t)} e^{-2w(t)} dt.$$

We say that a doubly indexed sequence  $\{r_{n,k} : n \in \mathbb{N}, k = 0, 1, \dots, \mu_n - 1\}$  is a biorthogonal sequence to the exponential system  $E_\Lambda$  in  $L^2_w$ , with  $\Lambda \in U$ , if

$$\int_{\gamma}^{\beta} r_{n,k}(t) t^l e^{\overline{\lambda_j} t} e^{-2w(t)} dt = \begin{cases} 1, & j = n, l = k, \\ 0, & j = n, l \in \{0, 1, \dots, \mu_n - 1\} \setminus \{k\}, \\ 0, & j \neq n, l \in \{0, 1, \dots, \mu_j - 1\}. \end{cases}$$

Recall now that in Lemma 3.5 we proved that the system  $E_\Lambda$  is minimal in  $L^2_w$ . Therefore, if we take an element  $p_{n,k}(x) := x^k e^{\lambda_n x}$  and let  $E_{\Lambda_{n,k}} := E_\Lambda \setminus \{p_{n,k}(x)\}$ , then there is some  $\varepsilon > 0$ , not necessarily the same for all elements of the system, so that

$$\inf_{g \in \overline{\text{span}}(E_{\Lambda_{n,k}})} \|p_{n,k} - g\|_{L^2_w} > \varepsilon.$$

In fact this lower bound is even larger. Let

$$D_{2,n,k} := \inf_{g \in \overline{\text{span}}(E_{\Lambda_{n,k}})} \|p_{n,k} - g\|_{L^2_w}.$$

Then the following result holds.

LEMMA 5.1. *There are positive constants  $M^*$  and  $\xi$ , independent of  $n \in \mathbb{N}$  and  $k = 0, 1, \dots, \mu_n - 1$ , so that*

$$D_{2,n,k} \geq M^* \exp\{\xi (\Re \lambda_n)^2\}. \tag{28}$$

*Proof.* Suppose that  $f \in \overline{\text{span}}(E_{\Lambda_{n,k}})$  in the space  $L_w^2$ . Hence for  $\varepsilon > 0$  there is an exponential polynomial  $P \in \text{span}(E_{\Lambda_{n,k}})$  such that

$$\|f - P\|_{L_w^2} < \varepsilon. \tag{29}$$

Let  $h_{n,k}$  be the function as in Lemma 3.4 with the upper bound (24). In combination with (1) this yields a positive constant  $N$  so that

$$\left( \int_{-\infty}^{\infty} |h_{n,k}(t)e^{w(t)}|^2 dt \right)^{\frac{1}{2}} < N \exp\{-\xi (\Re \lambda_n)^2\}. \tag{30}$$

Now since  $P \in \text{span}(E_{\Lambda_{n,k}})$  by (25) we get

$$\int_{-\infty}^{\infty} h_{n,k}(t)P(t) dt = 0.$$

Thus by (29) and (30), applying the Cauchy-Schwartz inequality gives

$$\left| \int_{-\infty}^{\infty} h_{n,k}(t)f(t) dt \right| = \left| \int_{-\infty}^{\infty} (h_{n,k}(t)e^{w(t)}) \left[ (f(t) - P(t))e^{-w(t)} \right] dt \right| \leq \varepsilon N \exp\{-\xi (\Re \lambda_n)^2\}.$$

Since  $\varepsilon$  is arbitrary we then have

$$\int_{-\infty}^{\infty} h_{n,k}(t)f(t) dt = 0.$$

By the above and (25) we get

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} h_{n,k}(t)t^k e^{\lambda_n t} dt = \int_{-\infty}^{\infty} h_{n,k}(t) \left( t^k e^{\lambda_n t} - f(t) \right) dt.$$

Thus

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} \left( h_{n,k}(t)e^{w(t)} \right) \left( (t^k e^{\lambda_n t} - f(t))e^{-w(t)} \right) dt.$$

By (30) and the Cauchy-Schwartz inequality we get

$$\sqrt{2\pi} \leq N \exp\{-\xi (\Re \lambda_n)^2\} \cdot \|p_{n,k} - f\|_{L_w^2}, \quad \text{where} \quad p_{n,k}(t) = t^k e^{\lambda_n t}.$$

Therefore, for any  $f \in \overline{\text{span}}(E_{\Lambda_{n,k}})$  in the space  $L_w^2$ , we have the lower bound

$$\|p_{n,k} - f\|_{L_w^2} \geq \frac{\sqrt{2\pi}}{N} \exp\{\xi (\Re \lambda_n)^2\}.$$

But then this is a lower bound for  $\inf_{f \in \overline{\text{span}}(E_{\Lambda_{n,k}})} \|p_{n,k} - g\|_{L_w^2}$  as well. Our proof is now complete.  $\square$

Next, from Hilbert space theory, we know that there exists a unique element in  $\overline{\text{span}}(E_{\Lambda_{n,k}})$  in  $L_w^2$ , that we denote by  $\phi_{n,k}$ , so that

$$\|p_{n,k} - \phi_{n,k}\|_{L_w^2} = D_{2,n,k}.$$

The function  $p_{n,k} - \phi_{n,k}$  is orthogonal to all the elements of the closed span of  $E_{\Lambda_{n,k}}$  in  $L_w^2$ , hence to  $\phi_{n,k}$  itself. It then follows that

$$\langle p_{n,k} - \phi_{n,k}, p_{n,k} - \phi_{n,k} \rangle = \langle p_{n,k} - \phi_{n,k}, p_{n,k} \rangle.$$

In other words, we have  $(D_{2,n,k})^2 = \langle p_{n,k} - \phi_{n,k}, p_{n,k} \rangle$ . If we now define

$$r_{n,k}(x) := \frac{p_{n,k}(x) - \phi_{n,k}(x)}{(D_{2,n,k})^2}. \tag{31}$$

it then follows that  $\{r_{n,k} : n \in \mathbb{N}, k = 0, 1, \dots, \mu_n - 1\}$  is biorthogonal to the system  $E_\Lambda$ , and since  $\phi_{n,k} \in \overline{\text{span}}(E_{\Lambda_{n,k}})$  in  $L_w^2$  then  $r_{n,k} \in \overline{\text{span}}(E_\Lambda)$  in  $L_w^2$ . In fact, it is the unique biorthogonal sequence to the system  $E_\Lambda$ , which belongs to its closed span in  $L_w^2$ . For, if there was another such biorthogonal sequence, call it  $\{t_{n,k}\}$ , then for all  $n \in \mathbb{N}$  and  $k \in \{0, 1, \dots, \mu_n - 1\}$  we would have

$$\langle r_{n,k} - t_{n,k}, p_{m,l} \rangle = 0 \quad \forall m \in \mathbb{N} \quad l = 0, 1, \dots, \mu_m - 1 \quad p_{m,l} = x^l e^{\lambda_m x}.$$

But this in turn implies that  $r_{n,k} - t_{n,k} = 0$  since the system  $E_\Lambda$  is complete in the closed span of  $E_\Lambda$ .

Next, we also claim that if  $\{u_{n,k}\}$  is any other sequence biorthogonal to the system  $E_\Lambda$ , then  $\|r_{n,k}\|_{L_w^2} \leq \|u_{n,k}\|_{L_w^2}$ . Indeed, choose an element  $u_{n,k}$  and write  $u_{n,k} = r_{n,k} + (u_{n,k} - r_{n,k})$ . We note that  $\langle u_{n,k} - r_{n,k}, f \rangle = 0$  for every  $f$  which belongs to the closed span of  $E_\Lambda$ , hence  $u_{n,k} - r_{n,k}$  belongs to the orthogonal complement of the closed span of  $E_\Lambda$  in  $L_w^2$ . Of course we also have  $\langle u_{n,k} - r_{n,k}, r_{n,k} \rangle = 0$ . Then we get

$$\|u_{n,k}\|_{L_w^2}^2 = \|r_{n,k}\|_{L_w^2}^2 + \|u_{n,k} - r_{n,k}\|_{L_w^2}^2 \geq \|r_{n,k}\|_{L_w^2}^2.$$

Next, observe that it follows from Lemma 5.1 and (31) that  $\|r_{n,k}\|_{L_w^2} \leq \frac{1}{M^*} e^{-\xi} (\Re \lambda_n)^2$ , thus obtaining an upper bound for these norms.

And finally, from our first result, Theorem 1.1, we know that each  $r_{n,k}$  extends analytically in the complex plane admitting a Taylor-Dirichlet series representation.

Overall, we proved the following result.

**THEOREM 5.1.** *Suppose that  $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^\infty$  belongs to the class  $U$ . Suppose also that*

$$\{r_{n,k} : n \in \mathbb{N}, k = 0, 1, \dots, \mu_n - 1\}$$

is the unique biorthogonal sequence to the system  $E_\Lambda$  in  $L_w^2$ , belonging to its closed span. Then there are positive constants  $Q$  and  $\xi$ , independent of  $n$  and  $k$ , so that

$$\|r_{n,k}\|_{L_w^2} \leq Qe^{-\xi(\Re\lambda_n)^2}, \quad \forall n \in \mathbb{N}, \quad k = 0, 1, \dots, \mu_n - 1.$$

Moreover, for each  $r_{n,k}$  there exists an entire function  $R_{n,k}$  so that  $r_{n,k}(x) = R_{n,k}(x)$  for almost all  $x \in \mathbb{R}$ , with  $R_{n,k}$  admitting a Taylor-Dirichlet series representation

$$R_{n,k}(z) = \sum_{j=1}^{\infty} \left( \sum_{l=0}^{\mu_n-1} c_{n,k,j,l} z^l \right) e^{\lambda_j z}, \quad c_{n,k,j,l} \in \mathbb{C}.$$

*Acknowledgements.* The author would like to thank the referee for the various remarks and for bringing article [6] to his attention.

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(Received July 26, 2016)

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