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THE POWER COLLECTION METHOD FOR CONNECTION RELATIONS: MEIXNER POLYNOMIALS

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Abstract. We introduce the power collection method for easily deriving connection relations for certain hypergeometric orthogonal polynomials in the (q-)Askey scheme. We summarize the full-extent to which the power collection method may be used. As an example, we use the power collection method to derive connection and connection-type relations for Meixner and Krawtchouk polynomials. These relations are then used to derive generalizations of generating functions for these orthogonal polynomials. The coefficients of these generalized generating functions are in general, given in terms of multiple hypergeometric functions. From derived generalized generating functions, we deduce corresponding contour integral and infinite series expressions by using orthogonality.

1. Introduction

Orthogonal polynomials are a collection of polynomial families such that any two different polynomials in that family are orthogonal to each other under some inner product. This relation can sometimes be expressed discretely for a sequence of orthogonal polynomials. For instance, given $\{P_n(x;\mathbf{a})\}$, $n \in \mathbb{N}_0$, with discrete weight $w_x \in \mathbb{C}$, $\mathbf{a}$ is a set of free parameters, and $r_n(\mathbf{a}) \in \mathbb{C}$, then one may have the following discrete orthogonality relation

$$\sum_{x=0}^{\infty} P_m(x;\mathbf{a})P_n(x;\mathbf{a})w_x(\mathbf{a}) = r_n(\mathbf{a})\delta_{m,n}.$$

By *connection* relations for orthogonal polynomials (see for instance [1, Lecture 7], [11, Section 13.3] and more recently [8, 9, 16, 19]), we mean a relation where the left-hand side is an orthogonal polynomial with argument x and a set of parameters $\mathbf{a}$, and the right-hand side is given by a finite sum over coefficients (which do not depend on x) multiplied by a product of that same polynomial with a set of different parameters $\mathbf{b}$, namely

$$P_n(x;\mathbf{a}) = \sum_{k=0}^{n} \alpha_{k,n}(\mathbf{a},\mathbf{b})P_k(x;\mathbf{b}).$$

By *connection-type* relations for orthogonal polynomials, we mean a relation where the left-hand side is an orthogonal polynomial with argument x and set of parameters $\mathbf{a}$,

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and the right-hand side is given by a finite sum over coefficients which in general may depend on x, multiplied by a product of that same polynomial with a set of different parameters $\mathbf{b}$, namely

$$P_n(x;\mathbf{a}) = \sum_{k=0}^{n} \beta_{k,n}(x;\mathbf{a},\mathbf{b}) P_k(x;\mathbf{b}).$$

Connection-type relations are not connection relations (nor are they unique) because the coefficients multiplying the orthogonal polynomials depend on the argument. For connection relations, the coefficients of the orthogonal polynomials must not depend on the argument. A generating function for an orthogonal polynomial is given as

$$f(x,t;\mathbf{a}) = \sum_{n=0}^{\infty} c_n(\mathbf{a}) P_n(x;\mathbf{a}) t^n, \tag{1}$$

where $x,t \in \mathbb{C}$, $|t| < 1$.

In this paper we discuss connection and connection-type relations, and generalizations of generating functions from these relations for a family of discrete hypergeometric orthogonal polynomials, namely the Meixner and Krawtchouk polynomials [15, Sections 9.10–11]. Note that we use the terminology that a double connection relation is a connection relation with two free parameters, and a triple connection relation is a connection relation with three free parameters.

The motivation for this paper is two fold, namely (1) to describe and elucidate the powerful and simple method (power collection method) for deriving connection and connection-type relations for (q-)hypergeometric orthogonal polynomials; and (2) to show that the generalized generating functions that we present in our paper, are similar to results recently appearing in the literature [13, Theorem 4.2].

The paper is organized as follows. In Section 2, some mathematical preliminaries which are used in our proofs are introduced. In Section 3, the power collection method for deriving connection relations is explained. Polynomials in which one can apply the power collection method are also listed. In Section 4, connection and connection-type relations are given for Meixner and Krawtchouk polynomials. In Section 5, generalizations of generating functions for Meixner and Krawtchouk polynomials are presented. In Section 6, infinite series expressions are given which are derived using orthogonality for Meixner and Krawtchouk polynomials.

2. Preliminaries: hypergeometric functions

Our generalizations of generating functions rely on Pochhammer symbols. The Pochhammer symbol, also called the shifted factorial, is a special function that is used to express coefficients of polynomials. They can be used to express binomial coefficients, coefficients of derivatives of polynomials, and are integral to the definition of hypergeometric functions. The Pochhammer symbol is defined for $a \in \mathbb{C}$, $n \in \mathbb{N}$, such that

$$(a)_0 := 1, \quad (a)_n := (a)(a+1)\cdots(a+n-1). \tag{2}$$

Note that we assume (and throughout this paper) that the empty product is unity. Define

$$\begin{aligned}\widehat{\mathbb{C}} &:= \{z \in \mathbb{C} : -z \notin \mathbb{N}_0\},\\ \mathbb{C}_0 &:= \{z \in \mathbb{C} : z \neq 0\},\\ \mathbb{C}_{0,1} &:= \{z \in \mathbb{C} : z \notin \{0,1\}\}.\end{aligned}$$

One has the following useful identities for Pochhammer symbols, namely for $n \in \mathbb{N}_0$,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{3}$$

$$\Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(-a+1)_n}, \tag{4}$$

where $a \in \widehat{\mathbb{C}}$, and for $k \in \mathbb{N}_0$, $a \in \mathbb{C}$, one has

$$(a)_{n+k} = (a)_n (a+n)_k = (a)_k (a+k)_n. \tag{5}$$

Another useful identity which we use is that for $n,k \in \mathbb{N}_0$, then

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!},$$

if $0 \leqslant k \leqslant n$, and zero otherwise.

Moreover, for many of the proofs in this paper, we will need the following inequalities for Pochhammer symbols [5, Lemma 12]. Let $j \in \mathbb{N}$, $k,n \in \mathbb{N}_0$, $z \in \mathbb{C}$, $u \in \{z \in \mathbb{C} : \Re z > 0\}$, $v \in [0,\infty)$, $w \in (-1,\infty)$. Then

$$|(u)_j| \geqslant (\Re u)(j-1)!, \tag{6}$$

$$\frac{(v)_n}{n!} \leqslant (1+n)^v, \tag{7}$$

$$(n+w)_k \leqslant \max\{1, 2^w\} \frac{(n+k)!}{n!}, \tag{8}$$

$$(z+k)_{n-k} \leqslant \frac{n!}{k!}(1+n)^{|z|}. \tag{9}$$

The generalized generating functions we present in this paper often have coefficients which can be expressed in terms of generalized hypergeometric functions. Generalized hypergeometric functions ${}_rF_s$ are special functions which can be represented by a hypergeometric series. These are solutions of a $\max(s+1,r)^{\text{th}}$ order differential equation with three regular singular points. The generalized hypergeometric function is defined as [15, (1.4.1)]

$${}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}. \tag{10}$$

For instance, we often take advantage of the binomial theorem [15, (1.5.1)] which can be expressed as

$${}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) = (1-z)^{-a}, \qquad |z| < 1. \tag{11}$$

The q-Pochhammer symbol (q-shifted factorial) is defined for $n \in \mathbb{N}_0$ by

$$(a;q)_0 := 1, \quad (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geqslant 1, \tag{12}$$

where $0 < q < 1$, $a \in \mathbb{C}$. The basic hypergeometric series is defined as

$${}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right) := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, \tag{13}$$

where we have used multi q-Pochhammer notation, namely

$$(a_1,\ldots,a_r;q)_k := \prod_{m=1}^{r}(a_m;q)_k.$$

We have also taken advantage of the q-binomial theorem [15, (1.11.1)]

$${}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix};q,z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Sometimes, the coefficients of our generalized generating functions are given in terms of double and triple hypergeometric functions. There exists a large classification of such functions. The versions of these functions which we encounter are given as follows. For double hypergeometric series we encounter the function F_1 which is an Appell series. These are hypergeometric series in two variables and are defined as [7, (16.13.1)]

$$F_1\left(a,b,b';c;x,y\right) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}. \tag{14}$$

We also encounter the function Φ_2, which is a Humbert hypergeometric series of two variables defined as [18, p. 25]

$$\Phi_2\left(\beta,\beta';\gamma;x,y\right) := \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}. \tag{15}$$

The function $F_D^{(3)}$, a hypergeometric function of three-variables, is a form of the triple Lauricella series defined as [18, p. 33]

$$F_D^{(3)}(a,b_1,b_2,b_3;c;x,y,z) := \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{16}$$

The function $\Phi_2^{(3)}$ is a confluent form of the triple Lauricella series defined as [18, p. 34]

$$\Phi_2^{(3)}(b_1,b_2,b_3;c;x,y,z) := \sum_{m,n,p=0}^{\infty} \frac{(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{17}$$

3. The power collection method and its orthogonal polynomials

In this section, we describe what we refer to as the power collection method. This method, which has been used extensively in the literature (see for example [12, 13, 14]), can be used to easily derive connection relations for generalized hypergeometric and basic hypergeometric orthogonal polynomials. For examples of where the power-collection method is used to derive duplication formulas, see [3, 20]. The method starts with a generating function for a (q-)hypergeometric orthogonal polynomial. For the power collection method to work, $f(x,t;\mathbf{a})$ in (1) must be in a particular elementary form, namely that it contains a simple (q-)binomial product which can be expanded using the (q-)binomial theorem. It is also furthermore crucial that if $\alpha \in \mathbf{a}$, then only the binomial term in the generating function may contain α. If this is the case, then the power collection method may be used to easily derive a connection relation for the free parameter α. Generalized generating functions for hypergeometric orthogonal polynomials (see for instance [2]) are produced by applying series rearragement to known generating functions using derived connection relations. Hence the power collection method is useful for obtaining identities such as these.

In the context of the generating function (1), consider free parameters $\alpha,\beta \in \mathbf{a}$, such that α,β are of the same type. The power collection method[1] proceeds by multiplying the (q-)binomial on the left-hand side of the generating function by a similar expression containing an alternate free parameter β instead of α. On the left-hand side, utilizing the binomial theorem and rearranging the nested series, produces the original generating function, however expressed in terms of β. If this method succeeds, by collecting terms corresponding to t in the resulting expression, the coefficients of the expansion produce a connection relation in terms of the free parameters α,β.

We now give an example of how the power collection method can be used for Meixner polynomials to obtain the well known connection relation

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}(\alpha-\beta)_{n-k}(\beta)_k M_k(x;\beta,c). \tag{18}$$

In the following elementary generating function for Meixner polynomials [15, (9.10.11)], the left-hand side is given in terms of a binomial expression, namely

$$\left(1-\frac{t}{c}\right)^x (1-t)^{-x-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n,$$

where $|t| < |c| < 1$. Multiplying the left-hand side by $(1-t)^{-\beta}/(1-t)^{-\beta}$, and expressing it in terms of the original generating function, produces

$$(1-t)^{\beta-\alpha}\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}M_n(x;\beta,c)t^n = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n.$$

[1]Note that this simple method was originally described to H. S. Cohl by Mourad Ismail. Ismail has also explained that this method is not new, and has been used previously in the literature.

Applying the binomial theorem (11) to the above expression yields

$$\sum_{k=0}^{\infty}\frac{(\alpha-\beta)_k}{k!}t^k\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}M_n(x;\beta,c)t^n=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n,$$

and after collecting terms associated with t produces

$$\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}(\beta)_k}{(n-k)!k!}M_k(x;\beta,c)=\sum_{n=0}^{\infty}t^n\frac{(\alpha)_n}{n!}M_n(x;\alpha,c).$$

If we rearrange this expression, one arrives aarrives at

$$\sum_{n=0}^{\infty}t^n\left(\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)-\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}(\beta)_k}{(n-k)!k!}M_k(x;\beta,c)\right)=0.$$

Since each term corresponding to t^n in the above expression is linearly independent, the connection relation (18) naturally follows.

This method is quite powerful and can be applied in many different contexts of basic and generalized hypergeometric orthogonal polynomials. Note that the continuous q-Hermite and discrete q-Hermite I & II polynomials are not displayed in the following list, even though these polynomials could potentially profit from use of the power collection method. The reason is that these polynomials contain no free parameters (other than q), and hence ordinary connection relations for these polynomials (not in terms of q), do not exist. Furthermore, for the continuous q-ultraspherical/Rogers polynomials, the generating function [15, (14.10.27)] may be used with the power collection method to produce the connection relation for these polynomials. However, the connection relation for these polynomials is well known. For the connection relation, see for instance [11, Section 13.3], and for generalized generating functions see [4].

We now provide a list of generalized and basic hypergeometric orthogonal polynomials in which one may apply the power collection method to easily obtain connection relations. We also display the generating function for these polynomials, which is the main vehicle for the method to work.

- *Continuous dual Hahn polynomials.*
  The relevant generating function is [15, (9.2.12)]

  $$(1-t)^{-c+ix}{}_2F_1\left(\begin{matrix}a+ix,b+ix\\ a+b\end{matrix};t\right)=\sum_{n=0}^{\infty}\frac{S_n(x^2;a,b,c)}{(a+b)_n n!}t^n.$$

  These polynomials have 3 free parameters and 5 known generating functions. Note that the parameters a, b, and c are symmetrical. The power collection method will produce 1 connection relation for each symmetric free parameter. Combining these connection relations produces 3 double connection relations and one triple connection relation, for a total of 7 connection relations.

- *Dual Hahn polynomials.*
  The relevant generating function is [15, (9.6.11)]

  $$(1-t)^{N-x}{}_2F_1\left(\begin{matrix}-x,-x-\delta\\ \gamma+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{n!}R_n(\lambda(x);\gamma,\delta,N)t^n,$$

where $\lambda(x) := x(x+\gamma+\delta+1)$. These polynomials have 3 free parameters and 4 known generating functions. The power collection method will produce 1 connection relation based on parameter N.

- *Bessel polynomials.*
  The relevant generating function is [15, (9.13.10)]

$$(1-2xt)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{a}\exp\left(\frac{2t}{1+\sqrt{1-2xt}}\right)=\sum_{n=0}^{\infty}\frac{y_n(x;a)}{n!}t^n.$$

  These polynomials have 1 free parameter and 2 known generating functions. The power collection method will produce 1 connection relation for the free parameter.

- *Charlier polynomials.*
  The relevant generating function is [15, (9.14.11)]

$$e^t\left(1-\frac{t}{a}\right)^x=\sum_{n=0}^{\infty}\frac{C_n(x;a)}{n!}t^n.$$

  These polynomials have 1 free parameter and 1 known generating function. The power collection method will produce 1 connection relation for the free parameter, but no generalized generating functions since the above generating function is the only known generating function for Charlier polynomials.

- *Continuous dual q-Hahn polynomials.*
  The relevant generating function is [15, (14.7.11)]

$${}_2\phi_1\left(\begin{matrix}ae^{i\theta},be^{i\theta}\\ ab\end{matrix};q,e^{-i\theta}t\right)=\sum_{n=0}^{\infty}\frac{p_n(x;a,b,c|q)}{(ab,q;q)_n}t^n,$$

  where $x=\cos\theta$. These polynomials have 3 free parameters and 4 known generating functions. Note that parameters a, b, and c are symmetrical. The power collection method will produce 1 connection relation for each symmetric free parameter. Combining these connection relations will produce 3 double connection relations and one triple connection relation, for a total of 7 connection relations.

- *Dual q-Hahn polynomials.*
  The relevant generating function is [15, (14.7.11)]

$${}_2\phi_1\left(\begin{matrix}q^{-x},\delta^{-1}q^{-x}\\ \gamma q\end{matrix};q,\gamma\delta q^{x+1}t\right)(q^{-N}t;q)_{N-x}=\sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}R_n(\mu(x);\gamma,\delta,N|q)t^n,$$

  where $\mu(x) := q^{-x}+\gamma\delta q^{x+1}$. These polynomials have 3 free parameters and 2 known generating functions. The power collection method will produce 1 connection relation based on parameter N.

- *Al-Salam-Chihara polynomials.*
  The relevant generating function is [15, (14.8.13)]

$$\frac{(at,bt;q)_\infty}{(e^{i\theta}t,e^{-i\theta}t;q)_\infty}=\sum_{n=0}^{\infty}\frac{Q_n(x;a,b|q)}{(q;q)_n}t^n,$$

  where $x=\cos\theta$. These polynomials have 2 free parameters and 4 known generating functions. Note that parameters a and b are symmetrical. The power collection method will produce 1 connection relation for each free parameter. Combining these connection relations will produce 1 double connection relation for a total of 3 connection relations.

- *q-Meixner-Pollaczek polynomials.*
  The relevant generating function is [15, (14.9.11)]

$$\left|\frac{(ae^{i\phi}t;q)_\infty}{(e^{i(\theta+\phi)}t;q)_\infty}\right|=\frac{(ae^{i\phi}t,ae^{-i\phi}t;q)_\infty}{(e^{i(\theta+\phi)}t,e^{-i(\theta+\phi)}t;q)_\infty}=\sum_{n=0}^{\infty}P_n(x;a,\phi|q)t^n,$$

  where $x=\cos(\theta+\phi)$. These polynomials have 2 free parameters and 2 known generating functions. The power collection method will produce 1 connection relation for each free parameter. Combining these connection relations will produce 1 double connection relation for a total of 3 connection relations.

- *Big q-Laguerre polynomials.*
  The relevant generating function is [15, (14.11.11)]

$$(bqt;q)_\infty\,{}_2\phi_1\left(\begin{matrix}aqx^{-1},0\\ aq\end{matrix};q,xt\right)=\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}P_n(x;a,b;q)t^n.$$

  These polynomials have 2 free parameters and 3 known generating functions. Note that parameters a and b are symmetrical. The power collection method will produce 1 connection relation for each free parameter. Combining these connection relations will produce 1 double connection relation for a total of 3 connection relations.

- *Affine q-Krawtchouk polynomials.*
  The relevant generating function is [15, (14.16.11)]

$$(q^{-N}t;q)_{N-x}\,{}_1\phi_1\left(\begin{matrix}q^{-x}\\ pq\end{matrix};q,pqt\right)=\sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}K_n^{\mathrm{Aff}}(q^{-x};p,N;q)t^n.$$

  These polynomials have 2 free parameters and 2 known generating functions. The power collection method will produce 1 connection relation based on parameter N.

- *Dual q-Krawtchouk polynomials.*
  The relevant generating function is [15, (14.17.11)]

$$(cq^{-N}t;q)_x(q^{-N}t;q)_{N-x}=\sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}K_n(\lambda(x);c,N|q)t^n,$$

where $\lambda(x) := q^{-x} + cq^{x-N}$. These polynomials have 2 free parameters and 1 known generating function. The power collection method will produce 1 connection relation for each free parameter. Combining these connection relations will produce 1 double connection relation for a total of 3 connection relations, but no generalized generating functions since the above generating function is the only known generating function for dual q-Krawtchouk polynomials.

- *Continuous big q-Hermite polynomials.*
  The relevant generating function is [15, (14.18.13)]

$$\frac{(at;q)_\infty}{(e^{i\theta}t, e^{-i\theta}t;q)_\infty} = \sum_{n=0}^{\infty} \frac{H_n(x;a|q)}{(q;q)_n} t^n,$$

  where $x = \cos\theta$. These polynomials have 1 free parameter and 3 known generating functions. The power collection method will produce 1 connection relation for the free parameter.

- *Al-Salam-Carlitz I polynomials.*
  The relevant generating function is [15, (14.24.11)]

$$\frac{(t,at;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{U_n^{(a)}(x;q)}{(q;q)_n} t^n.$$

  These polynomials have 1 free parameter and 1 known generating function. The power collection method will produce 1 connection relation for the free parameter, but no generalized generating functions since the above generating function is the only known generating function for Al-Salam-Carlitz I polynomials.

- *Al-Salam-Carlitz II polynomials.*
  The relevant generating function is [15, (14.25.11)]

$$\frac{(xt;q)_\infty}{(t,at;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} V_n^{(a)}(x;q) t^n.$$

  These polynomials have 1 free parameter and 2 known generating functions. The power collection method will produce 1 connection relation for the free parameter.

4. Connection and connection-type relations

The Meixner polynomials are defined as [15, (9.10.1)]

$$M_n(x;\alpha,c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \alpha \end{matrix}; 1 - \frac{1}{c}\right). \tag{19}$$

In this section we derive and discuss connection and connection-type relations for Meixner polynomials. For the entire paper, we assume that $x \in \mathbb{C}$, $n \in \mathbb{N}_0$. Even though the power collection method may be used to derive the following connection relations

for Meixner polynomials, these can also be found (with proofs) in Gasper (1974) [10, (5.2-5)].

THEOREM 1. *Let* $\alpha,\beta\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$. *Then*

$$M_n(x;\alpha,c)=\sum_{k=0}^{n}\binom{n}{k}\frac{(\beta)_k}{(\alpha)_k}\left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix}-n+k,k+\beta\\ k+\alpha\end{matrix};\frac{d(1-c)}{c(1-d)}\right)M_k(x;\beta,d). \tag{20}$$

By setting $\beta=\alpha$ in (20) one obtains the following specialized result. Let $\alpha\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$. Then

$$M_n(x;\alpha,c)=\left(\frac{c-d}{c(1-d)}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d(1-c)}{c-d}\right)^k M_k(x;\alpha,d). \tag{21}$$

Furthermore by setting $d=c$ in (20), and using the Gauss formula [7, (15.4.20)], one also has the following specialized result. Let $\alpha,\beta\in\widehat{\mathbb{C}}$, $c\in\mathbb{C}_{0,1}$. Then

$$M_n(x;\alpha,c)=\frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}(\alpha-\beta)_{n-k}(\beta)_k M_k(x;\beta,c). \tag{22}$$

REMARK 1. Note that even though Theorem 1 was originally stated in [10] for $\alpha>0$, $c\in(0,1)$, one can extend [10, (5.9–12)] analytically for $\alpha,\beta\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$, since, in such a case, one loses normality of the polynomials, i.e., $\deg M_n(x)<n$ for some n. However, formally, Theorem 1 remains true for $c=1$, and all β,d in the above domains.

We now derive a connection-type relation for Meixner polynomials corresponding to the parameter c, using the power collection method.

THEOREM 2. *Let* $\alpha\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$. *Then*

$$M_n(x;\alpha,c)=\frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k(x)_{n-k}}{d^{n-k}}\,{}_2F_1\left(\begin{matrix}-n+k,-x\\ -x+k-n+1\end{matrix};\frac{d}{c}\right)M_k(x;\alpha,d). \tag{23}$$

Proof. A generating function for Meixner polynomials is given as [15, (9.10.11)]

$$\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n,\qquad |t|<|c|<1. \tag{24}$$

The above connection-type relation (23) can be derived by starting with (24), and multiplying the left-hand side by $\left(1-\frac{t}{d}\right)^x\Big/\left(1-\frac{t}{d}\right)^x$, $|t|<|d|<1$. Then, the left-hand side becomes

$$\begin{aligned}&\left(1-\frac{t}{c}\right)^x\left(1-\frac{t}{d}\right)^{-x}\left(1-\frac{t}{d}\right)^x(1-t)^{-x-\alpha}\\ &\quad=\sum_{m=0}^{\infty}\frac{(-x)_m}{m!}\left(\frac{t}{c}\right)^m\sum_{s=0}^{\infty}\frac{(x)_s}{s!}\left(\frac{t}{d}\right)^s\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}M_k(x;\alpha,d)t^k,\end{aligned} \tag{25}$$

where the first two terms have been replaced using the binomial theorem (11), and the final two terms with the generating function (24) with c replaced by d. Let $s = n-k-m$, and collect the terms associated with t^n using (24) where the left-hand side has been re-expressed using (25). Then (23) follows using analytic continuation in c, d, and (3), (4) and (10). $\square$

We now derive an interesting connection-type relation for Meixner polynomials corresponding to free parameters α, c. The theorem below is not a connection relation because the coefficients multiplied by the Meixner polynomial depend on x.

THEOREM 3. *Let $\alpha,\beta \in \widehat{\mathbb{C}}$, $c,d \in \mathbb{C}_{0,1}$. Then*

$$M_n(x;\alpha,c) = \frac{(\alpha-\beta)_n}{(\alpha)_n}\sum_{k=0}^{n}\frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k} \times F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right)M_k(x;\beta,d), \tag{26}$$

where F_1 is given by (14).

Proof. We substitute the connection relation for the free parameter α (22) with the connection-type relation for the free parameter d (23) to obtain the result

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}k!\binom{n}{k}(\alpha-\beta)_{n-k}\sum_{m=0}^{k}\frac{(\beta)_m(x)_{k-m}}{m!(k-m)!d^{m-k}} \times {}_2F_1\left(\begin{matrix}-k+m,-x\\-x+m-k+1\end{matrix};\frac{d}{c}\right)M_m(x;\beta,d).$$

If we expand the hypergeometric, switch the order of summations twice, and use (3), (4), (10), (14), the result follows. $\square$

Krawtchouk polynomials are a particular case of Meixner polynomials. In fact, they are related in the following way

$$K_n(x;p,N) = M_n\left(x;-N,\frac{p}{p-1}\right). \tag{27}$$

Taking this into account, we can write them as a truncated hypergeometric series as [15, (9.11.1)]

$$K_n(x;p,N) := {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};\frac{1}{p}\right). \tag{28}$$

The following connection results for Krawtchouk polynomials can be found in [10, (5.9–10), (5.11–12)].

THEOREM 4. *Let $n,M,N \in \mathbb{N}_0$, $n \leqslant N \leqslant M$, $p,q \in \mathbb{C}_0$. Then*

$$K_n(x;p,N) = \sum_{k=0}^{n}\binom{n}{k}\frac{q^k(-M)_k}{p^k(-N)_k}\,{}_2F_1\left(\begin{matrix}-n+k,k-M\\k-N\end{matrix};\frac{q}{p}\right)K_k(x;q,M). \tag{29}$$

Setting $M=N$ in (29) one obtains the following connection result. Let $n,N\in\mathbb{N}_0$, $n\leqslant N$, $p,q\in\mathbb{C}_0$. Then

$$K_n(x;p,N)=\left(\frac{p-q}{p}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{q}{p-q}\right)^k K_k(x;q,N). \tag{30}$$

Furthermore by setting $d=c$ in (29) and using the Gauss formula [7, (15.4.20)], one obtains the following. Let $n,M,N\in\mathbb{N}_0$, $n\leqslant N\leqslant M$, $p,q\in\mathbb{C}_0$. Then

$$K_n(x;p,N)=\frac{1}{(-N)_n}\sum_{k=0}^{n}\binom{n}{k}(M-N)_{n-k}(-M)_k K_k(x;p,M). \tag{31}$$

REMARK 2. Observe that the results for Krawtchouk polynomials presented in this paper may also be obtained by starting with (19), setting the right values, and using the relation (27).

5. Generalized generating functions from connection(-type) relations

We now combine generating functions for Meixner and Krawtchouk polynomials with the above connection and connection-type relations to derive generalized generating functions. First we derive generalized generating functions for the Meixner polynomials.

THEOREM 5. *Let* $\alpha,\beta\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$, $x,t\in\mathbb{C}$. *Then*

$${}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)=\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\alpha)_n n!}\left(\frac{td(1-c)}{c(1-d)}\right)^n {}_1F_1\left(\begin{matrix}\beta+n\\ \alpha+n\end{matrix};\frac{-td(1-c)}{c(1-d)}\right)M_n(x;\beta,d). \tag{32}$$

Proof. Using the generating function for Meixner polynomials [15, (9.10.12)]

$$e^t\,{}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)=\sum_{n=0}^{\infty}M_n(x;\alpha,c)\frac{t^n}{n!} \tag{33}$$

and (20), we obtain

$$\begin{aligned}&e^t\,{}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)\\ &=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(\beta)_k}{(\alpha)_k}\left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix}-n+k,\beta+k\\ \alpha+k\end{matrix};\frac{d(1-c)}{c(1-d)}\right)M_k(x;\beta,d).\end{aligned} \tag{34}$$

Using for instance [17, Lemma 10, p. 56], we switch the order of the two summations in (34), and shift the n index by k. The resulting inner sum over n can be easily expressed, using series rearrangment, as

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}{}_2F_1\left(\begin{matrix}-n,\beta+k\\ \alpha+k\end{matrix};\frac{d(1-c)}{c(1-d)}\right)=e^t\,{}_1F_1\left(\begin{matrix}\beta+k\\ \alpha+k\end{matrix};\frac{-td(1-c)}{c(1-d)}\right),$$

where we have used (3), (4), (10). In order to justify reversing the summation symbols, it is enough to show that

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,d) \right| < \infty,$$

where $|M_k(x,\beta,d)| \leqslant K_1(1+k)^{\sigma_2} d^{-k}$, $a_n = t^n/n!$, hence $|a_n| \leqslant |t|^n/n!$, and

$$c_{k,n} = \sum_{s=0}^{n-k} \frac{(-1)^k (-n)_{s+k} (\beta)_{s+k}}{(\alpha)_{s+k} k! s!} \left(\frac{d(1-c)}{c(1-d)} \right)^{s+k},$$

where K_1 and σ_2 are positive constants not depending on n. Then since

$$\sum_{n=0}^{\infty} |a_n| \left| \sum_{k=0}^{n} c_{k,n} M_k(x;\beta,d) \right| \leqslant K_1 K_2 \sum_{n=0}^{\infty} \frac{(1+n)^{\sigma_1+\sigma_2+1}}{n!} \left| \frac{t}{c} \right|^n \left| \frac{1+d-2c}{1-d} \right|^n < \infty,$$

the result follows because all the sums connected with these coefficients converge. Note that in the final result we have replaced k by n. $\square$

A direct consequence of Theorem 5 with $c = d$, and [7, (13.2.39)] is given as follows. Let $x,t \in \mathbb{C}$, $\alpha,\beta \in \widehat{\mathbb{C}}$, $c \in \mathbb{C}_{0,1}$. Then

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c} \right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!} {}_1F_1\left(\begin{matrix} \alpha-\beta \\ \alpha+n \end{matrix}; t \right) M_n(x;\beta,c) t^n. \tag{35}$$

We now combine Meixner generating function (33) with the connection-type relation (26) to derive a generalized generating function.

THEOREM 6. *Let $\alpha \in \widehat{\mathbb{C}}$, $c,d \in \mathbb{C}_{0,1}$, $x,t \in \mathbb{C}$. Then*

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi_2\left(x, -x; \alpha+n; \frac{t}{c}, \frac{t}{d} \right) M_n(x;\alpha,d) t^n, \tag{36}$$

where Φ_2 is given by (15).

Proof. Using (33) and (23), we obtain

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c} \right) = \sum_{n=0}^{\infty} \frac{t^n}{(\alpha)_n} \sum_{k=0}^{n} \frac{(\alpha)_k (x)_{n-k}}{k!(n-k)! d^{n-k}} M_k(x;\alpha,d) {}_2F_1\left(\begin{matrix} -n+k, -x \\ -x+k-n+1 \end{matrix}; \frac{d}{c} \right). \tag{37}$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the hypergeometric, and use (3), (4), (10), and (15). We can justify the reversing the summation symbols since in this case

$$a_n = \frac{t^n}{n!}, \quad \text{and} \quad c_{k,n} = \binom{n}{k} \frac{(\alpha)_k (x)_{n-k}}{d^{n-k}} {}_2F_1\left(\begin{matrix} -n+k, -x \\ -x+k-n+1 \end{matrix}; \frac{d}{c} \right).$$

Therefore

$$\sum_{n=0}^{\infty}|a_n|\left|\sum_{k=0}^{n}c_{k,n}M_k(x;\alpha,d)\right|\leqslant K_3\sum_{n=0}^{\infty}\frac{(1+n)^{\sigma_3}}{n!}\left|\frac{t}{c}\right|^n,$$

where K_3 and σ_3 are positive constants not depending on n, then the result holds since all these sums connected with these coefficients converge. □

THEOREM 7. *Let* $\alpha,\beta\in\widehat{\mathbb{C}}$, $c,d\in\mathbb{C}_{0,1}$, $x,t\in\mathbb{C}$. *Then*

$$e^t\,{}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)=\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\alpha)_n n!}\Phi_2^{(3)}\left(x,-x,\alpha-\beta;\alpha+n;\frac{t}{c},\frac{t}{d},t\right)M_n(x;\beta,d)t^n, \tag{38}$$

where $\Phi_2^{(3)}$ *is given in (17).*

Proof. Using (33) and (26), we obtain

$$\begin{aligned}e^t\,{}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)&=\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{(\alpha-\beta)_n}{(\alpha)_n}\sum_{k=0}^{n}\frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k}M_k(x;\beta,d)\\&\quad\times F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right).\end{aligned} \tag{39}$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the Appell series, switch the order of summations two more times, and use (3), (4), (10), and (17). Indeed,

$$\sum_{n=0}^{\infty}|a_n|\left|\sum_{k=0}^{n}c_{k,n}M_k(x;\beta,d)\right|\leqslant K_4\sum_{n=0}^{\infty}\frac{(1+n)^{\sigma_4}}{n!}\left|\frac{t(c+d)}{cd}\right|^n<\infty,$$

where K_4 and σ_4 are positive constants not depending on n, then the result holds since all these sums connected with these coefficients can be rearranged in the desired way. □

We also have the connection relation with one free parameter given by (22). We now combine this connection relation with the above referenced generating functions to obtain new generalized generating functions for Meixner polynomials.

THEOREM 8. *Let* $c\in\mathbb{C}_{0,1}$, $\gamma,t\in\mathbb{C}$, $|t|<1$, $|t(1-c)|<|c(1-t)|$, $\alpha,\beta\in\widehat{\mathbb{C}}$. *Then*

$$(1-t)^{-\gamma}\,{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\beta)_n}{(\alpha)_n n!}\,{}_2F_1\left(\begin{matrix}\gamma+n,\alpha-\beta\\ \alpha+n\end{matrix};t\right)M_n(x;\beta,c)t^n. \tag{40}$$

Proof. Using the generating function for Meixner polynomials [15, (9.10.13)] and (22), we obtain

$$(1-t)^{-\gamma}\,{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_n t^n}{(\alpha)_n}\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}(\beta)_k}{(n-k)!k!}M_k(x;\beta,c).$$

If we switch the order of summations, shift the n variable by a factor of k and use (3), (4), (10), then the result follows. Indeed, in this case $a_n = t^n(\gamma)_n/n!$, therefore

$$|a_n| \leqslant |t|^n(1+n)^{|\gamma|}.$$

So, we have

$$\sum_{n=0}^{\infty}|a_n|\left|\sum_{k=0}^{n}c_{k,n}M_k(x;\beta,c)\right| \leqslant K_5\sum_{n=0}^{\infty}(1+n)^{\sigma_5}\left|\frac{t(1-c)}{c(1-t)}\right|^n,$$

where K_5 and σ_5 are positive constants not depending on n. Therefore if $|t|<1$ and $|t(1-c)|<|c(1-t)|$ the sum converges, then the result holds since all these sums connected with these coefficients can be rearranged in the desired way. □

THEOREM 9. *Let* $c,d \in \mathbb{C}_{0,1}$, $\gamma,t \in \mathbb{C}$, $|t| < \min\{1,|c(1-d)|/|1+d-2c|\}$, $\alpha,\beta \in \widehat{\mathbb{C}}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty}\frac{(\gamma)_n(\beta)_n}{(\alpha)_n n!}{}_2F_1\left(\begin{matrix}\gamma+n,\beta+n\\ \alpha+n\end{matrix};\frac{-dt(1-c)}{c(1-d)(1-t)}\right) \times\left(\frac{d(1-c)}{c(1-d)(1-t)}\right)^n M_n(x;\beta,d)t^n. \tag{41}$$

Proof. Using [15, (9.10.13)] and (20), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}t^n\sum_{k=0}^{n}\frac{(\beta)_k n!}{k!(n-k)!(\alpha)_k}\left(\frac{d(1-c)}{c(1-d)}\right)^k M_k(x;\beta,d) \times{}_2F_1\left(\begin{matrix}-n+k,\beta+k\\ \alpha+k\end{matrix};\frac{d(1-c)}{c(1-d)}\right).$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, switch the order of summations again, and use (3), (4), (10), then the result holds since all these sums connected with these coefficients converge (it is similar to the previous proof combined with the proof of Theorem 5) and can be rearranged in the desired way. □

Above, we have found a finite expansion of the Meixner polynomials with free parameter c in terms of Meixner polynomials with free parameter d (see the connection-type relation (23)). We now combine Meixner generating function [15, (9.10.13)] with that connection-type relation to derive a generalized generating function whose coefficient is an Appell F_1 double hypergeometric function.

THEOREM 10. *Let* $\alpha \in \widehat{\mathbb{C}}$, $\gamma,t \in \mathbb{C}$, $c,d \in \mathbb{C}_{0,1}$, $|t| < \min\{1,|c|\}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}F_1\left(\gamma+n,x,-x;\alpha+n;\frac{t}{c},\frac{t}{d}\right)M_n(x;\alpha,d)t^n. \tag{42}$$

Proof. Using [15, (9.10.13)] and (23), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_n t^n}{(\alpha)_n n!}\sum_{k=0}^{n}\frac{(\alpha)_k(x)_{n-k}}{k!(n-k)!d^{n-k}}M_k(x;\alpha,d)$$
$$\times{}_2F_1\left(\begin{matrix}-n+k,-x\\ -x+k-n+1\end{matrix};\frac{d}{c}\right).$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the hypergeometric, and use (3), (4), (10), and (14), then the result holds since all these sums connected with these coefficients converge (it is similar to the proof of Theorem 8 combined with the proof of Theorem 6) and can be rearranged in the desired way. □

THEOREM 11. *Let* $\alpha,\beta\in\widehat{\mathbb{C}}$, $\gamma\in\mathbb{C}$, $c,d\in\mathbb{C}_{0,1}$, $|t|<\min\{1,|cd|/|c+d|\}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)=\sum_{n=0}^{\infty}\frac{(\beta)_n(\gamma)_n}{(\alpha)_n n!}F_D^{(3)}\left(\gamma+n,x,-x,\alpha-\beta;\alpha+n;\frac{t}{c},\frac{t}{d},t\right)$$
$$\times M_n(x;\beta,d)t^n, \tag{43}$$

where $F_D^{(3)}$ *is given in (16).*

Proof. Using [15, (9.10.13)] and (26), we obtain

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_n t^n}{n!}\frac{(\alpha-\beta)_n}{(\alpha)_n}\sum_{k=0}^{\infty}\frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k}M_k(x;\beta,d)$$
$$\times F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right). \tag{44}$$

Switch the order of the summations based on n and k, shift the n variable by a factor of k, expand the Appell series, switch the order of summations two more times, and use (3), (4), (10), and (16), then the result holds since all these sums connected with these coefficients converge (it is similar to the proof of Theorem 8 combined with the proof of Theorem 12) and can be rearranged in the desired way. □

We have derived generalized generating functions for the free parameter c. However, since the coefficients of our connection-type relation is in terms of x, we cannot use the orthogonality relation to create new infinite sums. Note that the application of connection relations (22) and (23) to the rest of the known generating functions for Meixner polynomials [15, (9.10.11–13)] leave these generating functions invariant.

We now derive generalized generating functions for the Krawtchouk polynomials, where we will need a special notation for some of the generating functions. Therefore,

let $f \in C^\infty(\mathbb{C})$, $N \in \mathbb{N}_0$, $t \in \mathbb{C}$. Define the truncated Maclaurin expansion of f as (cf. [15, p. 6])

$$[f(t)]_N := \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} t^k.$$

THEOREM 12. *Let $p,q \in \mathbb{C}_0$, $M,N \in \mathbb{N}_0$, $N \leqslant M$, $x,t \in \mathbb{C}$. Then*

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{(-M)_n}{(-N)_n n!}\left(\frac{tq}{p}\right)^n \left[e^t {}_1F_1\left(\begin{matrix}n-M\\n-N\end{matrix};\frac{-tq}{p}\right)\right]_{N-n} K_n(x;q,M). \tag{45}$$

Proof. Using [15, (9.11.12)]

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{t^n}{n!} K_n(x;p,N), \tag{46}$$

and (30), we obtain

$$\begin{aligned}&\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N\\ &= \sum_{n=0}^{N} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-M)_k}{(-N)_k}\left(\frac{q}{p}\right)^k {}_2F_1\left(\begin{matrix}-n+k,k-M\\k-N\end{matrix};\frac{q}{p}\right) K_k(x;q,M).\end{aligned} \tag{47}$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, then switch the order of summations again and shift the n variable again, and use (3), (4), (10), the proof follows since all the series have finite number of terms. □

Letting $p = q$ in (45) yields the following result. Let $p \in \mathbb{C}_0$, $M,N \in \mathbb{N}_0$, $N \leqslant M$, $x,t \in \mathbb{C}$. Then

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{(-M)_n t^n}{(-N)_n n!}\left[{}_1F_1\left(\begin{matrix}M-N\\n-N\end{matrix};t\right)\right]_{N-n} K_n(x;p,M). \tag{48}$$

Furthermore, letting $M = N$ in (45) produces the following. Let $p,q \in \mathbb{C}_0$, $N \in \mathbb{N}_0$, $x,t \in \mathbb{C}$. Then

$$\left[e^t {}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N = \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{tq}{p}\right)^n \left[e^{t(1-q/p)}\right]_{N-n} K_n(x;q,N). \tag{49}$$

THEOREM 13. *Let $p,q \in \mathbb{C}_0$, $M,N \in \mathbb{N}_0$, $N \leqslant M$, $t,\gamma \in \mathbb{C}$. Then*

$$\begin{aligned}&\left[(1-t)^{-\gamma} {}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N\\ &= \sum_{n=0}^{N} \left(\frac{tq}{p}\right)^n \frac{(-M)_n(\gamma)_n}{(-N)_n n!}\left[(1-t)^{-\gamma-n} {}_2F_1\left(\begin{matrix}\gamma+n,n-M\\n-N\end{matrix};\frac{-qt}{p(1-t)}\right)\right]_{N-n} K_n(x;q,M).\end{aligned} \tag{50}$$

Proof. Using [15, (9.11.13)]

$$\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N=\sum_{n=0}^{N}\frac{(\gamma)_n t^n}{n!}K_n(x;p,N), \tag{51}$$

where $\gamma\in\mathbb{C}$, and (30), we obtain

$$\begin{aligned}&\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\gamma)_n}{n!}t^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-M)_k q^k}{(-N)_k p^k}\,{}_2F_1\left(\begin{matrix}-n+k,k-M\\k-N\end{matrix};\frac{q}{p}\right)K_k(x;q,M).\end{aligned} \tag{52}$$

If we switch the order of summations, shift the n variable by a factor of k, expand the hypergeometric, then switch the order of summations again and shift the n variable again, and use (3), (4), (10), the proof follows since all the series have finite number of terms. □

If we let $p=q$ in (50) and use [7, (15.8.1)], we obtain the following result. Let $p\in\mathbb{C}_0$, $M,N\in\mathbb{N}_0$, $N\leqslant M$, $x,t,\gamma\in\mathbb{C}$. Then

$$\begin{aligned}&\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(-M)_n(\gamma)_n t^n}{(-N)_n n!}\left[{}_2F_1\left(\begin{matrix}\gamma+n,M-N\\n-N\end{matrix};t\right)\right]_{N-n}K_n(x;p,M).\end{aligned}$$

If we let $M=N$ in (50) we obtain the following. Let $p,q\in\mathbb{C}_0$, $N\in\mathbb{N}_0$, $x,t,\gamma\in\mathbb{C}$. Then

$$\begin{aligned}&\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N\\&=\sum_{n=0}^{N}\frac{(\gamma)_n}{n!}\left(\frac{qt}{p}\right)^n\left[\left(1+t\left(\frac{q}{p}-1\right)\right)^{-\gamma-n}\right]_{N-n}K_n(x;q,N).\end{aligned}$$

Note that the application of connection relations (30) and (31) to the generating functions for Krawtchouk polynomials [15, (9.11.11–13)] leave these generating functions invariant.

6. Results using orthogonality

We have derived generalized generating functions for the free parameter α. We now combine this with the orthogonality relation for Meixner polynomials to produce new results from our generalized generating functions. The well-known orthogonality relation for Meixner polynomials for $n,m\in\mathbb{N}_0$, $\alpha>0$, $c\in(0,1)$ is [15, (14.25.2)]

$$\sum_{x=0}^{\infty}M_n(x;\alpha,c)M_m(x;\alpha,c)\frac{\Gamma(x+\alpha)c^x}{\Gamma(x+1)}=\kappa_n\delta_{m,n}, \tag{53}$$

where

$$\kappa_n = \frac{n!}{c^n(1-c)^\alpha(\alpha)_n}.$$

Note that this a particular case of a more general property of orthogonality fulfilled by Meixner polynomials (see [6, Proposition 9]).

PROPOSITION 1. *Let $m,n \in \mathbb{N}_0$, $\alpha \in \widehat{\mathbb{C}}$, $c \in \mathbb{C}\setminus[0,\infty)$. The orthogonality relation for Meixner polynomials can be given as*

$$\int_C M_n(z;\alpha,c)M_m(z;\alpha,c)w(z;\alpha,c)dz = \kappa_n\delta_{m,n}, \tag{54}$$

where

$$w(z;\alpha,c) := \Gamma(-z)\Gamma(z+\alpha)(-c)^z,$$

and C is a complex contour from $-\infty i$ to ∞i separating the increasing poles at $z \in \mathbb{N}_0$ from the decreasing poles at $z \in \{-\alpha,-\alpha-1,-\alpha-2,\ldots\}$.

In fact, observe that the case $c > 0$ cannot be considered by an integral of the form (54) since it diverges. However, when $|c| < 1$, (54) is rewritten on the form (see [21, Section 5.6] for details) presented in (53). With this result in mind, the following result and corresponding consequences hold.

THEOREM 14. *Let $t \in \mathbb{C}$, $\alpha,\beta \in \widehat{\mathbb{C}}$, $c \in \mathbb{C}\setminus[0,\infty)$. Then*

$$\begin{aligned}&\int_C {}_1F_1\left(\begin{matrix}-z\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)M_n(z;\beta,c)\Gamma(-z)\Gamma(z+\alpha)(-c)^z\,dz\\ &= \frac{t^n e^{-t}}{(1-c)^\beta(\alpha)_n c^n}\,{}_1F_1\left(\begin{matrix}\alpha-\beta\\ \alpha+n\end{matrix};t\right).\end{aligned} \tag{55}$$

Proof. From (35) we multiply both sides by $M_m(z;\beta,c)w(z;\beta,c)$, utilizing the orthogonality relation (54), produces the desired result. □

COROLLARY 1. *Let $t \in \mathbb{C}$, $\alpha,\beta > 0$, $c \in (0,1)$. Then*

$$\sum_{x=0}^{\infty} {}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)M_n(x;\beta,c)\frac{(\beta)_x c^x}{x!} = \frac{t^n e^{-t}}{(1-c)^\beta(\alpha)_n c^n}\,{}_1F_1\left(\begin{matrix}\alpha-\beta\\ \alpha+n\end{matrix};t\right). \tag{56}$$

COROLLARY 2. *Let $\alpha,\beta \in \widehat{\mathbb{C}}$, $\gamma \in \mathbb{C}$, $c,d \in \mathbb{C}\setminus[0,\infty)$, $t \in \mathbb{C}$, $|t| < \min\{1,|c(1-d)|/|1+d-2c|\}$. Then*

$$\begin{aligned}&\int_C {}_1F_1\left(\begin{matrix}-z\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)M_n(z;\beta,d)\Gamma(-z)\Gamma(z+\beta)(-d)^z\,dz\\ &= \frac{t^n(1-c)^n e^{-t}}{(1-d)^{n+\beta}(\alpha)_n c^n}\,{}_1F_1\left(\begin{matrix}\beta+n\\ \alpha+n\end{matrix};\frac{-dt(1-c)}{c(1-d)}\right).\end{aligned} \tag{57}$$

Proof. From (32) we multiply both sides by $M_m(z;\beta,c)w(z;\beta,c)$, utilizing the orthogonality relation (54). $\square$

COROLLARY 3. *Let* $t\in\mathbb{C}$, $\alpha,\beta>0$, $c,d\in(0,1)$. *Then*

$$\begin{aligned}&\sum_{x=0}^{\infty}{}_1F_1\left(\begin{matrix}-x\\ \alpha\end{matrix};\frac{t(1-c)}{c}\right)M_n(x;\beta,d)\frac{d^x(\beta)_x}{x!}\\ &=\frac{t^n(1-c)^n e^{-t}}{c^n(1-d)^{n+\beta}(\alpha)_n}\,{}_1F_1\left(\begin{matrix}\beta+n\\ \alpha+n\end{matrix};\frac{-dt(1-c)}{c(1-d)}\right).\end{aligned}\tag{58}$$

COROLLARY 4. *Let* $c\in\mathbb{C}\setminus[0,\infty)$, $t\in\mathbb{C}$, $|t|<1$, $|t(1-c)|<|c(1-t)|$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $\gamma\in\mathbb{C}$. *Then*

$$\begin{aligned}&\int_C{}_2F_1\left(\begin{matrix}\gamma,-z\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)M_n(z;\beta,c)\Gamma(z+\beta)(-c)^z\,dz\\ &=\frac{(1-t)^\gamma(\gamma)_n t^n}{(1-c)^\beta(\alpha)_n c^n}\,{}_2F_1\left(\begin{matrix}\alpha-\beta,\gamma+n\\ \alpha+n\end{matrix};t\right).\end{aligned}\tag{59}$$

Proof. From (40) we multiply both sides by $M_m(z;\beta,c)w(z;\beta,c)$, utilizing the orthogonality relation (54). $\square$

COROLLARY 5. *Let* $c\in(0,1)$, $t\in\mathbb{C}$, $|t|<1$, $|t(1-c)|<|c(1-t)|$, $\alpha,\beta>0$, $\gamma\in\mathbb{C}$. *Then*

$$\sum_{x=0}^{\infty}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)M_n(x;\beta,c)\frac{c^x(\beta)_x}{x!}=\frac{(1-t)^\gamma(\gamma)_n t^n}{(1-c)^\beta(\alpha)_n c^n}\,{}_2F_1\left(\begin{matrix}\alpha-\beta,\gamma+n\\ \alpha+n\end{matrix};t\right).\tag{60}$$

COROLLARY 6. *Let* $t\in\mathbb{C}$, $|t|<\min\{1,|c(1-d)|/|1+d-2c|\}$, $\alpha,\beta\in\widehat{\mathbb{C}}$, $\gamma\in\mathbb{C}$, $c,d\in\mathbb{C}\setminus[0,\infty)$. *Then*

$$\begin{aligned}&\int_C{}_2F_1\left(\begin{matrix}\gamma,-z\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)M_n(z;\beta,d)\Gamma(z+\beta)(-d)^z\,dz\\ &=\frac{(\gamma)_n}{(1-d)^{n+\beta}(\alpha)_n}\left(\frac{t(1-c)}{c(1-t)}\right)^n{}_2F_1\left(\begin{matrix}\gamma+n,\beta+n\\ \alpha+n\end{matrix};\frac{-dt(1-c)}{c(1-d)(1-t)}\right).\end{aligned}\tag{61}$$

Proof. From (41) we multiply both sides by $M_m(z;\beta,c)w(z;\beta,c)$, utilizing the orthogonality relation (53), produces the desired result. $\square$

COROLLARY 7. *Let* $c,d\in(0,1)$, $t\in\mathbb{C}$, $|t|<\min\{1,|cd|/|c+d|\}$, $\alpha,\beta>0$, $\gamma\in\mathbb{C}$. *Then*

$$\begin{aligned}&\sum_{x=0}^{\infty}{}_2F_1\left(\begin{matrix}\gamma,-x\\ \alpha\end{matrix};\frac{t(1-c)}{c(1-t)}\right)M_n(x;\beta,d)\frac{d^x(\beta)_x}{x!}\\ &=\frac{(\gamma)_n}{(1-d)^{n+\beta}(\alpha)_n}\left(\frac{t(1-c)}{c(1-t)}\right)^n{}_2F_1\left(\begin{matrix}\gamma+n,\beta+n\\ \alpha+n\end{matrix};\frac{-dt(1-c)}{c(1-d)(1-t)}\right).\end{aligned}\tag{62}$$

On the other hand, since the Krawtchouk polynomials satisfy the property of orthogonality

$$\sum_{x=0}^{N}\binom{N}{x}p^x(1-p)^{N-x}K_m(x;p,N)K_n(x;p,N)=\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{m,n},$$

the following identities follow with proofs given as above, which we omit.

COROLLARY 8. *Let* $p,q\in\mathbb{C}_0$, $M,N\in\mathbb{N}_0$, $N\leqslant M$, $t\in\mathbb{C}$. *Then*

$$\begin{aligned}&\sum_{x=0}^{M}\binom{M}{x}q^x(1-q)^{M-x}\left[e^t\,{}_1F_1\left(\begin{matrix}-x\\-N\end{matrix};-\frac{t}{p}\right)\right]_N K_n(x;q,M)\\&=\left(\frac{t(q-1)}{p}\right)^n\frac{1}{(-N)_n}\left[e^t\,{}_1F_1\left(\begin{matrix}n-M\\n-N\end{matrix};\frac{-tq}{p}\right)\right]_{N-n}.\end{aligned}$$

COROLLARY 9. *Let* $\gamma\in\mathbb{C}$, $p,q,\in\mathbb{C}_0$, $M,N\in\mathbb{N}_0$, $N\leqslant M$, $t\in\mathbb{C}$, $|t|<1$. *Then*

$$\begin{aligned}&\sum_{x=0}^{M}\binom{M}{x}q^x(1-q)^{M-x}\left[(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma,-x\\-N\end{matrix};\frac{t}{p(t-1)}\right)\right]_N K_n(x;q,M)\\&=\frac{(\gamma)_n}{(-N)_n}\left(\frac{t(q-1)}{p}\right)^n\left[(1-t)^{-\gamma-n}{}_2F_1\left(\begin{matrix}\gamma+n,n-M\\n-N\end{matrix};\frac{-qt}{p(1-t)}\right)\right]_{N-n}.\end{aligned}$$

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