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FURTHER RESULTS ON VALUE DISTRIBUTION OF L–FUNCTIONS

HARINA P. WAGHAMORE AND S. H. NAVEENKUMAR

Abstract. With the aid of weighted sharing we study the uniqueness of meromorphic functions concerning nonlinear differential polynomials that share a nonzero polynomial with the same of L-functions. Our results in the paper will improve, extend a results due to Fang Liu, Xiao-Min Li and Hong-Xun Yi [5].

1. Introduction

L-functions, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [25]. Value distribution of L-functions concerns distribution of the zeros of L-functions L and, more generally, the c-points of L, i.e., the roots of the equation $L(s) = c$, or the values in the preimage $L^{-1} = \{s \in \mathbb{C} : L(s) = c\}$, where and throughout the paper, s denotes the complex variable in the complex plane $\mathbb{C}$ and c denotes a complex value. L-functions can be analytically continued as meromorphic functions in $\mathbb{C}$.

It is well-known that a nonconstant meromorphic function in $\mathbb{C}$ is completely determined by five such preimages (cf. [8, 22, 28, 30], which is a famous theorem due to Nevanlinna and often referred to as Nevanlinna's uniqueness or unicity theorem. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. In terms of sharing values, two nonconstant meromorphic functions in $\mathbb{C}$ must be identically equal if they share five values IM, and one must be a Möbius transform of the other if they share four values CM.

Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, satisfying the following axioms (cf. [24, 25]): (i) Ramanujan hypothesis. $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$ (ii) Analytic continuation. There is a non-negative integer k such that $(s-1)^k L(s)$ is an entire function of finite order. (iii) Functional equation. L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

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where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geqslant 0$ and $|\omega| = 1$. (iv) Euler product hypothesis. $L(s) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

In this paper, a meromorphic function always mean a function which is meromorphic in the whole complex plane $\mathbb{C}$. We denote by $N_{k)}(r, \frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity $\leqslant k$, and by $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted.

Let z_0 be a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q \geqslant 1$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geqslant 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g), N_E^{1)}(r,a;g), \overline{N}_E^{(2}(r,a;g)$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share the value a with weight k then z_0 is an a-point of f with multiplicity $m(\leqslant k)$ if and only if it is an a-point of g with multiplicity $m(\leqslant k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0 \leqslant p < k$. Also we note that f,g share a value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively. We denote $\rho(f)$ for order of $f(z)$.

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

We first recall the following result due to Steuding [25], which actually holds without the Euler product hypothesis:

THEOREM A. ([25], p. 152) *If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.*

Later on, Li [17] proved the following result to deal with a question posed by Chung-Chun Yang (cf. [17]):

THEOREM B. ([17]) *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a nonconstant L-function L share a CM and b IM, then $L = f$.*

In 1997, Lahiri [12] posed the following question:

What can be said about the relationship between two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? In this direction, Fang [4] and Yang-Hua [28] respectively proved the following results:

THEOREM C. ([4]) *Let f and g be two nonconstant entire functions, and let n and k be two positive integers such that $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants, satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.*

THEOREM D. ([28]) *Let f and g be two nonconstant meromorphic functions, and let $n \geqslant 11$ be a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants, satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2017, Fang LIU, Xiao-Min LI and Hong-Xun YI proved the following results.

THEOREM E. ([5]) *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

THEOREM F. ([5]) *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

Now it is natural to ask the following question which is the motivation of the paper.

QUESTION. *Can a CM shared value be replaced by $(p(z), l)$ in Theorems E and F?*

In the paper, our main concern is to find the possible answer of the above question. The following are the main results of the paper.

THEOREM 1. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function and let n and k be two positive integers. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share $(p(z), l)$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$ and f and L share $(\infty, 0)$. Suppose one of the following conditions hold:*

a. $l \geqslant 3$ *and* $n > 3k+4$;

b. $l = 2$ *and* $n > 3k+6$;

c. $l = 1$ *and* $n > 3k+7$;

d. $l = 0$ *and* $n > 7k+11$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

THEOREM 2. *Let* f *be a nonconstant entire function in* $\mathbb{C}$, *let* L *be an nonconstant* L*-function, and let* n *and* k *be two positive integers. If* $(f^n)^{(k)}$ *and* $(L^n)^{(k)}$ *share* $(p(z), l)$, *where* $p(z)$ *be a nonzero polynomial with deg* $(p) = m$ *and* f *and* L *share* $(\infty, 0)$. *Suppose one of the following conditions hold:*

I. $l \geqslant 2$ *and* $n > 2k+4$;

II. $l = 1$ *and* $n > \frac{5k+9}{2}$;

III. $l = 0$ *and* $n > 5k+7$.

Then $f = tL$ *for a constant* t *satisfying* $t^n = 1$.

2. Some lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{2}$$

LEMMA 1. [27] *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, f)$ *for* $i = 0, 1, 2, \ldots, n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

LEMMA 2. [29] *Suppose that* f *is a nonconstant meromorphic function in the complex plane and* k *is a positive integer. Then*

$$N(r, 0; f^{(k)}) \leqslant N(r, 0; f) + k\overline{N}(r, \infty, f) + O(\log(T(r, f)) + \log r),$$

as $r \to \infty$, *outside of a possible exceptional set of finite linear measure.*

LEMMA 3. [33] *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{3}$$

$$N_p(r, 0; f^{(k)}) \leqslant k\overline{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f). \tag{4}$$

LEMMA 4. [14] *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leqslant k\overline{N}(r,\infty,f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geqslant k)+S(r,f).$$

LEMMA 5. *Suppose that f and g be two non-constant meromorphic functions. Let $F=[f^n]^{(k)}$, $G=[g^n]^{(k)}$, where n,k are positive integers. If f,g share ∞ IM and $V\equiv 0$, then $F\equiv G$.*

Proof. Suppose $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}\equiv A\Big(1-\frac{1}{G}\Big).$$

If z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_0)}=0$ and $\frac{1}{G(z_0)}=0$. So $A=1$ and hence $F\equiv G$. $\square$

LEMMA 6. [16] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$ for $i=1,2$. If $f_1^sf_2^t-1$ is not identically zero for arbitrary integers s and $t(|s|+|t|>0)$, then for any positive ε, we have*

$$N_0(r,1;f_1,f_2)\leqslant \varepsilon T(r)+S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function related to the common 1-points of f_1 and f_2 and $T(r)=T(r,f_1)+T(r,f_2)$, $S(r;f_1,f_2)=o(T(r))$ as $r\to\infty$ possibly outside a set of finite linear measure

LEMMA 7. [2] *Let f and g be two non-constant meromorphic functions sharing $(1,k_1)$, where $2\leqslant k_1\leqslant\infty$. Then*

$$\begin{aligned}\overline{N}(r,1;f|=2)+2\overline{N}(r,1;f|=3)+\ldots+(k_1-1)\overline{N}(r,1;f|=k_1)+k_1\overline{N}_L(r,1;f)\\ +(k_1+1)\overline{N}_L(r,1;g)+k_1\overline{N}_E^{(k_1+1}(r,1;g)\leqslant N(r,1;g)-\overline{N}(r,1;g).\end{aligned}\tag{5}$$

LEMMA 8. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)\\ +\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F)+S(r,G).\end{aligned}$$

LEMMA 9. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\ +\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G).\end{aligned}$$

LEMMA 10. [11] *Let f and g be two nonconstant meromorphic functions. If $(f^n)^{(k)} = (g^n)^{(k)}$ and $n > k+1$, then $f = tg$ for a constant t such that $t^n = 1$.*

LEMMA 11. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function, and let n and k be two positive integers with $H \not\equiv 0$. If $F = (f^n)^{(k)}$ and $G = (L^n)^{(k)}$ share $(1,k_1)$, and f and L share $(\infty,0)$, then*

$$(n-1)\overline{N}(r,\infty;f) \leqslant (k+1)\{T(r,f)+T(r,L)\} + \overline{N}_*(r,1;F,G) + O(\log r).$$

Proof. Suppose ∞ is an e.v.P. of f and L then the lemma follows immediately.

Next suppose ∞ is not an e.v.P. of f and L. Since $H \not\equiv 0$ from Lemma 5 we have $V \not\equiv 0$. We suppose that z_0 is a pole of f with multiplicity q and a pole of L with multiplicity r. Clearly z_0 is a pole of F with multiplicity $nq+k$ and a pole of G with multiplicity $nr+k$. Noting that f,L share $(\infty,0)$ from the definition of V it is clear that z_0 is a zero of V with multiplicity atleast $n+k-1$. Now using the Milloux theorem [[8], p. 55] and Lemma 1, we obtain from the definition of V that

$$m(r,V) = O(\log r).$$

Then by using Valiron-Mokhonko lemma (cf. [22]) and (4) we get

$$\begin{aligned}
&(n+k-1)\overline{N}(r,\infty;f)\\
&\leqslant N(r,0;V)\\
&\leqslant T(r,V)+O(1)\\
&\leqslant N(r,\infty;V)+m(r,0;V)+O(1)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant N_{k+1}(r,0;f^n)+k\overline{N}(r,\infty;f)+N_{k+1}(r,0;L^n)+k\overline{N}(r,\infty;L)+\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant (k+1)\overline{N}(r,0;f)+(k+1)\overline{N}(r,0;L)+k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+O(\log r).
\end{aligned} \tag{6}$$

This gives

$$(n-1)\overline{N}(r,\infty;f) \leqslant (k+1)\{T(r,f)+T(r,L)\} + \overline{N}_*(r,1;F,G) + O(\log r). \tag{7}$$

This completes the proof of the Lemma. □

LEMMA 12. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, let L be an nonconstant L-function and $F = \frac{(f^n)^{(k)}}{p(z)}$, $G = \frac{(L^n)^{(k)}}{p(z)}$, where $p(z)$ be a nonzero polynomial with deg $(p) = m$, n and k be two positive integers such that $n > 3k+2$. If f and L share $(\infty,0)$ and $H \equiv 0$ then either $[f^n]^{(k)}[L^n]^{(k)} \equiv p^2$ or $f^n \equiv L^n$.*

Proof. Since $H \equiv 0$, on integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{8}$$

where a,b are constants and $a \neq 0$. From (8) it is clear that F and G share $(1,\infty)$.

We now discuss the following three cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b=-1$, then from (8) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore,

$$\overline{N}(r,a+1;G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + O(\log r).$$

So in view of Lemma 3 and the second fundamental theorem we get

$$\begin{aligned}
nT(r,L) &\leqslant T(r,G) + N_{k+1}(r.0;L^n) - \overline{N}(r,0;G) + O(\log r)\\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}(r.0;L^n) - \overline{N}(r,0;G) + O(\log r)\\
&\leqslant \overline{N}(r,\infty;L) + N_{k+1}(r.0;L^n) + \overline{N}(r,\infty;f) + O(\log r)\\
&\leqslant (k+1)\overline{N}(r,0;L) + T(r,f) + O(\log r)\\
&\leqslant (k+1)T(r,L) + T(r,f) + O(\log r)\\
&\leqslant (k+2)T(r,L) + O(\log r).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,L)$ for $r \in I$. So for $r \in I$, we get a contradiction from above since $n > 3k+2$.

If $b \neq -1$, from (8) we obtain that

$$F - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So,

$$\overline{N}\left(r, \frac{(b-a)}{b}; G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + O(\log r).$$

Using Lemma 3 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$.

If $b=-1$, then from (8) we have

$$FG \equiv 1,$$

that is

$$[f^n]^{(k)}\,[L^n]^{(k)} \equiv p^2.$$

If $b \neq -1$, from (8) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+b}; G\right) = \overline{N}(r,0;F).$$

So in view of Lemma 3 and the second fundamental theorem we get

$$\begin{aligned}
nT(r,L) &\leqslant T(r,G)+N_{k+1}(r.0;L^n)-\overline{N}(r,0;G)+O(\log r)\\
&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\Big(r,\frac{1}{1+b};G\Big)\\
&\quad +N_{k+1}(r.0;L^n)-\overline{N}(r,0;G)+O(\log r)\\
&\leqslant \overline{N}(r,\infty;L)+(k+1)\overline{N}(r.0;L)+\overline{N}(r,0;F)+O(\log r)\\
&\leqslant (k+1)\overline{N}(r,0;L)+(k+1)\overline{N}(r.0;f)+k\overline{N}(r,\infty;f)+O(\log r)\\
&\leqslant (k+1)T(r,L)+(2k+1)T(r,f)+O(\log r).
\end{aligned}$$

So for $r\in I$ we have

$$nT(r,L)\leqslant (3k+2)T(r,L)+O(\log r),$$

which is a contradiction since $n>3k+2$.

Case 3. Let $b=0$. From (8) we obtain

$$F\equiv \frac{G+a-1}{a}. \tag{9}$$

If $a\neq 1$ then from (9) we obtain

$$\overline{N}(r,1-a;G)=\overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a=1$ and from (9) we obtain

$$F\equiv G.$$

Then by the Lemma 10 we have

$$f^n\equiv L^n. \quad \square$$

3. Proof of Theorem 1

Proof. Suppose that d is the degree of L. Then $d=2\sum_{i=1}^{k}\lambda_j$, where k and λ_j are respectively the positive integer and the positive real number in the axiom (iii) of the definition of L- function.

Then we have that

$$T(r,L)=\frac{d}{\Pi}r\log r+O(r) \tag{10}$$

(cf. [[25], p. 150]). Clearly, f and L are transcendental meromorphic functions (cf. [[5], p. 43]). Note that an L- function at most has one pole $z=1$ in the complex plane.

Let $F=\frac{(f^n)^{(k)}}{p(z)}$ and $G=\frac{(L^n)^{(k)}}{p(z)}$. It follows that F and G share $(1,l)$ except the zeros of $p(z)$ and f,g share $(\infty,0)$.

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. $l \geqslant 1$.

From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G with different multiplicities, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned} N(r,\infty;H) \leqslant & \overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2) \\ & +\overline{N}_0\left(r,0;F'\right)+\overline{N}_0\left(r,0;G'\right), \end{aligned} \tag{11}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $a(z_0) \neq 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1) \leqslant N(r,0;H) \leqslant N(r,\infty;H)+O(\log r). \tag{12}$$

While $l \geqslant 3$, using (11) and (12) we get

$$\begin{aligned} \overline{N}(r,1;F) & \leqslant N(r,1;F|=1)+\overline{N}(r,1;F|\geqslant 2) \\ & \leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}_*(r,1;F,G) \\ & \quad +\overline{N}(r,1;F|\geqslant 2)+\overline{N}_0\left(r,0;F'\right)+\overline{N}_0\left(r,0;G'\right)+O(\log r). \end{aligned} \tag{13}$$

Now in view of Lemmas 4 and 7 we get

$$\begin{aligned} & \overline{N}_0\left(r,0;G'\right)+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_*(r,1;F,G) \\ & \leqslant \overline{N}_0\left(r,0;G'\right)+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3)+\ldots+\overline{N}(r,1;F|=l) \\ & \quad +\overline{N}_E^{(l+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G) \\ & \leqslant \overline{N}_0\left(r,0;G'\right)-\overline{N}(r,1;F|=3)-\ldots-(l-2)\overline{N}(r,1;F|=l)-(l-1)\overline{N}_L(r,1;F) \\ & \quad -l\overline{N}_L(r,1;G)-(l-1)\overline{N}_E^{(l+1}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G) \\ & \leqslant \overline{N}_0\left(r,0;G'\right)+N(r,1;G)-\overline{N}(r,1;G)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G) \\ & \leqslant N(r,0;G^{'}|G\neq 0)-(l-2)\overline{N}_L(r,1;F)-(l-1)\overline{N}_L(r,1;G) \\ & \leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;G)-(l-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G) \\ & \leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;G)-\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G). \end{aligned} \tag{14}$$

Hence using (13), (14), Lemmas 3 and 11 we get from the second fundamental theorem

that

$$
\begin{aligned}
&nT(r,f)\\
&\leqslant T(r,F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)+O(\log r)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)-\overline{N}_0\left(r,0;F'\right)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+\overline{N}(r,0;F|\geqslant 2)\\
&\quad+\overline{N}(r,0;G|\geqslant 2)+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0\left(r,0;G'\right)-N_2(r,0;F)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;L)+N_{k+2}(r,0;f^n)+N_2(r,0;G)-\overline{N}_*(r,1;F,G)\\
&\quad-\overline{N}_L(r,1;G)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right)-\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant 2\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,0;f)+(k+2)\overline{N}(r,0;L)-\overline{N}_*(r,1;F,G)+O(\log r)\\
&\leqslant (k+2)\{T(r,f)+T(r,L)\}+\frac{2(k+1)}{(n-1)}\{T(r,f)+T(r,L)\}+O(\log r)\\
&\leqslant \left[k+2+\frac{2(k+1)}{n-1}\right]\{T(r,f)+T(r,L)\}+O(\log r).
\end{aligned}
\tag{15}
$$

In a similar way we can obtain

$$
nT(r,L)\leqslant\left[k+2+\frac{(k+1)^2}{n-1}\right]\{T(r,f)+T(r,L)\}+O(\log r). \tag{16}
$$

Adding (15) and (16), we get

$$
\left[n-2k-4-\frac{(k+3)(k+1)}{n-1}\right]\{T(r,f)+T(r,L)\}\leqslant O(\log r).
$$

Since the quantity in the third bracket can be written as

$$
\left[\frac{(n-1)^2-(2k+3)(n-1)-(k+3)(k+1)}{n-1}\right], \tag{17}
$$

by a simple computation one can easily verify that when

$$
n-1>3k+3>\frac{2k+3+\sqrt{(2k+3)^2+4(k+3)(k+1)}}{2}
$$

i.e., when $n>3k+4$ we get a contradiction from (17).

While $l\geqslant 2$, like (13), (14) and not using Lemma 11 in (15) we can deduce a contradiction when $n>3k+6$. So we omit the detail.

While $l=1$. From (3) we obtain

$$\begin{aligned}N_2(r,0;F) &\leqslant N_2(r,0;[f^n]^{(k)})+S(r,f)\\ &\leqslant T(r,[f^n]^{(k)})-nT(r,f)+N_{k+2}(r,0;f^n)+S(r,f)\\ &\leqslant T(r,F)-nT(r,f)+N_{k+2}(r,0;f^n)+O\{\log r\}+S(r,f).\end{aligned}$$

Which implies

$$nT(r,f)\leqslant T(r,F)+N_{k+2}(r,0;f^n)-N_2(r,0;F)+O\{\log r\}+S(r,f). \tag{18}$$

Using (18) and Lemma 8, we get

$$\begin{aligned}nT(r,f) &\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right)+O(\log r)\\ &\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\left\{N_{k+1}(r,0;f^n)+k\overline{N}(r,\infty;f)\right\}+N_{k+2}(r,0;L^n)\\ &\quad+(k+2)\overline{N}(r,0;f)+O(\log r)\\ &\leqslant \frac{5+k}{2}\overline{N}(r,\infty;f)+\frac{3k+5}{2}\overline{N}(r,0;f)+(k+2)\overline{N}(r,0;L)+O(\log r)\\ &\leqslant (2k+5)T(r,f)+(k+2)T(r,L)+O(\log r)\\ &\leqslant (3k+7)T(r)+O(\log r),\end{aligned} \tag{19}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$.

In a similar way we can obtain

$$nT(r,L)\leqslant (3k+7)T(r)+O(\log r). \tag{20}$$

Combining (19) and (20) we see that

$$nT(r)\leqslant (3k+7)T(r)+O(\log r), \tag{21}$$

i.e.,

$$(n-3k-7)T(r)\leqslant O(\log r). \tag{22}$$

Since $n>3k+7$, (22) leads to a contradiction.

Subcase 1.2. $l=0$. Using (18) and Lemma 9, we get

$$\begin{aligned}nT(r,f) &\leqslant 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;L)+2\overline{N}(r,0;F)+N_{k+2}(r,0;f^n)+N_2\left(r,0;(L^n)^{(k)}\right)\\ &\quad+\overline{N}\left(r,0;(L^n)^{(k)}\right)+O(\log r)\\ &\leqslant 4\overline{N}(r,\infty;f)+2N_{k+1}(r,0;f^n)+2k\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,0;f)\\ &\quad+N_{k+2}(r,0;L^n)+k\overline{N}(r,\infty;L)+N_{k+1}(r,0;L^n)+k\overline{N}(r,\infty;L)+O(\log r)\\ &\leqslant (2k+4)\overline{N}(r,\infty;f)+(3k+4)\overline{N}(r,0;f)+(2k+3)\overline{N}(r,0;L)+O(\log r)\\ &\leqslant (5k+8)T(r,f)+(2k+3)T(r,L)+O(\log r)\\ &\leqslant (7k+11)T(r)+O(\log r),\end{aligned} \tag{23}$$

where $T(r) = \max\{T(r,f), T(r,g)\}$.

In a similar way we can obtain

$$nT(r,L) \leqslant (7k+11)T(r) + O(\log r). \tag{24}$$

Combining (23) and (24) we see that

$$nT(r) \leqslant (7k+11)T(r) + O(\log r), \tag{25}$$

i.e.,

$$(n-7k-11)T(r) \leqslant O(\log r). \tag{26}$$

Since $n > 7k+11$, (26) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then by Lemma 12, we obtain either

$$(f^n)^{(k)}(L^n)^{(k)} \equiv p^2$$

or

$$f^n \equiv L^n.$$

We consider the following two cases:

Case 1. Suppose that $(f^n)^{(k)}(L^n)^{(k)} \equiv p^2$.

Then,

$$F_1 G_1 \equiv 1 \tag{27}$$

where

$$F_1 = \frac{(f^n)^{(k)}}{p(z)}, \quad G_1 = \frac{(L^n)^{(k)}}{p(z)}. \tag{28}$$

First of all, we prove that 0 is a Picard exceptional value of f and L. Indeed, suppose that $z_0 \notin (z : p(z) = 0)$ is a zero of f with multiplicity m. Then, by the view of (27) we can find that $z_0 = 1$ is a pole of L with multiplicity, say p_1, such that $mn - k = np_1 + k$, and so $(m - p_1)n = 2k$ and so we have $n \leqslant 2k$, which contradicts the assumption $n > 3k+4$. Similarly, we can prove that 0 is a Picard exceptional value of L. On the other hand, by (10) and (27), Valiron-Mokhonko lemma (cf. [22]), a result from Whittaker [[26], p. 82] and the definition of the order of a meromorphic function we have

$$\rho(f) = \rho(f^n) = \rho\left((f^n)^{(k)}\right) = \rho\left((L^n)^{(k)}\right) = \rho(L^n) = \rho(L) = 1. \tag{29}$$

Noting that L has at most one pole $z = 1$ in the complex plane, we have by (27), (29), Lemma 2 that

$$(n+k)\overline{N}(r,\infty;f) \leqslant N\left(r,0;(L^n)^{(k)}\right) \leqslant N(r,0;L^n) + k\overline{N}(r,\infty;L^n) + O(\log r) = O(\log r). \tag{30}$$

Therefore,

$$\overline{N}(r,\infty;f) + \overline{N}(r,\infty;L) \leqslant O(\log r). \tag{31}$$

Now we set

$$f_1 = \frac{(f^n)^{(k)}}{(L^n)^{(k)}}, \quad f_2 = \frac{(f^n)^{(k)} - 1}{(L^n)^{(k)} - 1}. \tag{32}$$

By (32) and the assumption that f and L are transcendental meromorphic functions. We have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by (32) we see that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share ∞ CM. Combining this with (27) we deduce that ∞ is a Picard exceptional value of f and L. Next we suppose that f_1 and f_2 are nonconstant meromorphic functions.

Then, by (27) and (32) we have

$$F_1 = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G_1 = \frac{1-f_2}{f_1 - f_2}. \tag{33}$$

By (33) we can find that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that $S(r) = o(T(r))$ and

$$\begin{aligned} T(r, F_1) &\leqslant 2(T(r, f_1) + T(r, f_2)) + S(r) \\ &\leqslant 8T(r, F_1) + S(r). \end{aligned} \tag{34}$$

These give $S(r, F_1) = S(r; f_1, f_2)$. Also we note that

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2),$$

for $i = 1, 2$.

We note that $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, since otherwise by the second fundamental theorem, F_1 will be a constant.

Also we see that

$$\overline{N}(r, -1; F_1) \leqslant N_0(r, 1; f_1, f_2).$$

Thus we have

$$T(r, f_1) + T(r, f_2) \leqslant 4N_0(r, 1; f_1, f_2) + S(r, F_1).$$

Then, by Lemma 6 there exists two mutually prime integers s and $t(|s| + |t| > 0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1 - 1}{G_1 - 1}\right]^t \equiv 1. \tag{35}$$

If either s or t is zero then we arrive at contradiction and so $st \neq 0$.

By (35) we consider the following two subcases:

Subcase 1.1. Suppose that $st < 0$, say $s > 0$ and $t < 0$, say $t = -t_1$, where t_1 is some Positive integer. Then, (35) can be written as

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{t_1}. \tag{36}$$

Let z_1 be a pole of F_1 of multiplicity p_1. Then from (36) we see that z_1 must be a zero of G_1 of multiplicity p_1. Now from (36) we get $2s = t_1$, which is impossible. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Subcase 1.2. Suppose that $st < 0$ or $st > 0$. Then by (35) we can see that F_1 and G_1 share ∞ CM. This together with (27) and (28) implies that ∞ is a Picard Exceptional value of f and L. Combining this with the obtained result that 0 is a Picard Exceptional value of f and L, we have

$$L(z) = e^{A_2 z + B_2}, \tag{37}$$

where $A_2 \neq 0$ and B_2 are constants. By (37) and Hayman [[8], p. 7] we have

$$T(r,L) = T(r, e^{A_2 z + B_2}) = \frac{|A_2| r}{\pi}(1 + o(1)). \tag{38}$$

Which contradicts (10).

Case 2. Suppose that $f^n = L^n$. Then, we have $f = tL$, where t is a constant satisfying $t^n = 1$.

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Proof. Noting that $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;L) = 0$, and proceeding in the like manner as the proof of Theorem 1 we obtain the proof of the Theorem 2. □

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Harina P. Waghamore
Department of Mathematics
Jnanabharathi Campus, Bangalore University
Bangalore-560 001, India
e-mail: `harinapw@gmail.com`

S. H. Naveenkumar
Department of Mathematics
Jnanabharathi Campus, Bangalore University
Bangalore-560 001, India
e-mail: `naveenkumarsh.220@gmail.com`