Journal of Classical Analysis
Volume 14, Number 2 (2019), 121–135 doi:10.7153/jca-2019-14-10

ON GENERALIZED GEOMETRIC DIFFERENCE OF SIX DIMENSIONAL ROUGH IDEAL CONVERGENT OF TRIPLE SEQUENCE DEFINED BY MUSIELAK–ORLICZ FUNCTION

A. ESI AND N. SUBRAMANIAN*

Abstract. We introduce a rough ideal convergent of triple sequence spaces defined by Musielak-Orlicz function, using an six dimensional infinite matrix, and a generalized geometric difference Zweier six dimensional matrix operator $B^p_{(abc)}$ of order p. We obtain some topological and algebraic properties of these spaces.

1. Introduction

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and let us denote the set $\{(m,n,k)\in K : m\leqslant u, n\leqslant v, k\leqslant w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{u,v,w\to\infty}\frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1-\delta(K)$ where $K^c = \mathbb{N}\backslash K$ is the complement of K. If $K_1\subseteq K_2$, then $\delta(K_1)\leqslant\delta(K_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x=(x_{mnk})$ such that $x_{mnk}\in\mathbb{R}, m,n,k\in\mathbb{N}$.

A triple sequence $x=(x_{mnk})$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st-lim\, x = 0$, provided that the set

$$\left\{(m,n,k)\in\mathbb{N}^3 : |x_{mnk},0|\geqslant\varepsilon\right\}$$

has natural density zero for any $\varepsilon>0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\varepsilon>0$, infinitely many terms of the sequence may remain outside the $\varepsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from

Mathematics subject classification (2010): 40F05, 40J05,40G05.
Keywords and phrases: Triple sequences, Wijsman rough convergence, strongly admissible ideal, cluster points, six dimensional matrix, geometric difference.
* Corresponding author.

© ELEMENT, Zagreb
Paper JCA-14-10

ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a subsequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \varepsilon\right\}$$

is different from zero for every $\varepsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geqslant M\right\}\right) = 0.$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [12], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [11] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X,ρ) be a metric space. For any non empty closed subsets A for which $A_{mnk} \subset X\left(m,n,k \in \mathbb{N}^3\right)$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A is the triple sequence $(d(x,A_{mnk}))$ is statistically convergent to $d(x,A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$lim_{r,s,t} \frac{1}{rst} \left|\{m \leqslant r, n \leqslant s, k \leqslant t : |d(x,A_{mnk}) - d(x,A)| \geqslant \varepsilon\}\right| = 0.$$

In this case, we write $St - lim_{m,n,k} A_{mnk} = A$ or $A_{mnk} \longrightarrow A\,(WS)$. The triple sequence (A_{mnk}) is bounded if $sup_{mnk} d(x,A_{mnk}) < \infty$ for each $x \in X$.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [13,14]*, *Esi et al. [3]*, *Dutta et al.*

[4],*Subramanian et al. [15]*, *Debnath et al. [5]*, *Aiyub et al. [2]* and Zweier sequence was introduced and investigated at the initial by *Fadile Karababa et al. [6]*, *Khan et al. [8]* many others.

Throughout the paper let β be a nonnegative real number.

A triple sequence $x=(x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k}\left|x_{mnk}\right|^{\frac{1}{m+n+k}}<\infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x=(x_{mnk})$ is called triple gai sequence if

$$((m+n+k)!\left|x_{mnk}\right|)^{\frac{1}{m+n+k}}\to 0 \quad \text{as} \quad m,n,k\to\infty.$$

The difference triple sequence space was introduced by Debnath et al. (see [5]) and is defined as

$$\begin{aligned}\Delta x_{mnk} =& x_{mnk}-x_{m,n+1,k}-x_{m,n,k+1}+x_{m,n+1,k+1}-x_{m+1,n,k}+x_{m+1,n+1,k}\\ &+x_{m+1,n,k+1}-x_{m+1,n+1,k+1},\\ \Delta^0 x_{mnk} =& \langle x_{mnk}\rangle .\end{aligned}$$

The generalized difference triple notion has the following binomial representation

$$B^p_{(abc)}=\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{k}\binom{m}{r}\binom{n}{s}\binom{k}{t}\mu^{(m-r)+(n-s)+(k-t)}\eta^{r+s+t}x_{(m-ar)(n-bs)(k-ct)}.$$

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A=\left(a^{ij\ell}_{mnk}\right),(m,n,k=1,2,3,\cdots.)$ be an six dimensional infinite matrix of complex numbers. We write $Ax=(A(x))$ if

$$A(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a^{ij\ell}_{mnk}x_{mnk} \tag{1}$$

converges. If $x=(x_{mnk})\in X\Rightarrow Ax=(A(x))\in Y$. We say that A defines a matrix transformation from $X\to Y$ and we denote it by $A:X\to Y$.

The triple sequence $y=(y_{mnk})$ which is frequently used as the Z transformation of the triple sequence $x=(x_{mnk})$ i.e., $y_{mnk}=\alpha x_{mnk}+(1-\alpha)x_{m-1,n-1,k-1}$, where $x_{m-1,n-1,k-1}=0;\ m,n,k\neq 0;\ 1<(m,n,k)<\infty$ and Z denotes the matrix $Z=\left(z_{\ell(mnk)}\right)$ defined by

$$z_{\ell(mnk)}=\begin{cases}\alpha, & \text{if } \ell=m=n=k,\\ 1-\alpha, & \text{if } \ell-1=m=n=k,\\ 0, & \text{otherwise.}\end{cases}$$

The Zweier sequence spaces Z as follows: $Z=\left\{x=(x_{mnk})\in w^3 : Z(x)\in X\right\}$.

In the area of non-Newtonian calculus pioneering work was carried out by Grossman and Katz which we call as multiplicative calculus. The operations of multiplicative

calculus are called as multiplicative derivative and multiplicative integral of different types of non-Newtonian calculi and its applications. An extension of multiplicative calculus to functions of complex variables.
Nowadays geometric calculus is an alternative to the usual calculus of Newton and Leibnitz. It provides differentiation and integration tools based on multiplication instead of addition. Almost all properties in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative caluculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, mainly problems of price elasticity, multiplicative growth etc. the use of multiplicative calculus is advocated instead of a traditional Newtonian one. Top know better about non-Newtonian calculus, we must have idea about different types of arithmetics and their generators.

1.1. $\alpha-$ generator and geometric real field

A generator is a one-to-one function whose domain is $\mathbb{R}$ (the set of all real numbers) and range is a set $A \subset \mathbb{R}$. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic, and exponential function generates geometric arithmetic. As a generator, we choose the function α such that whose basic algebraic operations are defined as follows:

(i) $\alpha-$ addition $x+y=\alpha\left[\alpha^{-1}(x)+\alpha^{-1}(y)\right]$,

(ii) $\alpha-$ subtraction $x-y=\alpha\left[\alpha^{-1}(x)-\alpha^{-1}(y)\right]$,

(iii) $\alpha-$ multiplication $x\times y=\alpha\left[\alpha^{-1}(x)\times\alpha^{-1}(y)\right]$,

(iv) $\alpha-$ division $x/y=\alpha\left[\alpha^{-1}(x)/\alpha^{-1}(y)\right]$,

(v) $\alpha-$ order $x<y\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y)$,

for $x,y\in A$, where A is a range of the function α.

If we choose exp as an $\alpha-$ generator defined by $\alpha(z)=Inz$ and $\alpha-$ arithmetic turns out to geometric arithmetic:

(i) $\alpha-$ addition $x\oplus y=\alpha\left[\alpha^{-1}(x)+\alpha^{-1}(y)\right]=e^{[Inx+Iny]}=x.y$, geometric addition;

(ii) $\alpha-$ subtraction $x\ominus y=\alpha\left[\alpha^{-1}(x)-\alpha^{-1}(y)\right]=e^{[Inx-Iny]}=x\div y, y\neq 0$, geometric subtraction;

(iii) $\alpha-$ multiplication $x\odot y=\alpha\left[\alpha^{-1}(x)\times\alpha^{-1}(y)\right]=e^{[Inx\times Iny]}=x^{Iny}$, geometric multiplication;

(iv) $\alpha-$ division $x\oslash y=\alpha\left[\alpha^{-1}(x)/\alpha^{-1}(y)\right]=e^{[Inx\div Iny]}=x^{\frac{1}{Iny}}, y\neq 1$, geometric division.

It is obvious that $In(x) < In(y)$ if $x < y$ for $x, y \in \mathbb{R}^+$. That is, $x < y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$. So, without loss of generality, we use $x < y$ instead of the geometric order $x < y$.

Defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows:

(i) $\mathbb{Z}(G) = \{e^x : x \in \mathbb{Z}\}$,

(ii) $\mathbb{R}(G) = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \setminus \{0\}$,

(iii) $\mathbb{C}(G) = \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}$.

REMARK 1. $(\mathbb{R}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e, since:

(i) $(\mathbb{R}(G), \oplus, \odot)$ is a geometric additive abelian group with geometric zero 1,

(ii) $(\mathbb{R}(G) \setminus 1, \odot)$ is a geometric multiplicative abelian group with geometric identity e,

(iii) $\odot$ is distributive over $\oplus$.

But $(\mathbb{C}(G), \oplus, \odot)$ is not a field, however, geometric binary operation $\odot$ is not associative in $\mathbb{C}(G)$.

1.2. Geometric limit

Geometric limit of a positive valued function defined in a positive interval is same to the ordinary limit. Here, we defined geometric limit of a function with the help of geometric arithmetic as follows:

A function f, which is positive in a given positive interval, is said to tend to the limit $l > 0$ as x tends to $a \in \mathbb{R}$, if, corresponding to any arbitrary chosen number $\varepsilon > 1$ and r be a positive real number, however samml (but greater than 1), there exists a positive number $\delta > 1$, such that

$$1 < |f(x) \ominus l|^G < \varepsilon,$$

for all values of x for which $1 < |x \ominus a|^G < \delta$. We write

$$G - lim_{x \to a} f(x) = l \quad \text{or} \quad f(x) G - l.$$

Here, $|x \ominus a|^G < \delta \Rightarrow \left|\frac{x}{a}\right|^G < \delta \Rightarrow \frac{1}{\delta} < \frac{x}{a} < \delta \Rightarrow \frac{a}{\delta} < x < a\delta$. Similarly, $|f(x) \ominus l|^G < \varepsilon \Rightarrow \frac{l}{\varepsilon} < f(x) < l\varepsilon$. Thus, $f(x) G - l$ means that for any given positive real number $\varepsilon > 1$, no matter however closer to 1, $\exists$ a finite number $\delta > 1$ such that $f(x) \left(\frac{l}{\varepsilon}, l\varepsilon\right)$ for every $x \in \left(\frac{a}{\delta}, a\delta\right)$. It is to be note that lengths of the open intervals $\left(\frac{a}{\delta}, a\delta\right)$ and $\left(\frac{l}{\varepsilon}, l\varepsilon\right)$ decreases as δ and ε respectively decreases to 1, $f(x)$ becomes closer and closer to l, as well as x becomes closer and closer to a as δ decreases to 1. Hence,

l is also the ordinary limit of $f(x)$. i.e. $f(x)\,G-l \Rightarrow f(x) \to l$. In other words, we say that $G-$ limit and ordinary limit are same for bipositive functions whose functional values as well as arguments are positive in the given interval only difference is that in geometric calculus we approach the limit geometrically, but in ordinary calculus we approach the limit linearly.
A function f is said to rough tend to limit l as x tends to a from the left, if for each $\varepsilon > 1$ and r be a positive number (however small), there exists $\delta > 1$ such that $|f(x) \ominus l|^G < r+\varepsilon$ when $\frac{a}{\delta} < x < a$. In symbols, we then write

$$G-lim_{x\to a} f(x) = l \quad \text{or} \quad f(a-1) = l.$$

Similarly, a function f is said to rough tend to limit l as x tends to a from the right, if for each $\varepsilon > 1$ (however small), there exists $\delta > 1$ such that $|f(x) \ominus l|^G < r+\varepsilon$ when $a < x < a\delta$. In symbols, we then write

$$G-lim_{x\to a+} f(x) = l \quad \text{or} \quad f(a+1) = l.$$

If $f(x)$ is negative valued in a given interval, it will be said to rough tend to a limit $l < 0$ if for $\varepsilon > 1, \exists \delta > 1$ such that $f(x) \in \left(l\varepsilon, \frac{l}{\varepsilon}\right)$ whenever $x \in \left(\frac{a}{\delta}, a\delta\right)$.

1.3. Geometric continuity

A function f is said to be geometric continuous at $x = a$ if:

(i) $f(a)$ i.e., the value of $f(x)$ at $x = a$, is a definite number,

(ii) the geometric-limit of the function $f(x)$ as xGa exists and is equal to $f(a)$.

Alternatively, a function f is said to rough Geometric-continuous at $x = a$, if for arbitrarily chosen $\varepsilon > 1$, however small, there exists a number $\delta > 1$ such that

$$lim_{x\to a} \frac{f(x)}{f(a)} = 1.$$

It is easy to prove that

$$w(G) = \{(x_{mnk}) : x_{mnk} \in \mathbb{R}(G) \text{ for all } m,n,k \in \mathbb{N}\}$$

is a vector space over $\mathbb{R}(G)$ with respect to the algebraic operations $\oplus$ addition and $\odot$ multiplication:

$$\oplus : w(G) \times w(G) \to w(G), \quad (x,y) \to x \oplus y = (x_{mnk}) \oplus (y_{mnk}) = (x_{mnk} y_{mnk}),$$
$$\odot : \mathbb{R}(G) \times w(G) \to w(G), \quad (\alpha y) \to \alpha \odot y = \alpha \odot (y_{mnk}) = \left(\alpha^{In y_{mnk}}\right),$$

where $x = (x_{mnk}), y = (y_{mnk}) \in w(G)$. Then the definitions follow.

2. Definitions and preliminaries

DEFINITION 1. An Orlicz function ([see [7]) is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x+y)\leqslant M(x)+M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([9]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g=(g_{mnk})$ defined by

$$g_{mnk}(v)=sup\{|v|u-(f_{mnk})(u):u\geqslant 0\},m,n,k=1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [10]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f=\left\{x\in w^3:I_f(|x_{mnk}|)^{1/m+n+k}\to 0\ \ as\ \ m,n,k\to\infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}f_{mnk}(|x_{mnk}|)^{1/m+n+k},x=(x_{mnk})\in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an extended real number.

DEFINITION 2. A triple sequence $x=(x_{mnk})$ of real numbers is said to be statistically convergent to $l\in\mathbb{R}^3$ if for any $\varepsilon>0$ we have $d(A(\varepsilon))=0$, where

$$A(\varepsilon)=\left\{(m,n,k)\in\mathbb{N}^3:|x_{mnk}-l|\geqslant\varepsilon\right\}.$$

DEFINITION 3. A triple sequence $x=(x_{mnk})$ is said to be statistically convergent to $l\in\mathbb{R}^3$, written as $st-limx=l$, provided that the set

$$\left\{(m,n,k)\in\mathbb{N}^3:|x_{mnk}-l|\geqslant\varepsilon\right\}$$

has natural density zero, for every $\varepsilon>0$.

In this case, l is called the statistical limit of the sequence x.

DEFINITION 4. Let $x=(x_{mnk})_{m,n,k\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a triple sequence in a metric space $(X,|.,.|)$ and r be a non-negative real number. A triple sequence $x=(x_{mnk})$ is said to be $r-$convergent to $l\in X$, denoted by $x\to^r l$, if for any $\varepsilon>0$ there exists $N_\varepsilon\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that for all $m,n,k\geqslant N_\varepsilon$ we have

$$|x_{mnk}-l|<r+\varepsilon.$$

In this case l is called an $r-$ limit of x.

REMARK 2. We consider $r-$ limit set x which is denoted by LIM_x^r and is defined by

$$LIM_x^r = \{l \in X : x \to^r l\}.$$

DEFINITION 5. A triple sequence $x = (x_{mnk})$ is said to be $r-$ convergent if $LIM_x^r \neq \phi$ and r is called a rough convergence degree of x. If $r = 0$ then it is ordinary convergence of triple sequence.

DEFINITION 6. Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be $r-$ statistically convergent to l, denoted by $x \to^{r-st_3} l$, if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geqslant r + \varepsilon\right\}.$$

In this case l is called $r-$ statistical limit of x. If $r = 0$ then it is ordinary statistical convergent of triple sequence.

DEFINITION 7. A class I of subsets of a nonempty set X is said to be an ideal in X provided:

(i) $\phi \in I$;

(ii) $A, B \in I$ implies $A \cup B \in I$;

(iii) $A \in I, B \subset A$ implies $B \in I$.

I is called a nontrivial ideal if $X \notin I$.

DEFINITION 8. A nonempty class F of subsets of a nonempty set X is said to be a filter in X. Provided:

(i) $\phi \in F$;

(ii) $A, B \in F$ implies $A \cap B \in F$;

(iii) $A \in F, A \subset B$ implies $B \in F$.

DEFINITION 9. I is a non trivial ideal in X, $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \backslash A \text{ for some } A \in I\}$$

is a filter on X, called the filter associated with I.

DEFINITION 10. A non trivial ideal I in X is called admissible if $\{x\} \in I$, for each $x \in X$.

REMARK 3. If I is an admissible ideal, then usual convergence in X implies I convergence in X.

REMARK 4. If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.

DEFINITION 11. Let $x=(x_{mnk})$ be a triple sequence in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be rough ideal convergent or $rI-$ convergent to l, denoted by $x \to^{rI} l$, if for any $\varepsilon > 0$ we have

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geqslant r + \varepsilon\right\} \in I.$$

In this case l is called $rI-$ limit of x and a triple sequence $x=(x_{mnk})$ is called rough $I-$ convergent to l with r as roughness of degree. If $r=0$ then it is ordinary $I-$ convergent.

REMARK 5. Generally, a triple sequence $y=(y_{mnk})$ is not $I-$ convergent in usual sense and $|x_{mnk} - y_{mnk}| \leqslant r$ for all $(m,n,k) \in \mathbb{N}^3$ or

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - y_{mnk}| \geqslant r\right\} \in I,$$

for some $r > 0$. Then the triple sequence $x=(x_{mnk})$ is $rI-$ convergent.

REMARK 6. It is clear that $rI-$ limit of x is not necessarily unique.

DEFINITION 12. Consider $rI-$ limit set of x, which is denoted by

$$I - LIM_x^r = \left\{L \in X : x \to^{rI} l\right\},$$

then the triple sequence $x=(x_{mnk})$ is said to be $rI-$ convergent if $I - LIM_x^r \neq \phi$ and r is called a rough $I-$ convergence degree of x.

DEFINITION 13. A triple sequence $x=(x_{mnk}) \in X$ is said to be $I-$ analytic if there exists a positive real number M such that

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geqslant M\right\} \in I.$$

DEFINITION 14. A point $L \in X$ is said to be an $I-$ accumulation point of a triple sequence $x=(x_{mnk})$ in a metric space (X,d) if and only if for each $\varepsilon > 0$ the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : d(x_{mnk}, l) = |x_{mnk} - l| < \varepsilon\right\} \notin I.$$

We denote the set of all $I-$ accumulation points of x by $I(\Gamma_x)$.

DEFINITION 15. Let β be a non-negative real number. A triple sequence $x=(x_{mnk}) \in X$ is said to be geometric difference of rough I_q- convergent $\bar{0}$ if for all $q \in Q$ and all $\varepsilon > 0$,

$$\left\{(m,n,k) \in \mathbb{N}^3 : q_G\left(\Delta x_{mnk}, \bar{0}\right) \geqslant \beta + \varepsilon\right\} \in I. \tag{2}$$

In this case we can write $I_q - lim_G \Delta x_{mnk} = \bar{0}$. We denote

$$I_q = \left\{\left\{(m,n,k) \in \mathbb{N}^3 : q_G\left(\Delta x_{mnk}, \bar{0}\right) \geqslant \beta + \varepsilon\right\} \in I\right\}. \tag{3}$$

Note: If the triple sequence of X is Hausdorff, then the limit of rough ideal convergent sequence is unique.

DEFINITION 16. Let β be a non-negative real number. A triple sequence $x = (x_{mnk}) \in X$ is said to be geometric difference of rough $B^p_{(abc)}(I_q) -$ convergent to $\bar{0} \in X$ if for all $q_p \in Q$ and all $\varepsilon > 0$,

$$\left\{(m,n,k) \in \mathbb{N}^3 : q_G\left(B^p_{(abc)}\Delta x_{mnk}, \bar{0}\right) \geqslant \beta + \varepsilon\right\} \in I. \tag{4}$$

In this case we can write $I_q - lim_G B^p_{(abc)}(\Delta x_{mnk}) = \bar{0}$. We denote

$$B^p_{(abc)}(I_q) = \left\{\left\{(m,n,k) \in \mathbb{N}^3 : q_G\left(B^p_{(abc)}\Delta x_{mnk}, \bar{0}\right) \geqslant \beta + \varepsilon\right\} \in I\right\}, \tag{5}$$

where

$$B^p_{(abc)}\Delta x_{mnk} = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{k}\binom{m}{r}\binom{n}{s}\binom{k}{t}\alpha^{(m-r)+(n-s)+(k-t)}(1-\alpha)^{r+s+t}\Delta x_{(m-ar)(n-bs)(k-ct)}.$$

DEFINITION 17. Let β be a non-negative real number and f be a Musielak Orlicz function. A triple geometric difference of rough sequence $x = (x_{mnk}) \in w^I\left(B^p_{(abc)}, f\right)$ if and only if there exists $\bar{0} \in X$ such that for $q_G \in Q$ and for every $\varepsilon > 0$,

$$\left\{(i,j,\ell) \in \mathbb{N}^3 : \frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}\left[f_{mnk}\left(q_G\left(B^p_{(abc)}\Delta x_{mnk}, \bar{0}\right)\right)\right] \geqslant \beta + \varepsilon\right\} \in I \tag{6}$$

when (2.5) holds we write $\Delta x_{mnk} \to \bar{0}\left(w^I\left(B^p_{(abc)}, f\right)\right)$.

2.1. Zweier ideal triple sequence in a locally convex space

Let I be an admissible ideal of $\mathbb{N}$ and $A = \left(a^{ij\ell}_{mnk}\right)$ be an six dimensional infinite matrix. Let f be a Musielak-Orlicz function and $w(G,X)$ denotes the space of all $X-$ valued triple geometric difference sequence spaces of rough. For each $\varepsilon > 0$ for all $q_G \in Q$, we define the following rough triple geometric difference sequence spaces of rough:

$$\left[Z_0\left(G, A, B^p_{(abc)}, f, q\right)\right]_I = \Bigg\{x = (\Delta x_{mnk}) \in w(G,X) : \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a^{ij\ell}_{mnk}\left[f_{mnk}\left(q_G\left(B^p_{(abc)}\Delta x_{mnk}\right)\right)\right] \geqslant \beta + \varepsilon\right\} \in I\Bigg\},$$

$$\left[Z_\infty\left(G, A, B^p_{(abc)}, f, q\right)\right]_I = \Bigg\{x = (x_{mnk}) \in w(G,X) : \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a^{ij\ell}_{mnk}\left[f_{mnk}\left(q_G\left(B^p_{(abc)}\Delta x_{mnk}\right)\right)\right] \geqslant K\right\} \in I\Bigg\}.$$

The condition (2.5) procides a definition of triple geometric difference sequence of rough ideal of locally convex space.

3. Main results

THEOREM 1. (Main) *Let β be non-negative real number, $A=\left(a_{mnk}^{ij\ell}\right)$ be a six dimensional regular matrix and f be a Musielak Orlicz function. Then the triple geometric difference sequence of rough $(\Delta x_{mnk})\to\bar{0}\,(w\,(f,A))\Rightarrow(\Delta x_{mnk})\to\bar{0}\left(B^{p}_{(abc)}\,(I_q)\,(A)\right)$.*

Proof. Let $q_G\in Q$. Assume that $(\Delta x_{mnk})\to\bar{0}\,(w\,(f,A))$ we have

$$_G lim_{\ell\to\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B^{p}_{(abc)}\Delta x_{mnk},\bar{0}\right)\right)\right]=0. \tag{7}$$

Let $\varepsilon>0$ be given. We define

$$K\,(\beta+\varepsilon)=\left\{(m,n,k)\in\mathbb{N}^3:q_G\left(B^{p}_{(abc)}\Delta x_{mnk},\bar{0}\right)\geqslant\beta+\varepsilon\right\} \tag{8}$$

and we write

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B^{p}_{(abc)}\Delta x_{mnk},\bar{0}\right)\right)\right]\\ &=\sum_{(mnk)\in K(\beta+\varepsilon)}a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B^{p}_{(abc)}\Delta x_{mnk},\bar{0}\right)\right)\right]\\ &\quad+\sum_{(mnk)\notin K(\beta+\varepsilon)}a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B^{p}_{(abc)}\Delta x_{mnk},\bar{0}\right)\right)\right]\\ &\geqslant\left(\sum_{(mnk)\in K(\beta+\varepsilon)}a_{mnk}^{ij\ell}\right)\left[f_{mnk}\,(\beta+\varepsilon)\right].\end{aligned}$$

Then we have $(\Delta x_{mnk})\to\bar{0}\left(B^{p}_{(abc)}\,(I_q)\,(A)\right)$. □

THEOREM 2. *Let β be non-negative real number, $A=\left(a_{mnk}^{ij\ell}\right)$ be a six dimensional regular matrix and f be a Musielak Orlicz function. If the triple geometric difference sequence of rough $(\Delta x_{mnk})\in\Lambda^3\left(B^{p}_{(abc)}\right)$ and $(\Delta x_{mnk})\to\bar{0}\left(B^{p}_{(abc)}\,(I_q)\,(A)\right)$, then $(\Delta x_{mnk})\to\bar{0}\,(w\,(f,A))$.*

Proof. Suppose that $(\Delta x_{mnk})\to\Lambda^3\left(B^{p}_{(abc)}\right)$ and $(\Delta x_{mnk})\to\bar{0}\left(B^{p}_{(abc)}\,(I_q)\,(A)\right)$. Then there is a set $K\in\bar{0}\left(B^{p}_{(abc)}\,(I_q)\right)$ such that

$$_G lim_{(mnk)\in K}q_G\left(B^{p}_{(abc)}\Delta x_{mnk}^{1/m+n+k}\right)=0. \tag{9}$$

Now

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B_{(abc)}^{p}\Delta x_{mnk}^{1/m+n+k}\right)\right)\right] \\
&= \sum_{(mnk)\in K(\beta+\varepsilon)} a_{mnk}^{ij\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}\Delta x_{mnk}^{1/m+n+k}\right)\right)\right] \\
&\quad + \sum_{(mnk)\notin K(\beta+\varepsilon)} a_{mnk}^{ij\ell}\left[f_{mnk}\left(q_G\left(B_{(abc)}^{p}\Delta x_{mnk}^{1/m+n+k}\right)\right)\right] \\
&\quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mnk}^{ij\ell}\chi_K(mnk)\left[f_{mnk}\left(q_G\left(B_{(abc)}^{p}\Delta x_{mnk}^{1/m+n+k}\right)\right)\right] \\
&\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mnk}^{ij\ell}\chi_{K^c}(mnk)\left[f_{mnk}\left(q_G\left(B_{(abc)}^{p}\Delta x_{mnk}^{1/m+n+k}\right)\right)\right].
\end{aligned}$$

If we consider the six dimensional regular matrix of $A, K^c B_{(abc)}^{p}(I_q)$ and analyticness of triple geometric difference sequence of rough (Δx_{mnk}) right side tends to zero. Hence $(\Delta x_{mnk}) \in \bar{0}(w(f,A))$. $\square$

The proof of the following theorems is easy to verify. Therefore we omit the details.

THEOREM 3. *$\left[Z_0\left(G,A,B_{(abc)}^{p},f,q\right)\right]_I$ and $\left[Z_\infty\left(G,A,B_{(abc)}^{p},f,q\right)\right]_I$ are linear spaces.*

THEOREM 4. *Let $f=(f_{mnk})$ and $g=(g_{mnk})$ be two Musielak-Orlicz functions of triple geometric difference sequence spaces of rough, then the following holds:*

$$\left[Z_0\left(G,A,B_{(abc)}^{p},f,q\right)\right]_I \bigcap \left[Z_0\left(G,A,B_{(abc)}^{p},g,q\right)\right]_I \subseteq \left[Z_0\left(G,A,B_{(abc)}^{p},f+g,q\right)\right]_I.$$

THEOREM 5. *Let $f=(f_{mnk})$ and $g=(g_{mnk})$ be two Musielak-Orlicz functions of triple geometric difference sequence spaces of rough, then the following holds:*

$$\left[Z_0\left(G,A,B_{(abc)}^{p},g,q\right)\right]_I \subseteq \left[Z_0\left(G,A,B_{(abc)}^{p},fg,q\right)\right]_I.$$

THEOREM 6. *The inclusions*

$$\left[Z_0\left(G,A,B_{(abc)}^{p-1},f,q\right)\right]_I \subset \left[Z_0\left(G,A,B_{(abc)}^{p},f,q\right)\right]_I,$$

are strict for $p \geqslant 1$. In general

$$\left[Z_0\left(G,A,B_{(abc)}^{j},f,q\right)\right]_I \subset \left[Z_0\left(G,A,B_{(abc)}^{p},f,q\right)\right]_I,$$

for $j=0,1,2,\cdots p-1$ and the inclusions are strict.

EXAMPLE 1. Let $A=[C,1,1,1], f_{mnk}(\Delta x)=\Delta x$, for all $x\in[0,\infty),(m,n,k)\in\mathbb{N}$. Consider a triple geometric difference sequence spaces of rough $x=(\Delta x_{mnk})=(m^p n^p k^p)$. Then $x=(\Delta x_{mnk})\in\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I$ but does not belong to $\left[Z_0\left(G,A,B^{p-1}_{(abc)},f,q\right)\right]_I$ because $B^p_{(abc)}\Delta x_{mnk}=0$ and $B^{p-1}_{(abc)}\Delta x_{mnk}=(-\alpha)^{p-1}(p-1)!$.

THEOREM 7. *Let $f=(f_{mnk})$ be a Musielak-Orlicz function of triple geometric difference sequence spaces of rough, then the following statements are equivalent:*

(i) $\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I\subseteq\left[Z_0\left(G,A,B^p_{(abc)},q\right)\right]_I$;

(ii) $\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I\subseteq\left[Z_\infty\left(G,A,B^p_{(abc)},f,q\right)\right]_I$;

(iii) $inf_\ell\sum_{(m,n,k)=1}^{\ell}a^{ij\ell}_{mnk}[f_{mnk}(t)]>0\ (t>0)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii). Suppose $\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I\subseteq\left[Z_\infty\left(G,A,B^p_{(abc)},f,q\right)\right]_I$. We assume that (iii) does not hold. Then for some $t>0$

$$inf_\ell\sum_{(m,n,k)=1}^{\ell}a^{ij\ell}_{mnk}[f_{mnk}(t)]>0=0.$$

We can choose an index sequence (ℓ_j) of positive integer such that

$$\sum_{(m,n,k)=1}^{\ell_j}a^{ij\ell}_{mnk}[f_{mnk}(abc)]>\frac{1}{j},\quad j=1,2,3,\cdots. \tag{10}$$

Define a triple geometric difference sequence spaces of rough $x=(\Delta x_{mnk})$ by

$$B^p_{(abc)}\Delta x_{mnk}=\begin{cases}j, & \text{if } 1\leqslant(m,n,k)\leqslant\ell_j,\\ 0, & \text{if}\,(m,n,k)>\ell_j.\end{cases}$$

Then by equation (3.4) we have triple geometric difference sequence spaces of rough $x=(\Delta x_{mnk})\in\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I$, but $x=(\Delta x_{mnk})\notin\left[Z_\infty\left(G,A,B^p_{(abc)},f,q\right)\right]_I$ which contradicts (ii). Hence (iii) must hold.
(iii)$\Rightarrow$ (i). Let (iii) hold and $x\in\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I$. Then for every $\varepsilon>0$ we have

$$\left\{\sum_{(m,n,k=1)}^{\ell}a^{ij\ell}_{mnk}\left[f_{mnk}\left(q_G\left(B^p_{(abc)}\Delta x_{mnk}\right)\right)\right]\geqslant\beta+\varepsilon\right\}\in I. \tag{11}$$

Suppose that $x\notin\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I$. Then for some integer $\varepsilon_0>0$ we have

$$\left\{\sum_{(m,n,k=1)}^{\ell}a^{ij\ell}_{mnk}\left[f_{mnk}\left(q\left(B^p_{(abc)}\Delta x_{mnk}\right)\right)\right]\geqslant\beta+\varepsilon_0\right\}\notin I. \tag{12}$$

Therefore we have

$$[f_{mnk}(\beta+\varepsilon_0)] \leqslant \left[f_{mnk}\left(q\left(B^p_{(abc)}\Delta x_{mnk}\right)\right)\right] \tag{13}$$

and consequently, by the relation (3.6), we have

$$inf_\ell \sum_{(m,n,k)=1}^{\ell} a^{ij\ell}_{mnk}[f_{mnk}(\beta+\varepsilon_0)] = 0, \tag{14}$$

which contradicts (iii). Hence $\left[Z_0\left(G,A,B^p_{(abc)},f,q\right)\right]_I \subseteq \left[Z_0\left(G,A,B^p_{(abc)},q\right)\right]_I$. $\square$

Acknowledgement. The authors are extremely grateful to the anonymous referees for their valuable suggestion and constructive comments for the improvement of the manuscript.

REFERENCES

[1] M. AIYUB, A. ESI AND N. SUBRAMANIAN, *The triple entire difference ideal of fuzzy real numbers over fuzzy $p-$metric spaces defined by Musielak Orlicz function*, Journal of Intelligent & Fuzzy Systems, **33** (2017), 1505–1512.

[2] S. AYTAR, *Rough statistical convergence*, Numer. Funct. Anal. Optim., **29** (2008), No. 3, 291–303.

[3] A. ESI AND E. SAVAS, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Appl. Math. Inf. Sci., **9** (2015), No. 5, 2529–2534.

[4] S. DEBNATH, B. SARMA AND B. C. DAS,*Some generalized triple sequence spaces of real numbers*, J. Nonlinear Anal. Optim., **6** (2015), No. 1, 71–79.

[5] A. J. DUTTA, A. ESI AND B. C. TRIPATHY, *Statistically convergent triple sequence spaces defined by Orlicz function*, J. Math. Anal., **4**, (2013), No. 2, 16–22.

[6] Y. FADILE KARABABA AND A. ESI, *On some strong zweier convergent sequence spaces*, Acta Univ. Apulensis Math. Inform., **29** (2012), 9–15.

[7] P. K. KAMTHAN AND M. GUPTA, *Sequence Spaces and Series, Lecture notes, Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 1981.

[8] V. A. KHAN, K. EBADULLAH, A. ESI, N. KHAN AND M. SHAFIQ, *On paranorm Zweier I-convergent sequence spaces*, J. Math., **2013**, Article ID 653501, 6pp.

[9] J. LINDENSTRAUSS AND L. TZAFRIRI, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379–390.

[10] J. MUSIELAK, *Orlicz Spaces, Lectures Notes in Mathematics 1034*, Springer-Verlag, 1983.

[11] S. K. PAL, D. CHANDRA AND S. DUTTA, *Rough ideal Convergence*, Hacet. J. Math. Stat., **42** (2013), No. 6, 633–640.

[12] H. X. PHU, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22** (2001), 201–224.

[13] A. SAHINER, M. GURDAL AND F. K. DUDEN, *Triple sequences and their statistical convergence*, Selcuk J. Appl. Math.,**8** (2007), No. 2, 49–55.

[14] A. SAHINER AND B. C. TRIPATHY, *Some I related properties of triple sequences*, Selcuk J. Appl. Math., **9** (2008), No. 2, 9–18.
[15] N. SUBRAMANIAN AND A. ESI, *The generalized tripled difference of χ^3 sequence spaces*, Global Journal of Mathematical Analysis, **3** (2015), No. 2, 54–60.

(Received August 8, 2018)

A. Esi
Department of Mathematics
Adiyaman University
02040 Adiyaman, Turkey
e-mail: aesi23@hotmail.com

N. Subramanian
School of Arts Sciences and Humanities
Department of Mathematics, SASTRA Deemed to be University
Thanjavur-613 401,India
e-mail: nsmaths@gmail.com