

A NOTE ON q -MKZ OPERATORS USING GENERATING FUNCTIONS

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Abstract. This paper is based on new generalization of q -analogue of the MKZ type operators using generating functions. We study approximation properties of the proposed operator using Korovkin type theorem. Further, we estimate the rate of convergence of these operators by using the modulus of continuity. In the last, we introduce and establish the uniform convergence of 2D-generalization of the q -MKZ operators using generating functions.

1. Introduction

The classical MKZ operators are defined in [8] as follows:

$$M_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k (1-x)^{n+1}, \text{ if } x \in [0, 1),$$

$$M_n(f; 1) = f(1), \text{ if } x = 1, n \in \mathbb{N},$$

for $f \in C[0, 1]$. Later, Cheney and Sharma [3] modified these operators as the Bernstein power series to discuss the monotonicity properties. In 2005, Altin, Dögrü and Taşdelen [2] introduced a new version of MKZ operators using generating function as follows:

$$L_n(f; x) = \frac{1}{h_n(x, t)} \sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n} + b_n}\right) C_{k,n}(t) x^k$$

for $0 \leq \frac{a_{k,n}}{a_{k,n} + b_n} \leq A$, $A \in (0, 1)$, where $\{h_n(x, t)\}_{n \in \mathbb{N}}$ is a generating function defined in terms of the sequence of functions $\{C_{k,n}(t)\}_{k \in \mathbb{N}_0}$ as follows:

$$h_n(x, t) = \sum_{k=0}^{\infty} C_{k,n}(t) x^k.$$

The authors have studied the approximation behavior of the sequence of the operators $\{L_n\}$, under the following conditions:

- (1) $h_n(x, t) = (1-x)h_{n+1}(x, t)$,
- (2) $b_n C_{k,n+1}(t) = a_{k+1,n} C_{k+1,n}(t)$,
- (3) $b_n \rightarrow \infty$, $\frac{b_{n+1}}{b_n} \rightarrow 1$ and $b_n \neq 0$, $\forall n \in \mathbb{N}$,
- (4) $C_{k,n}(t) \geq 0$, $\forall t \in \mathbb{I} \subset \mathbb{R}$,
- (5) $a_{k+1,n} = a_{k,n+1} + \phi_n$, $|\phi_n| \leq m < \infty$ and $a_{0,n} = 0$.

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During last two decades, approximation by using quantum calculus has been an interest area for many researchers [1]. In 2000, T. Trif [6] introduced the q -MKZ operators for $f \in C[0, 1]$. Further, with a slight modification in these operators, Döğru and Duman [5] defined the q -MKZ operators for $f \in C[0, a]$, $a \in (0, 1)$, as follows:

$$M_{n,q}(f;x) = \prod_{s=0}^n (1 - q^s x) \sum_{k=0}^{\infty} f\left(\frac{q^n [k]_q}{[n+k]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k, \quad q \in (0, 1), n \in \mathbb{N}.$$

2. Construction of the operator

Motivated by [2], here, we introduce a generalization of q -MKZ operators based on some generating function and study their approximation properties.

For $q \in (0, 1)$ and $\{h_{n,q}(x, t)\}_{n \in \mathbb{N}}$ the sequence of generating functions for the sequence of functions $\{C_{k,n}^q(t)\}_{k \in \mathbb{N}_0}$ such that

$$h_{n,q}(x, t) = \sum_{k=0}^{\infty} C_{k,n}^q(t) x^k.$$

For $A \in \mathbb{R}$ and $A \in (0, 1)$, we consider a generalization of q -MKZ operators using generating functions as follows:

$$\tilde{M}_{n,q}(f;x) = \frac{1}{h_{n,q}(x, t)} \sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n} + b_n}\right) C_{k,n}^q(t) x^k \tag{1}$$

for $0 \leq \frac{a_{k,n}}{a_{k,n} + b_n} \leq A$, under the following assumptions:

$$(1) \quad h_{n,q}(x, t) = (1 - q^{n+1}x)h_{n+1,q}(x, t), \tag{2}$$

$$(2) \quad q^n b_{n+1} C_{k,n+1}^q(t) = a_{k+1,n} C_{k+1,n}^q(t), \tag{3}$$

$$(3) \quad \lim_{q \rightarrow 1, n \rightarrow \infty} b_n \rightarrow \infty, \frac{b_{n+1}}{b_n} \rightarrow 1 \text{ and } b_n \neq 0, \forall n \in \mathbb{N}, \tag{4}$$

$$(4) \quad C_{k,n}^q(t) \geq 0, \forall t \in \mathbb{I} \subset \mathbb{R}, \tag{5}$$

$$(5) \quad a_{k+1,n} = a_{k,n+1} + \phi_n, |\phi_n| \leq M < \infty \text{ and } a_{0,n} = 0. \tag{6}$$

REMARK 1. By taking $h_{n,q}(x, t) = \frac{1}{(1-x)_q^{n+1}}$, $a_{k,n} = q^n [k]_q$, $C_{k,n}^q(t) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$, $b_n = [n]_q$, the assumptions (3)–(6) are satisfied and proposed operators will be converted into q -MKZ operators considered by Döğru and Duman [5].

REMARK 2. By taking $q = 1$, the operators $\{\tilde{M}_{n,q}\}$ turn out to be MKZ operators given by Altin et al. [2].

LEMMA 1. *The moments of the generalized operators $\tilde{M}_{n,q}$ for the test functions $e_i = \left(\frac{s}{1-s}\right)^i$, $i = 0, 1, 2$ are as follows:*

$$\begin{aligned} \tilde{M}_{n,q}(1;x) &= 1, \\ \tilde{M}_{n,q}\left(\frac{s}{1-s};x\right) &= \frac{b_{n+1}}{b_n} \frac{q^n x}{(1-q^{n+1}x)}, \\ \tilde{M}_{n,q}\left(\frac{s^2}{(1-s)^2};x\right) &= q^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q^{n+1}x)(1-q^{n+2}x)} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q^n x}{(1-q^{n+1}x)}. \end{aligned}$$

Proof. For $i = 0$, result is trivial.

For $i = 1$, using the definition of the new generalized operator, the moment can be obtained directly as follows:

$$\begin{aligned} \tilde{M}_{n,q}\left(\frac{s}{1-s};x\right) &= \frac{1}{b_n} \frac{1}{h_{n,q}(x,t)} \sum_{k=1}^{\infty} a_{k,n} C_{k,n}^q(t) x^k \\ &= \frac{1}{b_n} \frac{x}{h_{n,q}(x,t)} \sum_{k=0}^{\infty} a_{k+1,n} C_{k+1,n}^q(t) x^k \\ &= \frac{q^n b_{n+1}}{b_n} \frac{x}{h_{n,q}(x,t)} \sum_{k=0}^{\infty} C_{k,n+1}^q(t) x^k \\ &= \frac{b_{n+1}}{b_n} \frac{q^n x}{1-q^{n+1}x} \frac{1}{h_{n+1,q}(x,t)} \sum_{k=0}^{\infty} C_{k,n+1}^q(t) x^k \\ &= \frac{b_{n+1}}{b_n} \frac{q^n x}{1-q^{n+1}x} \end{aligned}$$

Finally for $i = 2$, we have following computations:

$$\begin{aligned} &\tilde{M}_{n,q}\left(\frac{s^2}{(1-s)^2};x\right) \\ &= \frac{1}{(b_n)^2} \frac{1}{h_{n,q}(x,t)} \sum_{k=1}^{\infty} (a_{k,n})^2 C_{k,n}^q(t) x^k \\ &= \frac{1}{(b_n)^2} \frac{1}{h_{n,q}(x,t)} \sum_{k=1}^{\infty} a_{k,n} q^n b_{n+1} C_{k-1,n+1}^q(t) x^k \\ &= q^n \frac{b_{n+1}}{(b_n)^2} \frac{1}{h_{n,q}(x,t)} \sum_{k=0}^{\infty} a_{k+1,n} C_{k,n+1}^q(t) x^{k+1} \\ &= q^n \frac{b_{n+1}}{(b_n)^2} \frac{x}{h_{n,q}(x,t)} \sum_{k=0}^{\infty} (a_{k,n+1} + \phi_n) C_{k,n+1}^q(t) x^k \\ &= q^n \frac{b_{n+1}}{(b_n)^2} \frac{x}{h_{n,q}(x,t)} \left(\sum_{k=1}^{\infty} a_{k,n+1} C_{k,n+1}^q(t) x^k + \phi_n \sum_{k=0}^{\infty} C_{k,n+1}^q(t) x^k \right) \end{aligned}$$

$$\begin{aligned}
 &= q^n \frac{b_{n+1}}{(b_n)^2} \frac{x}{h_{n,q}(x,t)} \left(q^{n+1} b_{n+2} \sum_{k=1}^{\infty} C_{k-1,n+2}^q(t) x^k + \phi_n \sum_{k=0}^{\infty} C_{k,n+1}^q(t) x^k \right) \\
 &= q^n \frac{b_{n+1}}{(b_n)^2} \frac{x}{h_{n,q}(x,t)} \left(q^{n+1} b_{n+2x} \sum_{k=0}^{\infty} C_{k,n+2}^q(t) x^k + \phi_n \sum_{k=0}^{\infty} C_{k,n+1}^q(t) x^k \right) \\
 &= q^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q^{n+1}x)(1-q^{n+2}x)} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q^n x}{(1-q^{n+1}x)}. \quad \square
 \end{aligned}$$

REMARK 3. For $q \in (0, 1)$ one can observe that $\lim_{n \rightarrow \infty} [n]_{p,q} = 1/(1-q)$. In order to study the convergence properties of the operator, we take $q_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} q_n = 1$, so that $\lim_{n \rightarrow \infty} \frac{1}{[n]_{q_n}} = 0$. Such a sequence can always be constructed. For example, we may consider $q_n = 1 - 1/n, 1 - 1/n^2$. Clearly in each case $\lim_{n \rightarrow \infty} \frac{1}{[n]_{q_n}} = 0$.

We consider W_ω as a subspace of the space $C[0, A]$, $A \in (0, 1)$ satisfying the condition:

$$|f(s) - f(x)| \leq \omega \left(f, \left| \frac{s}{1-s} - \frac{x}{1-x} \right| \right), \quad \forall x, s \in [0, A],$$

where ω is the modulus of continuity.

For the space of function W_ω Altin, Dögrü and Taşdelen [2] proved the following result:

THEOREM 1. ([2]) Let $A_n : W_\omega \rightarrow C[0, A]$ be the sequence of positive linear operators satisfying the condition

$$\lim_{n \rightarrow \infty} \left\| A_n \left(\left(\frac{s}{(1-s)} \right)^i ; x \right) - \left(\frac{x}{(1-x)} \right)^i \right\|_{C[0,A]} = 0, \quad i = 0, 1, 2.$$

Then

$$\lim_{n \rightarrow \infty} \|A_n(f) - f\|_{C[0,A]} = 0,$$

for all $f \in W_\omega$ and $\|\cdot\|$ is supremum norm on the space $C[0, A]$.

THEOREM 2. Let $\{q_n\}$ be the sequence as defined by Remark 3. Then for all $f \in W_\omega$, the operators $M_{n,q_n}(f; x)$ converges uniformly to f .

Proof. By using Theorem 1, it is sufficient to prove that the following equation holds:

$$\lim_{n \rightarrow \infty} \left\| \tilde{M}_{n,q_n} \left(\left(\frac{s}{(1-s)} \right)^i ; x \right) - \left(\frac{x}{(1-x)} \right)^i \right\|_{C[0,A]} = 0, \quad \text{for } i = 0, 1, 2.$$

By using definition of the operator \tilde{M}_{n,q_n} and Lemma 1, result for $i = 0$ is obvious.

For $i = 1$, we have

$$\left\| \widetilde{M}_{n,q_n} \left(\frac{s}{1-s}; x \right) - \frac{x}{1-x} \right\|_{C[0,A]} = \sup_{x \in [0,A]} \left| \frac{b_{n+1}}{b_n} \frac{q_n^n x}{1-q_n^{n+1}x} - \frac{x}{1-x} \right|.$$

Using the assumption (5) as $n \rightarrow \infty$, we get

$$\left\| \widetilde{M}_{n,q_n} \left(\frac{s}{1-s}; x \right) - \frac{x}{1-x} \right\|_{C[0,A]} = 0.$$

Finally, for $i = 2$, we have

$$\begin{aligned} & \left\| \widetilde{M}_{n,q} \left(\frac{s^2}{(1-s)^2}; x \right) - \frac{x^2}{(1-x)^2} \right\|_{C[0,A]} \\ &= \sup_{x \in [0,A], 0 < q_n < 1} \left| \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{q_n^{2n+1} x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} - \frac{x^2}{(1-x)^2} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{1-q_n^{n+1}x} \right| \\ &\leq \sup_{x \in [0,A]} \left| \left(\frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} - 1 \right) \frac{x^2}{(1-x)^2} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{x}{1-x} \right|. \end{aligned}$$

Using the assumptions (5) and (6) as $n \rightarrow \infty$, we get

$$\left\| \widetilde{M}_{n,q_n} \left(\frac{s^2}{(1-s)^2}; x \right) - \frac{x^2}{(1-x)^2} \right\|_{C[0,A]} = 0. \quad \square$$

3. Rate of convergence

THEOREM 3. *Let $\{q_n\}$ be the sequence as defined by Remark 3. Then for all $f \in W_\omega$, we have*

$$\|\widetilde{M}_{n,q_n}(f;x) - f\|_{C[0,A]} \leq (1 + \sqrt{T})\omega(f, \delta_n),$$

here,

$$\begin{aligned} T &= \max_{x \in [0,A]} \left\{ \frac{x^2}{(1-x)^2}, \frac{x}{1-x} \right\}, \\ \delta_n &= \left[1 - 2 \frac{b_{n+1}}{b_n} + \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} + \frac{\phi_n}{b_n} \frac{b_{n+1}}{b_n} \right]^{\frac{1}{2}}. \end{aligned}$$

Proof. For all $x, s \in [0, A]$ and $f \in W_\omega$, we have

$$\begin{aligned} |\widetilde{M}_{n,q_n}(f;x) - f| &\leq \widetilde{M}_{n,q_n}(|f(s) - f(x)|; x) \\ &\leq \widetilde{M}_{n,q_n} \left(\omega \left(f, \left| \frac{s}{1-s} - \frac{x}{1-x} \right| \right); x \right) \end{aligned}$$

$$\begin{aligned}
&\leq \tilde{M}_{n,q_n} \left(\omega(f, \delta_n) \left(1 + \frac{\left| \frac{s}{1-s} - \frac{x}{1-x} \right|}{\delta_n} \right); x \right) \\
&= \omega(f, \delta_n) \tilde{M}_{n,q_n} \left(\left(1 + \frac{\left| \frac{s}{1-s} - \frac{x}{1-x} \right|}{\delta_n} \right); x \right) \\
&= \omega(f, \delta_n) \left[1 + \frac{1}{\delta_n} \left(\frac{1}{h_{n,q_n}(x,t)} \sum_{k=0}^{\infty} \left| \frac{a_{k,n}}{b_n} - \frac{x}{1-x} \right| C_{k,n}^{q_n}(t) x^k \right) \right].
\end{aligned}$$

By Cauchy-Schwarz inequality and Lemma 1, we have

$$\begin{aligned}
|\tilde{M}_{n,q_n}(f;x) - f| &\leq \omega(f, \delta_n) \left[1 + \frac{1}{\delta_n} \left(\frac{1}{h_{n,q_n}(x,t)} \sum_{k=0}^{\infty} \left(\frac{a_{k,n}}{b_n} - \frac{x}{1-x} \right)^2 C_{k,n}^{q_n}(t) x^k \right)^{\frac{1}{2}} \right] \\
&= \omega(f, \delta_n) \left[1 + \frac{1}{\delta_n} (S_{n,q_n}(x,t))^{\frac{1}{2}} \right],
\end{aligned}$$

where

$$\begin{aligned}
S_{n,q_n}(x,t) &= \frac{1}{h_{n,q_n}(x,t)} \sum_{k=0}^{\infty} \left(\frac{a_{k,n}}{b_n} - \frac{x}{1-x} \right)^2 C_{k,n}^{q_n}(t) x^k \\
&= \tilde{M}_{n,q_n} \left(\frac{s^2}{(1-s)^2}; x \right) - \frac{2x}{1-x} \tilde{M}_{n,q_n} \left(\frac{s}{1-s}; x \right) + \frac{x^2}{(1-x)^2} \\
&= \left(q_n^{2n+1} \frac{b_{n+2} b_{n+1}}{b_n b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} \right. \\
&\quad \left. + \frac{b_{n+1} \phi_n}{b_n b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} - \frac{2x}{1-x} \frac{b_{n+1}}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} + \frac{x^2}{(1-x)^2} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{x \in [0,A], 0 < q_n < 1} S_{n,q_n}(x,t) &= \sup_{x \in [0,A], 0 < q_n < 1} \left(q_n^{2n+1} \frac{b_{n+2} b_{n+1}}{b_n b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} \right. \\
&\quad \left. + \frac{b_{n+1} \phi_n}{b_n b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} - \frac{2x}{1-x} \frac{b_{n+1}}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} + \frac{x^2}{(1-x)^2} \right) \\
&\leq \sup_{x \in [0,A], 0 < q_n < 1} \left(q_n^{2n+1} \frac{b_{n+2} b_{n+1}}{b_n b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} \right. \\
&\quad \left. + \frac{b_{n+1} \phi_n}{b_n b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} + \frac{x^2}{(1-x)^2} \right) \\
&\leq \left(1 + \frac{b_{n+2} b_{n+1}}{b_n b_n} \right) \frac{x^2}{(1-x)^2} + \frac{\phi_n b_{n+1}}{b_n b_n} \frac{x}{1-x} \\
&\leq T \left[1 + \frac{b_{n+2} b_{n+1}}{b_n b_n} + \frac{\phi_n b_{n+1}}{b_n b_n} \right] \\
&= T \delta_n^2,
\end{aligned}$$

here,

$$T = \max_{x \in [0,A]} \left\{ \frac{x^2}{(1-x)^2}, \frac{x}{1-x} \right\},$$

$$\delta_n = \left[1 + \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} + \frac{\phi_n}{b_n} \frac{b_{n+1}}{b_n} \right]^{\frac{1}{2}}.$$

Finally taking supremum norm, we have

$$\begin{aligned} \|\tilde{M}_{n,q_n}(f;x) - f\|_{C[0,A]} &\leq \omega(f, \delta_n) \left[1 + \frac{1}{\delta_n} \left(\sup_{x \in [0,A]} S_{n,q_n}(x,t) \right)^{\frac{1}{2}} \right] \\ &\leq (1 + \sqrt{T})\omega(f, \delta_n). \end{aligned}$$

Hence the Theorem. \square

Further, we employ the elements of the modified Lipschitz class $\tilde{Lip}_M(\alpha)$ as considered by Altin et al. [2] to estimate the rate of convergence of the operators $\{\tilde{M}_{n,q}\}$.

A function $f \in C[0,A]$ is said to belong to modified Lipschitz class function, i.e. $f \in \tilde{Lip}_M(\alpha)$ if

$$|f(s) - f(x)| \leq M \left| \frac{s}{1-s} - \frac{x}{1-x} \right|^\alpha, \quad \forall s, x \in [0,A],$$

where $M > 0$ and $0 < \alpha \leq 1$.

REMARK 4. For $t, s \in [0,A]$, $0 < A < 1$, we have

$$\left| \frac{t}{1-t} - \frac{s}{1-s} \right| = \left| \frac{t-s}{(1-t)(1-s)} \right| > |t-s|. \tag{7}$$

This implies

$$|t-s|^\alpha < \left| \frac{t}{1-t} - \frac{s}{1-s} \right|^\alpha,$$

which proves that $Lip_M(\alpha) \subset \tilde{Lip}_M(\alpha)$.

THEOREM 4. Let $\{q_n\}$ be the sequence as defined by Remark 3. Then for all $f \in \tilde{Lip}_M(\alpha)$, we have

$$\|\tilde{M}_{n,q_n}(f) - f\|_{C[0,A]} \leq MT^{\frac{\alpha}{2}} \delta_n^\alpha,$$

where T and δ_n are same as in Theorem 3.

Proof. Let $f \in \tilde{Lip}_M(\alpha)$ and $0 < \alpha \leq 1$. By linearity and monotonicity of the operators \tilde{M}_{n,q_n} , we have

$$\begin{aligned} |\tilde{M}_{n,q_n}(f;x) - f| &\leq \tilde{M}_{n,q_n}(|f(s) - f(x)|;x) \\ &\leq \tilde{M}_{n,q_n} \left(M \left| \frac{s}{1-s} - \frac{x}{1-x} \right|^\alpha ;x \right), \quad \forall x, s \in [0,A] \\ &= M \frac{1}{h_{n,q_n}(x,t)} \sum_{k=0}^{\infty} \left| \frac{a_{k,n}}{b_n} - \frac{x}{1-x} \right|^\alpha C_{k,n}^{q_n}(t) x^k \end{aligned}$$

By using Hölder’s inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ in above inequality, we have

$$\begin{aligned}
 |\tilde{M}_{n,q_n}(f;x) - f| &\leq M \left[\frac{1}{h_{n,q_n}(x,t)} \sum_{k=0}^{\infty} \left(\frac{a_{k,n}}{b_n} - \frac{x}{1-x} \right)^2 C_{k,n}^{q_n}(t) x^k \right]^{\frac{\alpha}{2}} \\
 &= MS_{n,q_n}(x,t)^{\frac{\alpha}{2}}.
 \end{aligned}$$

Now taking the supremum on both sides of the above equation, we have

$$\begin{aligned}
 \|\tilde{M}_{n,q_n}(f) - f\|_{C[0,A]} &\leq M \sup_{x \in [0,A]} S_{n,q_n}(x,t)^{\frac{\alpha}{2}} \\
 &= MT^{\frac{\alpha}{2}} \delta_n^\alpha. \quad \square
 \end{aligned}$$

4. 2D-generalization of the operators $\tilde{M}_{n,q}$

In this section, we give 2-dimensional generalization of the operators $\tilde{M}_{n,q}$ and obtain the moments for these operators as given below.

Let $0 < q_n < 1$ for all $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} q_n = 1$ and A, B be a real numbers with $0 < A, B < 1$, we have rectangular domain

$$D = \{(x, y); 0 < x \leq A, 0 < y \leq B\}.$$

Now, for all real valued continuous functions f defined on D and considering the assumptions made for the operators $\{\tilde{M}_{n,q}\}$, we define

$$\begin{aligned}
 \tilde{M}_{n,m}^{q_n,q_m}(f;x,y) &= \frac{1}{h_{n,q_n}(x,t_1)} \frac{1}{h_{m,q_m}(y,t_2)} \\
 &\times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n} + q^k b_n}, \frac{a_{j,m}}{a_{j,m} + b_m}\right) I_{n,m}^{q_n,q_m}(t_1, t_2) x^k y^j, \quad (8)
 \end{aligned}$$

where

$$I_{n,m}^{q_n,q_m}(t_1, t_2) = C_{k,n}^{q_n}(t_1) C_{j,m}^{q_m}(t_2).$$

THEOREM 5. Let $\{\tilde{M}_{n,m}^{q_n,q_m}\}$ be the sequence of operators defined by (8), we have

- (i) $\tilde{M}_{n,m}^{q_n,q_m}(1;x,y) = 1,$
- (ii) $\tilde{M}_{n,m}^{q_n,q_m}\left(\frac{s_1}{1-s_1};x,y\right) = \frac{b_{n+1}}{b_n} \frac{q_n^n x}{1-q_n^{n+1}x},$
- (iii) $\tilde{M}_{n,m}^{q_n,q_m}\left(\frac{s_2}{1-s_2};x,y\right) = \frac{b_{m+1}}{b_m} \frac{q_m^m y}{1-q_m^{m+1}y},$

$$\begin{aligned}
 (iv) \quad \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1^2}{(1-s_1)^2}; x, y \right) &= q_n^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} \\
 &\quad + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)}, \\
 (v) \quad \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_2^2}{(1-s_2)^2}; x, y \right) &= q_m^{2m+1} \frac{b_{m+2}}{b_m} \frac{b_{m+1}}{b_m} \frac{y^2}{(1-q_m^{m+1}x)(1-q_m^{m+2}y)} \\
 &\quad + \frac{b_{m+1}}{b_m} \frac{\phi_m}{b_m} \frac{q_m^m y}{(1-q_m^{m+1}y)}. \\
 (vi) \quad \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1^2}{(1-s_1)^2} + \frac{s_2^2}{(1-s_2)^2}; x, y \right) &= q_n^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} \\
 &\quad + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} + q_m^{2m+1} \frac{b_{m+2}}{b_m} \frac{b_{m+1}}{b_m} \frac{y^2}{(1-q_m^{m+1}y)(1-q_m^{m+2}y)} \\
 &\quad + \frac{b_{m+1}}{b_m} \frac{\phi_m}{b_m} \frac{q_m^m y}{(1-q_m^{m+1}y)}.
 \end{aligned}$$

Proof. Using the moment estimates obtained for the operators $\widetilde{M}_{n,q}$ in Lemma-1, the proof can be given as below:

$$\begin{aligned}
 (i) \quad \widetilde{M}_{n,m}^{q_n,q_m}(1; x, y) &= \frac{1}{h_{n,q_n}(x, t_1)} \sum_{k=0}^{\infty} C_{k,n}^{q_n}(t_1) x^k * \frac{1}{h_{m,q_m}(y, t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \\
 &= 1.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1}{1-s_1}; x, y \right) &= \frac{1}{h_{n,q_n}(x, t_1)} \sum_{k=0}^{\infty} \frac{a_{k,n}}{b_n} C_{k,n}^{q_n}(t_1) x^k * \frac{1}{h_{m,q_m}(y, t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \\
 &= \left(\frac{1}{b_n} \frac{1}{h_{n,q_n}(x, t_1)} \sum_{k=0}^{\infty} a_{k,n} C_{k,n}^{q_n}(t_1) x^k \right) * \left(\frac{1}{h_{m,q_m}(y, t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \right) \\
 &= \frac{b_{n+1}}{b_n} \frac{q_n^n x}{1-q_n^{n+1}x}.
 \end{aligned}$$

Further, we have

$$\begin{aligned}
 (iv) \quad \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1^2}{(1-s_1)^2}; x, y \right) &= \frac{1}{h_{n,q_n}(x, t_1)} \sum_{k=0}^{\infty} \left(\frac{a_{k,n}}{b_n} \right)^2 C_{k,n}^{q_n}(t_1) x^k * \frac{1}{h_{m,q_m}(y, t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \\
 &= \left(\frac{1}{b_n^2} \frac{1}{h_{n,q_n}(x, t_1)} \sum_{k=0}^{\infty} a_{k,n}^2 C_{k,n}^{q_n}(t_1) x^k \right) * \left(\frac{1}{h_{m,q_m}(y, t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \right) \\
 &= q_n^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)}.
 \end{aligned}$$

Similarly, one can give the proof for (iii) and (v). Now, we have

$$\begin{aligned}
 (vi) \quad & \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1^2}{(1-s_1)^2} + \frac{s_2^2}{(1-s_2)^2}; x, y \right) \\
 &= \frac{1}{h_{n,q_n}(x,t_1)} \sum_{k=0}^{\infty} \left(\frac{a_{k,n}}{b_n} \right)^2 C_{k,n}^{q_n}(t_1) x^k * \frac{1}{h_{m,q_m}(y,t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \\
 &+ \frac{1}{h_{n,q_n}(x,t_1)} \sum_{k=0}^{\infty} C_{k,n}^{q_n}(t_1) x^k * \frac{1}{h_{m,q_m}(y,t_2)} \sum_{j=0}^{\infty} \left(\frac{a_{j,m}}{b_m} \right)^2 C_{j,m}^{q_m}(t_2) y^j \\
 &= \left(\frac{1}{b_n^2} \frac{1}{h_{n,q_n}(x,t_1)} \sum_{k=0}^{\infty} a_{k,n}^2 C_{k,n}^{q_n}(t_1) x^k \right) * \left(\frac{1}{h_{m,q_m}(y,t_2)} \sum_{j=0}^{\infty} C_{j,m}^{q_m}(t_2) y^j \right) \\
 &+ \left(\frac{1}{h_{n,q_n}(x,t_1)} \sum_{k=0}^{\infty} C_{k,n}^{q_n}(t_1) x^k \right) * \left(\frac{1}{b_m^2} \frac{1}{h_{m,q_m}(y,t_2)} \sum_{j=0}^{\infty} a_{j,m}^2 C_{j,m}^{q_m}(t_2) y^j \right) \\
 &= q_n^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} \\
 &+ q_m^{2m+1} \frac{b_{m+2}}{b_m} \frac{b_{m+1}}{b_m} \frac{y^2}{(1-q_m^{m+1}y)(1-q_m^{m+2}y)} + \frac{b_{m+1}}{b_m} \frac{\phi_m}{b_m} \frac{q_m^m y}{(1-q_m^{m+1}y)}. \quad \square
 \end{aligned}$$

Let ω be the full modulus of continuity as given in [4]. Using the definition of W_ω as given by Altin et al. [2], we consider W_ω^2 as the space of real valued continuous functions $f \in C(D)$, satisfying the following condition:

$$|f(s_1, s_2) - f(x, y)| \leq \omega \left(f, \min \left\{ \left| \frac{s_1}{1-s_1} - \frac{x}{1-x} \right|, \left| \frac{s_2}{1-s_2} - \frac{y}{1-y} \right| \right\} \right), \quad \forall (x, y), (s_1, s_2) \in D.$$

Now, we have the following theorem to establish the uniform convergence of the operators $\{\widetilde{M}_{n,m}^{q_n,q_m}\}$.

THEOREM 6. *If $f \in W_\omega^2$ and $\lim_{n \rightarrow \infty} q_n = 1$, the equality*

$$\lim_{n,m \rightarrow \infty} \|\widetilde{M}_{n,m}^{q_n,q_m}(f; x, y) - f(x, y)\|_{C(D)} = 0$$

holds.

Proof. By using the moment estimates obtained in the Theorem 5, we have

$$\begin{aligned}
 & \|\widetilde{M}_{n,m}^{q_n,q_m}(1; x, y) - 1\|_{C(D)} = 0, \\
 & \left\| \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_1}{(1-s_1)}; x, y \right) - \frac{x}{(1-x)} \right\|_{C(D)} = \sup_{(x,y) \in D, 0 < q_n < 1} \left| \frac{b_{n+1}}{b_n} \frac{q_n^n x}{1-q_n^{n+1}x} - \frac{x}{1-x} \right|, \\
 & \left\| \widetilde{M}_{n,m}^{q_n,q_m} \left(\frac{s_2^i}{(1-s_2)^i}; x, y \right) - \frac{y^i}{(1-y)^i} \right\|_{C(D)} = \sup_{(x,y) \in D, 0 < q_m < 1} \left| \frac{b_{m+1}}{b_m} \frac{q_m^m y}{1-q_m^{m+1}y} - \frac{y}{1-y} \right|,
 \end{aligned}$$

$$\begin{aligned} & \left\| \widetilde{M}_{n,m}^{q_n, q_m} \left(\frac{s_1^2}{(1-s_1)^2} + \frac{s_2^2}{(1-s_2)^2}; x, y \right) - \left(\frac{x^2}{(1-x)^2} + \frac{y^2}{(1-y)^2} \right) \right\|_{C(D)} \\ &= \sup_{(x,y) \in D, 0 < q_n, q_m < 1} \left| q_n^{2n+1} \frac{b_{n+2}}{b_n} \frac{b_{n+1}}{b_n} \frac{x^2}{(1-q_n^{n+1}x)(1-q_n^{n+2}x)} - \frac{x^2}{(1-x)^2} \right. \\ & \quad + \frac{b_{n+1}}{b_n} \frac{\phi_n}{b_n} \frac{q_n^n x}{(1-q_n^{n+1}x)} + q_m^{2m+1} \frac{b_{m+2}}{b_m} \frac{b_{m+1}}{b_m} \frac{y^2}{(1-q_m^{m+1}y)(1-q_m^{m+2}y)} \\ & \quad \left. - \frac{y^2}{(1-y)^2} + \frac{b_{m+1}}{b_m} \frac{\phi_m}{b_m} \frac{q_m^m y}{(1-q_m^{m+1}y)} \right|. \end{aligned}$$

Now, by applying the Korovkin's type theorem given in [7], we have the desired result. \square

Conflict of interest. The authors declare that they have no conflict of interest.

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