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RECURRENCE RELATIONS FOR THE MOMENTS OF DISCRETE SEMICLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. We study recurrence relations satisfied by the moments $\nu_n(z)$ of a linear functional L whose first moment satisfies a differential equation (in z) with polynomial coefficients.

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