

ON MULTIPLE q -LAGUERRE POLYNOMIALS

P. NJIONOU SADJANG, J. C. MÚNLÚEM MOUNCHAROU
 AND SALIFOU MBOUTNGAM*

Abstract. We study q -Laguerre multiple orthogonal polynomials. These polynomials are orthogonal with respect to q -analogues of Laguerre weight functions. We focus our attention on their structural properties. Raising and lowering operators as well as Rodrigues-type formulas are obtained and their explicit representations are given. A high-order linear q -difference equation with polynomial coefficients is deduced. Moreover, we obtain the nearest neighbor recurrence relation using a q -analogue of the theorem 23.1.11 by M.E.H. Ismail in [12].

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to $r \in \mathbb{N}$ measures μ_1, \dots, μ_r [2, 11, 18]. Throughout this paper r will always be the number of weights. Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite-Padé approximation of a system of r (Markov) functions (see for instance [9, 10, 16]).

There are two types of multiple orthogonal polynomials, for more details about these two kinds we refer the reader to [2] and [12, Chapter 23]. In the present paper we only consider multiple orthogonal polynomials of type II. Let $\vec{n} = (n_1, n_2, \dots, n_r)$ be a vector of r nonnegative integers, which is called a multi-index with length $|\vec{n}| := n_1 + n_2 + \dots + n_r$. Furthermore let $\Omega_1, \dots, \Omega_r$ be the supports of the r measures. A multiple orthogonal polynomial $P_{\vec{n}}$ of type II with respect to the multi-index \vec{n} , is a (nontrivial) polynomial of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\int_{\Omega_j} P_{\vec{n}}(x)x^k d\mu_j(x) = 0, \quad k = 0, \dots, n_j - 1, \quad j = 1, \dots, r. \quad (1)$$

In the literature, one can find many examples of multiple orthogonal polynomials with respect to positive measures on the real line which have the same flavour as the classical orthogonal polynomials. In the continuous case where the measures can be written as $d\mu_j(x) = w_j(x)dx$, with w_j the weight function of the measure μ_j , there are multiple Hermite, multiple Laguerre I and II, Jacobi-Piñeiro, multiple Bessel, Jacobi-Angeleso, Jacobi-Laguerre and Laguerre-Hermite polynomials, see [2, 11, 18] and the

Mathematics subject classification (2020): 33C45, 33D45.

Keywords and phrases: Multiple q -orthogonal polynomials, multiple q -Laguerre polynomials, Rodrigues formula, q -difference equation, nearest neighbor recurrence relation.

* Corresponding author.

references therein. Some classical discrete examples are multiple Charlier, multiple Krawtchouk, multiple Meixner I and II and multiple Hahn introduced [4] and their difference equations are studied in [15]. Multiple Wilson polynomials and multiple Meixner-Pollaczek polynomials are defined and studied respectively in [7] and [8]. The multiple q -Charlier orthogonal polynomials and their structural properties are studied in [6]. In [21], the authors introduced two kinds of the multiple Little q -Jacobi polynomials and gave their explicit expressions. But neither q -difference equation nor recurrence relation is given. Recently, in [20], the authors use special transformations to obtain explicit expressions of multiple Askey-Wilson, multiple continuous dual q -Hahn, and multiple Al-Salam-Chihara polynomials from the multiple Little q -Laguerre and the multiple Little q -Jacobi polynomials. But this transformation does not make it possible to obtain their structural properties.

Since it is known that q -Laguerre polynomials (at least their explicit representation) can be obtained as limit case from Little q -Jacobi polynomials, in this paper the main point will be structural properties of multiple q -Laguerre polynomials. Rodrigues formulas, partial basic hypergeometric representation, explicit representation, high-order linear q -difference equation and the nearest neighbor recurrence relation are discussed.

The structure of the paper is as follows. Section 2, we recall some basic results on q -calculus. In Section 3, we will define the multiple q -Laguerre polynomials and obtain raising operators and then the Rodrigues-type formula using the r orthogonality conditions. In Section 4, an $(r + 1)$ -order q -difference equation is obtained. Section 5 deals with the $(r + 2)$ -term recurrence relations.

2. Preliminary definitions and results

The q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$ are polynomials on the lattice $\{q^k, k = 0, 1, 2, \dots\}$, where $0 < q < 1$. They fulfill the following discrete orthogonality relation [14, p. 522]

$$\int_0^\infty \frac{x^\alpha}{(-x; q)_\infty} L_m^{(\alpha)}(q^k; q) L_n^{(\alpha)}(q^k; q) d_q x = \frac{1 - q}{2} \frac{(q, -q^{\alpha+1}, -q^{-\alpha}; q)_\infty}{(q^{\alpha+1}, -q, -q; q)_\infty} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \delta_{mn},$$

$\alpha > -1.$ (2)

where we use the Jackson q -integral [13]

$$\int_0^\infty f(t) d_q t = (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n), \quad 0 < q < 1.$$
 (3)

Here $(a; q)_n$ stands for the q -Pochhammer symbol and defined by:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \text{ for } n \in \mathbb{N},$$

and $(a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n$, provided that the limit exists. Observe that

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad n \in \mathbb{N}_0, \quad |q| < 1.$$
 (4)

The q -Laguerre polynomials have hypergeometric representation [14]

$$\begin{aligned} L_n^{(\alpha)}(x; q) &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \middle| q; -q^{n+\alpha+1}x \right) \\ &= \frac{1}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \middle| q; q^{n+\alpha+1} \right); \end{aligned} \quad (5)$$

where

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} \frac{z^n}{(q; q)_n},$$

with $(a_1, \dots, a_r)_n := (a_1; q)_n \cdots (a_r; q)_n$.

Using the transformation formula [14, Eq. (1.8.14)]

$$(aq^{-n}; q)_n = (a^{-1}q; q)_n (-a)^n q^{-n-\binom{n}{2}}, \quad a \neq 0,$$

for the specific case $a = 1$

$$(q^{-n}; q)_n = (-1)^n q^{-n-\binom{n}{2}} (q; q)_n, \quad (6)$$

it is not difficult to see that the monic q -Laguerre polynomials are given by

$$\mathcal{L}_n^{(\alpha)}(x; q) = (-1)^n \frac{(q; q)_n}{q^{n(n+\alpha)}} L_n^{(\alpha)}(x; q). \quad (7)$$

They satisfy the *lowering operation*

$$D_q \mathcal{L}_n^{(\alpha)}(x; q) = q^{1-n} \frac{1-q^n}{1-q} \mathcal{L}_{n-1}^{(\alpha+1)}(qx; q), \quad (8)$$

and the *raising operation*

$$D_q \left[w(x, \alpha; q) \mathcal{L}_n^{(\alpha)}(x; q) \right] = \frac{-q^{n+\alpha}}{1-q} w(x, \alpha-1; q) \mathcal{L}_{n+1}^{(\alpha-1)}(x; q), \quad (9)$$

where $w(x, \alpha; q) = \frac{x^\alpha}{(-x; q)_\infty}$ and D_q is the Hahn derivative defined by [13, 14]

$$D_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & x = 0; \end{cases}$$

with the assumption that f is differentiable at 0.

D_q fulfills the power derivative rule [19]

$$\begin{aligned} D_q^n f(x) &= \frac{1}{(1-q)^n x^n} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2} - (n-1)k} f(q^k x) \\ &= \frac{1}{(1-q)^n x^n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k f(q^k x). \end{aligned} \quad (10)$$

where the q -binomial coefficient is defined for positive integers n, k , as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{11}$$

Repeated application of the raising operator gives the *Rodrigues formula*

$$D_q^n w(x, \alpha + n; q) = \frac{q^{n(n+\alpha)}}{(q-1)^n} w(x, \alpha; q) \mathcal{L}_n^{(\alpha)}(x; q). \tag{12}$$

REMARK 1. It is well known that the q -Laguerre and the Little q -Laguerre are related by [14]

$$P_n(x, q^{-\alpha}; q^{-1}) = \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} L_n^{(\alpha)}(-x; q),$$

which for monic polynomials reads

$$p_n(x, q^{-\alpha}; q^{-1}) = \mathcal{L}_n^{(\alpha)}(-x; q). \tag{13}$$

If we change q to q^{-1} , x to $-x$ and take into account that $\frac{1}{(x, q^{-1})_\infty} = (qx; q)_\infty$, $|x| < 1$, then (12) is transformed to the Rodrigues formula for Little q -Laguerre/Wall polynomials in [14, Equation (14.20.10)].

REMARK 2. Taking $a = (1 - q)\mu$ and $x = q^s$, the Little q -Laguerre/Wall polynomials defined in [14, formula (14.20.1)] by:

$$P_n(x, a; q) = \frac{1}{(a^{-1}q^{-n}; q)_n} {}_2\phi_0 \left(\begin{matrix} q^{-n}, x^{-1} \\ q^{\alpha+1} \end{matrix} \middle| q; \frac{x}{a} \right), \tag{14}$$

become the q -Charlier polynomials $C_{q,n}^\mu(s)$ given in [1, formula (85), page 317], i.e ${}_2\phi_0 \left(\begin{matrix} q^{-n}, q^{-s} \\ - \end{matrix} \middle| q; \frac{q^s}{(1-q)\mu} \right)$. This connection establishes a relationship between the q -Laguerre polynomials and the q -Charlier polynomials on the lattice $x(s) = q^s$ due to (13).

Based on these two previous remarks, we will show how algebraic properties of the multiple q -Charlier polynomials studied in [6] are equivalent to those of multiple q -Laguerre polynomials in section 2.

REMARK 3. Notice that the monic multiple q -Charlier polynomials $C_{q,\bar{n}}^{\bar{\mu}}(s)$ on the lattice $y(s) = \frac{q^s - 1}{q - 1}$ studied in [6] and the monic multiple q -Charlier polynomials $C_{\bar{n}}^{\bar{\mu}}(s; q)$ on lattice $x(s) = q^s$ are connected by $C_{\bar{n}}^{\bar{\mu}}(s; q) = (q - 1)^{|\bar{n}|} C_{q,\bar{n}}^{\bar{\mu}}(s)$.

Recall that for any complex number a the q -number $[a]_q$ is defined by [14]: $[a]_q = \frac{1 - q^a}{1 - q}$, $q \neq 1$.

In section 2, we use the following result on the determinant of a particular matrix of Cauchy-type involving q -numbers.

Let C be the matrix defined by

$$C = \begin{pmatrix} \frac{1}{[n_1]_q} & \frac{1}{[n_1 + \alpha_1 - \alpha_2]_q} & \cdots & \frac{1}{[n_1 + \alpha_1 - \alpha_r]_q} \\ \frac{1}{[n_2 + \alpha_2 - \alpha_1]_q} & \frac{1}{[n_2]_q} & \cdots & \frac{1}{[n_2 + \alpha_2 - \alpha_r]_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[n_r + \alpha_r - \alpha_1]_q} & \frac{1}{[n_r + \alpha_r - \alpha_2]_q} & \cdots & \frac{1}{[n_r]_q} \end{pmatrix}. \tag{15}$$

It can be inductively proved by row and column operations using [5, Lemma 3.2] (see also [15, Lemma 2.8]), that the determinant of C is given by:

$$\det C = \frac{q^{\sum_{t=2}^r \sum_{s=t}^r n_s r-1} \prod_{t=1}^r \prod_{s=t+1}^r [\alpha_s - \alpha_t]_q [n_t - n_s + \alpha_t - \alpha_s]_q}{\prod_{t=1}^r \prod_{s=1}^r [n_s + \alpha_s - \alpha_t]_q}. \tag{16}$$

3. Multiple q -Laguerre polynomials

Multiple q -Laguerre polynomials $\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$\int_0^\infty x^k w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2, \dots, r, \tag{17}$$

where $\alpha_1, \dots, \alpha_r > -1$. Observe that all the measures are orthogonality measures for q -Laguerre polynomials with different parameters α_j . All the multi-indices will be normal when we impose the condition that $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, then all the measures are absolutely continuous with respect to $w(x, 0; q) d_q x$ and the system of functions

$$x^{\alpha_1}, x^{\alpha_1+1}, \dots, x^{\alpha_1+n_1-1}, x^{\alpha_2}, x^{\alpha_2+1}, \dots, x^{\alpha_2+n_2-1}, \dots, x^{\alpha_r}, x^{\alpha_r+1}, \dots, x^{\alpha_r+n_r-1}$$

is a Chebyshev system on $(0, \infty)$. Thus the measures $(w(x, \alpha_1; q) d_q x, \dots, w(x, \alpha_r; q) d_q x)$ form an AT-system, which implies that all the multi-indices are normal [18, Theorem 4.3].

REMARK 4. The link between $w(x, \alpha_i; q)$ and the weight functions $v_q^{H_i}(s)$ in [6] (see page 10) makes clear that the orthogonality conditions (17) are equivalent to the

orthogonality conditions (3.5) in [6] for q -Charlier multiple orthogonal polynomials. Therefore, q -Laguerre multiple orthogonal polynomials are q -Charlier multiple polynomials studied in [6].

There are r raising operators for these multiple orthogonal polynomials.

THEOREM 1. *Suppose that $\alpha_1, \dots, \alpha_r > 1$, with $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, then*

$$D_q \left[w(x, \alpha_j, q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) \right] = \frac{-q^{\alpha_j + |\vec{n}|}}{1 - q} w(x, \alpha_j - 1; q) \mathcal{L}_{\vec{n} + \vec{e}_j}^{\vec{\alpha} - \vec{e}_j}(x; q), \tag{18}$$

for $1 \leq j \leq r$, where $\vec{e}_1 = (1, 0, 0, \dots, 0), \dots, \vec{e}_r = (0, \dots, 0, 0, 1)$, are the standard unit vectors.

Proof. We have

$$\begin{aligned} D_q \left[w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) \right] &= \frac{w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) - w(qx, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(qx; q)}{(1 - q)x} \\ &= w(x, \alpha_j - 1; q) \frac{\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) - q^{\alpha_j} (1 + x) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(qx; q)}{1 - q}. \end{aligned}$$

$\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) - q^{\alpha_j} (1 + x) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(qx; q)$ is a polynomial of degree $|\vec{n}| + 1$ with the leading coefficient $-q^{\alpha_j + |\vec{n}|}$. Henceforth

$$D_q \left[w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) \right] = \frac{-q^{\alpha_j + |\vec{n}|}}{1 - q} w(x, \alpha_j - 1; q) Q_{|\vec{n}|+1}(x; q),$$

where $Q_{|\vec{n}|+1}(x; q)$ is a monic polynomial of degree $|\vec{n}| + 1$. We will show that $Q_{|\vec{n}|+1}(x; q)$ satisfies the multiple orthogonality conditions (17) of $\mathcal{L}_{\vec{n} + \vec{e}_j}^{\vec{\alpha} - \vec{e}_j}(x; q)$. Since all $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, the uniqueness of the monic multiple orthogonal polynomials will lead us to the conclude that $Q_{|\vec{n}|+1}(x; q) = \mathcal{L}_{\vec{n} + \vec{e}_j}^{\vec{\alpha} - \vec{e}_j}(x; q)$.

Using the formula of q -integration by parts for q -integral [13], we get:

$$\begin{aligned} &\frac{-q^{\alpha_j + |\vec{n}|}}{1 - q} \int_0^\infty x^k w(x, \alpha_j - 1; q) Q_{|\vec{n}|+1}(x; q) d_q x \\ &= \frac{-q^{\alpha_j + |\vec{n}|}}{1 - q} q^{-k} \int_0^\infty (qx)^k w(x, \alpha_j - 1; q) Q_{|\vec{n}|+1}(x; q) d_q x \\ &= -q^k [k]_q \int_0^\infty x^{k-1} w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x \\ &= 0, \quad k = 0, 1, \dots, n_j. \end{aligned}$$

For the other components α_i ($i \neq j$) of $\vec{\alpha}$, we have

$$\begin{aligned} & \frac{-q^{\alpha_j+|\vec{m}|}}{1-q} \int_0^\infty x^k w(x, \alpha_i; q) \mathcal{Q}_{|\vec{m}|+1}(x; q) d_q x \\ &= \frac{-q^{\alpha_j+|\vec{m}|}}{1-q} q^{-k-\alpha_i+\alpha_j-1} \int_0^\infty (qx)^{k+\alpha_i-\alpha_j+1} w(x, \alpha_j-1; q) \mathcal{Q}_{|\vec{m}|+1}(x; q) d_q x \\ &= \frac{[k+\alpha_i-\alpha_j+1]_q}{q^{k+\alpha_i-\alpha_j+1}} \int_0^\infty x^{k+\alpha_i-\alpha_j} w(x, \alpha_j; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x \\ &= \frac{[k+\alpha_i-\alpha_j+1]_q}{q^{k+\alpha_i-\alpha_j+1}} \int_0^\infty x^k w(x, \alpha_i; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x = 0, \end{aligned}$$

$k = 0, 1, \dots, n_i - 1$. Hence, all the orthogonality conditions for $\mathcal{L}_{\vec{n}+\vec{e}_j}^{\vec{\alpha}-\vec{e}_j}(x; q)$ are satisfied. \square

In the next proposition, we give a partial Rodrigues formula for $\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q)$.

PROPOSITION 1. The polynomials $\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q)$ satisfy the following equations

$$D_q^{n_j} [w(x, \alpha_j; q) \mathcal{L}_{\vec{m}}^{\vec{\alpha}}(x; q)] = \frac{q^{n_j(\alpha_j+|\vec{m}|)}}{(q-1)^{n_j}} w(x, \alpha_j-n_j; q) \mathcal{L}_{\vec{m}+n_j\vec{e}_j}^{\vec{\alpha}-n_j\vec{e}_j}(x; q), \quad (19)$$

$$D_q^{n_j} [w(x, \alpha_j; q)] = \frac{q^{n_j\alpha_j}}{(q-1)^{n_j}} w(x, \alpha_j-n_j; q) \mathcal{L}_{n_j\vec{e}_j}^{\vec{\alpha}-n_j\vec{e}_j}(x; q), \quad (20)$$

for $j = 1, \dots, r$.

Proof. If we apply the raising operator (18) for α_j recursively n_j times to $\mathcal{L}_{\vec{m}}^{\vec{\alpha}}(x; q)$, then

$$\begin{aligned} D_q^{n_j} [w(x, \alpha_j, q) \mathcal{L}_{\vec{m}}^{\vec{\alpha}}(x; q)] &= \left(\frac{-q^{\alpha_j+|\vec{m}|}}{1-q} \right) \left(\frac{-q^{(\alpha_j-1)+|\vec{m}+\vec{e}_j|}}{1-q} \right) \times \\ &\dots \times \left(\frac{-q^{(\alpha_j-n_j+1)+|\vec{m}+(n_j-1)\vec{e}_j|}}{1-q} \right) \\ &\times w(x, \alpha_j-n_j; q) \mathcal{L}_{\vec{m}+n_j\vec{e}_j}^{\vec{\alpha}-n_j\vec{e}_j}(x; q). \end{aligned}$$

By taking $\vec{m} = \vec{0}$ in (19), (20) follows. This ends the proof of the proposition. \square

Note that if in (20), one replaces α_j by α_j+n_j , it follows that

$$D_q^{n_j} [w(x, \alpha_j+n_j; q)] = \frac{q^{n_j(\alpha_j+n_j)}}{(q-1)^{n_j}} w(x, \alpha_j; q) \mathcal{L}_{n_j\vec{e}_j}^{\vec{\alpha}}(x; q), \quad j = 1, \dots, r, \quad (21)$$

which is the Rodrigues formula for the polynomials $\mathcal{L}_{n_j\vec{e}_j}^{\vec{\alpha}}(x; q)$, $j = 1, \dots, r$.

THEOREM 2. *The multiple q -Laguerre polynomials have the following Rodrigues formula representation*

$$\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) = \frac{(q-1)^{|\vec{n}|}(-x; q)_{\infty}}{q^{\sum_{i=1}^r n_i} \left(\alpha_i + \sum_{j=1}^i n_j \right)} \prod_{j=1}^r \left(x^{-\alpha_j} D_q^{n_j} x^{\alpha_j + n_j} \right) \frac{1}{(-x; q)_{\infty}}. \tag{22}$$

Proof. We write relation (20) with $j = 1$ and multiply both sides of the resulting equation by $\frac{w(x, \alpha_2; q)}{w(x, \alpha_1 - n_1; q)} = x^{\alpha_2 - \alpha_1 + n_1}$, to get

$$x^{\alpha_2 - \alpha_1 + n_1} D_q^{n_1} [w(x, \alpha_1; q)] = \frac{q^{n_1 \alpha_1}}{(q-1)^{n_1}} w(x, \alpha_2; q) \mathcal{L}_{n_1 \vec{e}_1}^{\vec{\alpha} - n_1 \vec{e}_1}(x; q).$$

Next, applying (19) with $j = 2$, we get

$$\begin{aligned} D_q^{n_2} [x^{\alpha_2 - \alpha_1 + n_1} D_q^{n_1} [w(x, \alpha_1; q)]] \\ = \frac{q^{n_1 \alpha_1}}{(q-1)^{n_1}} \times \frac{q^{n_2(n_1 + \alpha_2)}}{(q-1)^{n_2}} w(x, \alpha_2 - n_2; q) \mathcal{L}_{n_1 \vec{e}_1 + n_2 \vec{e}_2}^{\vec{\alpha} - n_1 \vec{e}_1 - n_2 \vec{e}_2}(x; q). \end{aligned}$$

Continuing this process we arrive at

$$\begin{aligned} D_q^{n_r} (x^{\alpha_r - \alpha_{r-1} + n_{r-1}} D_q^{n_{r-1}}) (x^{\alpha_{r-1} - \alpha_{r-2} + n_{r-2}} D_q^{n_{r-2}}) \dots (x^{\alpha_2 - \alpha_1 + n_1} D_q^{n_1}) w(x, \alpha_1; q) \\ = \frac{q^{n_1 \alpha_1}}{(q-1)^{n_1}} \times \frac{q^{n_2(n_1 + \alpha_2)}}{(q-1)^{n_2}} \times \dots \times \frac{q^{n_r(n_1 + n_2 + \dots + n_{r-1} + \alpha_r)}}{(q-1)^{n_r}} w(x, \alpha_r - n_r; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha} - \vec{n}}(x; q). \end{aligned}$$

Then we replace each α_j by $\alpha_j + n_j$ to obtain

$$\begin{aligned} (D_q^{n_r} x^{\alpha_r + n_r}) (x^{-\alpha_{r-1}} D_q^{n_{r-1}} x^{n_{r-1}}) \dots (x^{-\alpha_1} D_q^{n_1}) w(x, \alpha_1 + n_1; q) \\ = \frac{q^{\sum_{i=1}^r n_i} \left(\alpha_i + \sum_{j=1}^i n_j \right)}{(q-1)^{|\vec{n}|}} w(x, \alpha_r; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q). \end{aligned}$$

Hence the required expression is indeed proved. \square

As consequence, we have a non-terminated basic hypergeometric representation and explicit expressions using finite sums which are limited cases of the multiple Little q -Jacobi polynomials of the first kind $P_{\vec{n}}(x, \vec{\alpha}, b; q)$ with the normalization $P_{\vec{n}}(0, \vec{\alpha}, b; q) = 1$, ($b = q^{\beta}$), when using the substitution $x \rightarrow -b^{-1}q^{-1}x$ and then taking the limit $b \rightarrow -\infty$ (see [21, Equation 2.7]).

COROLLARY 1. *The multiple q -Laguerre polynomials have the following basic hypergeometric representation*

$$\begin{aligned} \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) = (-x; q)_{\infty} \frac{\prod_{j=1}^r (q^{\alpha_j + 1}; q)_{n_j}}{q^{\sum_{i=1}^r n_i} \left(\alpha_i + \sum_{j=1}^i n_j \right)} {}_{r+1}\phi_r \left(\begin{matrix} q^{\alpha_1 + n_1 + 1}, \dots, q^{\alpha_r + n_r + 1}, 0 \\ q^{\alpha_1 + 1}, \dots, q^{\alpha_r + 1} \end{matrix} \middle| q; -x \right); \\ |x| < 1. \end{aligned}$$

PROPOSITION 2. The multiple q -Laguerre polynomials are given by the relation

$$\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) = K_{q, \vec{n}} \sum_{k_1=0}^{n_1} \dots \sum_{k_r=0}^{n_r} \sum_{m=0}^{|\vec{k}|} \left(\prod_{j=1}^r q^{k_j(n_j+\alpha_j+1)} \frac{(q^{-n_j}; q)_{k_j}}{(q; q)_{k_j}} \right) \left[\begin{matrix} \vec{k} \\ m \end{matrix} \right]_q q^{\binom{m}{2}} x^m, \quad (23)$$

$$\begin{aligned} &= K_{q, \vec{n}} \sum_{k_1=0}^{n_1} \dots \sum_{k_r=0}^{n_r} \prod_{j=1}^r \frac{(q^{-n_j}; q)_{k_j} (q^{\alpha_j + \sum_{i=1}^{j-1} k_i + 1}; q)_{n_j}}{(q; q)_{k_j} (q^{\alpha_j + \sum_{i=1}^{j-1} k_i + 1}; q)_{k_j}} \\ &\quad \times q^{\binom{k_j}{2} + k_j(n_j + \alpha_j + 1)} q^{\sum_{1 \leq i < j \leq r} k_i k_j} x^{|\vec{k}|}, \end{aligned} \quad (24)$$

where $K_{q, \vec{n}} = (-1)^{|\vec{n}|} q^{-\sum_{i=1}^r n_i (\alpha_i + \sum_{j=1}^i n_j)}$.

Proof. The proof of the second expression is the same as the proof of the explicit expression of the multiple Little q -Jacobi polynomials of the first kind. (see [21, Theorem 2.3] or [20, Theorem 3.2]).

Let us now prove (23). We apply (10) recursively to the Rodrigues formula (22) then, setting

$$A_{\vec{n}, q} = \frac{(q-1)^{|\vec{n}|}}{q^{\sum_{i=1}^r n_i (\alpha_i + \sum_{j=1}^i n_j)}},$$

we have

$$\begin{aligned} \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) &= (-x; q)_{\infty} A_{\vec{n}, q} \prod_{j=1}^r \left(x^{-\alpha_j} D_q^{n_j} x^{\alpha_j + n_j} \right) \frac{1}{(-x; q)_{\infty}} \\ &= (-x; q)_{\infty} A_{\vec{n}, q} \prod_{j=2}^r \left(x^{-\alpha_j} D_q^{n_j} x^{\alpha_j + n_j} \right) \left[x^{-\alpha_1} D_q^{n_1} \frac{x^{\alpha_1 + n_1}}{(-x; q)_{\infty}} \right] \\ &= \frac{(-x; q)_{\infty} A_{\vec{n}, q}}{(1-q)^{n_1}} \sum_{k_1=0}^{n_1} q^{k_1(n_1 + \alpha_1 + 1)} \frac{(q^{-n_1}; q)_{k_1}}{(q; q)_{k_1}} \prod_{j=2}^r \left(x^{-\alpha_j} D_q^{n_j} x^{\alpha_j + n_j} \right) \frac{1}{(-q^{k_1} x; q)_{\infty}} \\ &= \frac{(-x; q)_{\infty} A_{\vec{n}, q}}{(1-q)^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \dots \sum_{k_r=0}^{n_r} q^{k_1(n_1 + \alpha_1 + 1)} \frac{(q^{-n_1}; q)_{k_1}}{(q; q)_{k_1}} \dots q^{k_r(n_r + \alpha_r + 1)} \frac{(q^{-n_r}; q)_{k_r}}{(q; q)_{k_r}} \\ &\quad \times \frac{1}{(-q^{|\vec{k}|} x; q)_{\infty}}. \end{aligned}$$

We end by using the relation [14, Eq. 1.9.8]

$$\frac{(-x; q)_{\infty}}{(-q^{|\vec{k}|} x; q)_{\infty}} = (-x; q)_{|\vec{k}|} = \sum_{m=0}^{|\vec{k}|} \left[\begin{matrix} |\vec{k}| \\ m \end{matrix} \right]_q q^{\binom{m}{2}} x^m. \quad \square$$

As mentioned in Remark 1 we establish a connection with the Rodrigues formula of the multiple q -Charlier polynomials.

REMARK 5. Changing q to q^{-1} and x to $-x$, the Rodrigues formula (22) is transformed into

$$\begin{aligned} \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(-x; q^{-1}) &= (1-q)^{|\vec{n}|} q^{-|\vec{n}| + \sum_{i=1}^r n_i} \binom{\alpha_i + \sum_{j=1}^i n_j}{i} (x; q^{-1})_{\infty} \\ &\quad \times \prod_{j=1}^r \left(x^{-\alpha_j} D_{q^{-1}}^{n_j} x^{\alpha_j + n_j} \right) \frac{1}{(x; q^{-1})_{\infty}} \\ &= (-1)^{|\vec{n}|} (q-1)^{|\vec{n}|} q^{-|\vec{n}| + \sum_{i=1}^r n_i} \binom{\alpha_i + \sum_{j=1}^i n_j}{i} \frac{1}{(qx; q)_{\infty}} \\ &\quad \times \prod_{j=1}^r \left(x^{-\alpha_j} D_{q^{-1}}^{n_j} x^{\alpha_j + n_j} \right) (qx; q)_{\infty}, \end{aligned}$$

which is the Rodrigues formula for the monic multiple Little q -Laguerre polynomials $p_{\vec{n}}(x, \vec{\alpha}; q)$ found in [20]. Thus multiple q -Laguerre polynomials and multiple Little q -Laguerre polynomials are related by the following formula

$$p_{\vec{n}}(x, \vec{\alpha}; q) = \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(-x; q^{-1}). \tag{25}$$

Hence the multiple Little q -Laguerre polynomials satisfy the q -difference equation

$$D_{q^{-1}} [w(x, \alpha_j; q) p_{\vec{n}}(x, \vec{\alpha}; q)] = \frac{q^{1-\alpha_j-|\vec{n}|}}{q-1} w(x, \alpha_j - 1; q) p_{\vec{n} + \vec{e}_j}(x, \vec{\alpha} - \vec{e}_j; q), \tag{26}$$

for $1 \leq j \leq r$, with $w(x, \alpha_j; q) = (qx; q)_{\infty} x^{\alpha_j}$, and the Rodrigues formula

$$p_{\vec{n}}(x, \vec{\alpha}; q) = \frac{(1-q)^{|\vec{n}|}}{q^{|\vec{n}| - \sum_{i=1}^r n_i} \binom{\alpha_i + \sum_{j=1}^i n_j}{i}} \frac{1}{(qx; q)_{\infty}} \prod_{j=1}^r \left(x^{-\alpha_j} D_{q^{-1}}^{n_j} x^{\alpha_j + n_j} \right) (qx; q)_{\infty}. \tag{27}$$

We now take $q^{\alpha_j} = (1-q)\mu_j$ for $1 \leq j \leq r$ in the expression of $p_{\vec{n}}(x, \vec{\alpha}; q)$ and use the following observations:

$$w(x, \alpha; q) = (q; q)_{\infty} v_q^{\mu}(s); \quad \nabla x(s+1/2) = (q-1) \nabla y(s+1/2);$$

$$D_{q^{-1}} = q^{1/2} \frac{\nabla}{\nabla x(s+1/2)},$$

where $\nabla f(s) = \Delta f(s-1) = f(s) - f(s-1)$; $v_q^{\mu}(s) = \frac{\mu^s}{\Gamma_q(s+1)}$ and Γ_q is the q -Gamma function given by,

$$\Gamma_q(s) = \begin{cases} f(s; q) = \frac{(q; q)_{\infty}}{(1-q)^{s-1} (q^s; q)_{\infty}}, & 0 < q < 1 \\ q^{\frac{(s-1)(s-2)}{2}} f(s; q^{-1}), & q > 1. \end{cases} \tag{28}$$

The q -Charlier multiple polynomials is given by

$$C_{q,\vec{n}}^{\vec{\mu}}(s) = \frac{\mathcal{L}_{\vec{n}}^{\vec{\mu}}(-q^s; q^{-1})}{(q-1)^{|\vec{n}|}}. \tag{29}$$

They satisfy the raising operations

$$\frac{\nabla}{\nabla x(s+1/2)} \left[v_q^{\mu_j}(s) C_{q,\vec{n}}^{\vec{\mu}}(s) \right] = -\frac{q^{1/2}}{\mu_j q^{|\vec{n}|} (q-1)} v_q^{\mu_{j,1/q}}(s) C_{q,\vec{n}+\vec{e}_j}^{\vec{\mu}}(s) \tag{30}$$

for $1 \leq j \leq r$, with $\vec{\mu} = (\mu_1, \dots, \mu_r)$, $\vec{\mu}_{j,1/q} = (\mu_1, \dots, \mu_j/q, \dots, \mu_r)$ and the Rodrigues formula

$$C_{q,\vec{n}}^{\vec{\mu}}(s) = (1-q)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{j=1}^r \mu_j^{n_j} \right) \left(\prod_{i=1}^r q^{n_i \sum_{j=i}^r n_j} \right) \Gamma_q(s+1) \\ \times \prod_{j=1}^r \left(\mu_j^{-s} \nabla^{n_j} (q^{n_j} \mu_j)^s \right) \frac{1}{\Gamma_q(s+1)}; \tag{31}$$

where $\nabla := \frac{\nabla}{\nabla x(s+1/2)}$.

Equations (30) and (31) are receptively equivalent to equations (4.2) and (4.8) in [6].

4. Difference equations for the multiple q -Laguerre polynomials

The idea here is to find a lowering operator that can be combined with the raising operators (18) to derive a q -difference equation of order $r + 1$. This idea was applied in [12] for multiple Hermite polynomials and in [15] for multiple Charlier and Meixner polynomials. In order to prove the next theorem, we need the following lemmas.

LEMMA 1. *The multiple q -Laguerre polynomials fulfil*

$$\int_0^\infty x^{n_k-1} w(qx, \alpha_k + 1; q) D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x \\ = \frac{q^{\alpha_k+1}}{1-q} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x; \tag{32}$$

and for $i = 1, \dots, r$,

$$\int_0^\infty x^{n_k-1} w(qx, \alpha_k + 1; q) \mathcal{L}_{\vec{n}-\vec{e}_i}^{\vec{\alpha}+\vec{e}_i}(qx; q) d_q x \\ = \theta_{k,i} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x; \tag{33}$$

where $\theta_{k,i} = \frac{q^{|\vec{n}|}}{1-q} \frac{q^{\alpha_k}}{[n_k + \alpha_k - \alpha_i]_q}$.

Proof. We use the following relation

$$D_q w(x, \alpha; q) = \frac{q^\alpha(1+x) - 1}{q - 1} w(x, \alpha - 1; q), \tag{34}$$

to obtain

$$D_q [x^{n_k-1} w(x, \alpha_k + 1; q)] = \frac{x^{n_k-1} [q^{n_k+\alpha_k}(1+x) - 1]}{q - 1} w(x, \alpha_k; q).$$

Then, using the q -integration by parts together with the orthogonality condition of $\mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q)$, we obtain:

$$\begin{aligned} \int_0^\infty x^{n_k-1} w(qx, \alpha_k + 1; q) D_q \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x &= q^{1-n_k} \int_0^\infty (qx)^{n_k-1} w(qx, \alpha_k + 1; q) D_q \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= -q^{1-n_k} \int_0^\infty D_q [x^{n_k-1} w(x, \alpha_k + 1; q)] \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= \frac{q^{\alpha_k+1}}{1-q} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x. \end{aligned}$$

Thus (32) holds.

On the other hand, using the raising operator and again the q -integration by parts, we get for $i = 1, \dots, r$,

$$\begin{aligned} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x &= \int_0^\infty x^{n_k+\alpha_k-\alpha_i} w(x, \alpha_i; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= \frac{q-1}{q^{\alpha_i+|\bar{n}|}} \int_0^\infty x^{n_k+\alpha_k-\alpha_i} D_q [w(x, \alpha_i + 1; q) \mathcal{L}_{\bar{n}-\bar{e}_i}^{\bar{\alpha}+\bar{e}_i}(x; q)] d_q x \\ &= -\frac{(q-1)[n_k + \alpha_k - \alpha_i]_q}{q^{\alpha_i+|\bar{n}|}} \int_0^\infty x^{n_k+\alpha_k-\alpha_i-1} w(qx, \alpha_i + 1; q) \mathcal{L}_{\bar{n}-\bar{e}_i}^{\bar{\alpha}+\bar{e}_i}(qx; q) d_q x \\ &= \frac{q^{-\alpha_k}(1-q)[n_k + \alpha_k - \alpha_i]_q}{q^{|\bar{n}|}} \int_0^\infty x^{n_k-1} w(qx, \alpha_k + 1; q) \mathcal{L}_{\bar{n}-\bar{e}_i}^{\bar{\alpha}+\bar{e}_i}(qx; q) d_q x. \end{aligned}$$

which is (33). \square

The multiple q -Laguerre polynomials satisfy the following q -derivative rule

THEOREM 3. *The multiple q -Laguerre polynomials satisfy*

$$D_q \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) = \sum_{j=1}^r d_j \mathcal{L}_{\bar{n}-\bar{e}_j}^{\bar{\alpha}+\bar{e}_j}(qx; q), \tag{35}$$

with

$$d_j = \frac{q^{1-|\vec{n}|} \prod_{s=1}^r [n_s + \alpha_j - \alpha_t]_q}{\prod_{t=1, t \neq j}^r [\alpha_j - \alpha_t]_q \prod_{t=j+1}^r [\alpha_t - \alpha_j]_q} \times \sum_{i=1}^r \frac{(-1)^{i+j} q^{\xi(\vec{n}, \vec{\alpha}, i)} \prod_{t=1}^r [n_i + \alpha_i - \alpha_t]_q}{[n_i + \alpha_i - \alpha_j]_q \prod_{t=1, t \neq i}^{r-1} [n_t - n_i + \alpha_t - \alpha_i]_q \prod_{t=i+1}^r [n_i - n_t + \alpha_i - \alpha_t]_q}; \tag{36}$$

where $\xi(\vec{n}, \vec{\alpha}, 1) = -\sum_{s=2}^r n_s$ and for $2 \leq i \leq r$

$$\xi(\vec{n}, \vec{\alpha}, i) = \sum_{s=2}^{i-1} (\alpha_s - \alpha_i) - (i-1)n_i - \sum_{s=i+1}^r n_s.$$

In particular for $r = 2$, $\vec{n} = (n, m)$ and $\vec{\alpha} = (\alpha, \beta)$, (35) reduces to

$$D_q \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x; q) = q^{1-m-n} \frac{[n]_q [m + \beta - \alpha]_q}{[\beta - \alpha]_q} \mathcal{L}_{(n-1,m)}^{(\alpha+1,\beta)}(qx; q) + q^{1-m-n} \frac{[m]_q [n + \alpha - \beta]_q}{[\alpha - \beta]_q} \mathcal{L}_{(n,m-1)}^{(\alpha,\beta+1)}(qx; q).$$

Proof. We denote by \mathcal{P} the linear subspace of polynomials of degree $\leq |\vec{n}| - 1$ which are orthogonal to all polynomials of degree $\leq n_j - 2$ with respect to $w(qx, \alpha_j + 1, q)$, $j = 1, \dots, r$. It is easy to see that each polynomial $\mathcal{L}_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(qx; q)$ belongs to \mathcal{P} .

Using integration by parts together with the relation (34) and finally the orthogonality conditions of $\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q)$ we get:

$$\begin{aligned} \int_0^\infty x^k w(qx, \alpha_j + 1; q) D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x &= q^{-k} \int_0^\infty (qx)^k w(qx, \alpha_j + 1; q) D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x \\ &= -\frac{q^{-k}}{q-1} \int_0^\infty x^k \left[q^{\alpha_j+k+1} (1+x) - 1 \right] w(x, \alpha_j; q) \\ &\quad \times \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) d_q x \\ &= 0, \quad k = 0, \dots, n_j - 2. \end{aligned}$$

Since $D_q \mathcal{L}_{\vec{n}}^{(\vec{\alpha})}(x; q)$ is a polynomial of degree $|\vec{n}| - 1$, this shows that $D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) \in \mathcal{P}$. In the next step, we will show that $D_q \mathcal{L}_{\vec{n}}^{(\vec{\alpha})}(x; q)$ can be represented by a linear combination of polynomials $\mathcal{L}_{\vec{n}-\vec{e}_j}^{(\vec{\alpha}+\vec{e}_j)}(qx; q)$, $j = 1, \dots, r$.

All indices being normal, any polynomial of \mathcal{P} can be written with $|\vec{n}|$ coefficients and $(|\vec{n}| - r)$ conditions are imposed on \mathcal{P} ; so $\dim \mathcal{P} \leq r$.

Assume that

$$\sum_{j=1}^r d_j \mathcal{L}_{\bar{n}-\bar{e}_j}^{\bar{\alpha}+\bar{e}_j}(qx; q) = 0; \tag{37}$$

where all the d_j are constants. Multiplying (37) by x^{n_k-1} and integrating with respect to $w(qx, \alpha_k + 1, q)$, we get

$$\sum_{j=1}^r d_j \int_0^\infty x^{n_k-1} w(qx, \alpha_k + 1, q) \mathcal{L}_{\bar{n}-\bar{e}_j}^{\bar{\alpha}+\bar{e}_j}(qx; q) d_q x = 0.$$

Using (33) leads to

$$\left(\frac{q^{|\bar{n}|+\alpha_k}}{q-1} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \right) \sum_{j=1}^r \frac{1}{[n_k + \alpha_k - \alpha_j]_q} d_j = 0, \quad k = 1, 2, \dots, r;$$

and since $\frac{q^{|\bar{n}|+\alpha_k}}{q-1} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \neq 0$, we obtain

$$\sum_{j=1}^r \frac{1}{[n_k + \alpha_k - \alpha_j]_q} d_j = 0, \quad k = 1, 2, \dots, r;$$

which can be written $Cd^T = 0$, where C is given by (15) and $d = (d_1, \dots, d_r)$.

Since, $\det C \neq 0$ as given in (16), then $d_j = 0$ for $j = 1, \dots, r$.

Hence we can write $D_q \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q)$ as linear combination of polynomials $\mathcal{L}_{\bar{n}-\bar{e}_j}^{\bar{\alpha}+\bar{e}_j}(qx; q)$, $j = 1, \dots, r$.

To complete the proof, we compute the coefficients of the linear combination. We suppose that $n_j \geq 1$, $j = 1, \dots, r$ and

$$D_q \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) = \sum_{j=1}^r d_j \mathcal{L}_{\bar{n}-\bar{e}_j}^{\bar{\alpha}+\bar{e}_j}(qx; q).$$

If we multiply both sides by x^{n_k-1} and integrate with respect to $w(qx, \alpha_k + 1; q)$, $k = 1, \dots, r$, then we get after using Lemma 1,

$$\sum_{j=1}^r d_j \theta_{k,j} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x = \frac{q^{\alpha_k+1}}{1-q} \int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x$$

which gives, since $\int_0^\infty x^{n_k} w(x, \alpha_k; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \neq 0$,

$$\sum_{j=1}^r \theta_{k,j} d_j = \frac{q^{\alpha_k+1}}{1-q} \Leftrightarrow \sum_{j=1}^r \frac{1}{[n_k + \alpha_k - \alpha_j]} \left(q^{|\bar{n}|-1} d_j \right) = 1, \quad k = 1, 2, \dots, r.$$

This last relation can be written as:

$$C\tilde{d} = e^T,$$

where $e = (1, \dots, 1)$; $\tilde{d}_j = q^{|\tilde{m}|-1}d_j$.

We have $\det C_j = \sum_{i=1}^r (-1)^{i+j} \det C_{ij}$, where C_{ij} is the (i, j) -minor of C , and using again (16),

$$\det C_{1j} = \frac{q^{\sum_{t=2, t \neq 2s=t, s \neq 2}^r \sum_{n_s}^r} \prod_{t=1, t \neq j, s=t+1, s \neq j}^{r-1} \prod_{t=1, s \neq j}^r [\alpha_s - \alpha_t]_q \prod_{t=1, t \neq 1, s=t+1, s \neq 1}^{r-1} \prod_{t=1, s \neq 1}^r [n_t - n_s + \alpha_t - \alpha_s]_q}{\prod_{s=1, s \neq 1}^r \prod_{t=1, t \neq j}^r [n_s + \alpha_s - \alpha_t]_q},$$

and for $2 \leq i \leq r$,

$$\det C_{ij} = q^{\sum_{t=2, t \neq i, s=t, s \neq i}^r \sum_{n_s}^r + \sum_{s=2}^{i-1} (\alpha_s - \alpha_i)} \times \frac{\prod_{t=1, t \neq j, s=t+1, s \neq j}^{r-1} \prod_{t=1, s \neq j}^r [\alpha_s - \alpha_t]_q \prod_{t=1, t \neq i, s=t+1, s \neq i}^{r-1} \prod_{t=1, s \neq i}^r [n_t - n_s + \alpha_t - \alpha_s]_q}{\prod_{s=1, s \neq i}^r \prod_{t=1, t \neq j}^r [n_s + \alpha_s - \alpha_t]_q}.$$

Putting together these last two relations we obtain

$$\frac{\det C_{ij}}{\det C} = q^{\xi(\vec{n}, \vec{\alpha}, i)} \frac{\prod_{s=1}^r [n_s + \alpha_j - \alpha_t]_q}{\prod_{t=1, t \neq j}^r [\alpha_j - \alpha_t]_q \prod_{t=j+1}^r [\alpha_t - \alpha_j]_q \prod_{t=1}^r [n_i + \alpha_i - \alpha_t]_q} \times \frac{[n_i + \alpha_i - \alpha_j]_q \prod_{t=1, t \neq i}^{r-1} [n_t - n_i + \alpha_t - \alpha_i]_q \prod_{t=i+1}^r [n_i - n_t + \alpha_i - \alpha_t]_q}{[n_i + \alpha_i - \alpha_j]_q \prod_{t=1, t \neq i}^{r-1} [n_t - n_i + \alpha_t - \alpha_i]_q \prod_{t=i+1}^r [n_i - n_t + \alpha_i - \alpha_t]_q},$$

with $\xi(\vec{n}, \vec{\alpha}, 1) = -\sum_{s=2}^r n_s$ and $\xi(\vec{n}, \vec{\alpha}, i) = \sum_{s=2}^{i-1} (\alpha_s - \alpha_i) - (i-1)n_i - \sum_{s=i+1}^r n_s$.

Finally $d_j = \frac{q^{1-|\vec{n}|}}{\det C} \sum_{i=1}^r (-1)^{i+j} \det C_{ij}$ which is equivalent to (36). \square

REMARK 6. In (37), if we take $m = 0$ or $n = 0$ then we obtain respectively

$$D_q \mathcal{L}_{(n,0)}^{(\alpha,\beta)}(x; q) = q^{1-n} \frac{1 - q^n}{1 - q} \mathcal{L}_{(n-1,0)}^{(\alpha+1,\beta)}(qx; q)$$

and

$$D_q \mathcal{L}_{(0,m)}^{(\alpha,\beta)}(x; q) = q^{1-m} \frac{1 - q^m}{1 - q} \mathcal{L}_{(0,m-1)}^{(\alpha,\beta+1)}(qx; q),$$

which both are equivalent to (8), since $\mathcal{L}_{(0,m)}^{(\alpha,\beta)}(x; q) = \mathcal{L}_m^{(\beta)}(x; q)$ and $\mathcal{L}_{(n,0)}^{(\alpha,\beta)}(x; q) = \mathcal{L}_n^{(\alpha)}(x; q)$.

Combining raising operators (18) and the lowering operator (35), we find a linear differential equation of order $r + 1$ for the multiple q -Laguerre polynomials.

THEOREM 4. *The multiple q -Laguerre polynomials satisfy*

$$\begin{aligned} & \left[\prod_{i=1}^r (x^{-\alpha_i} D_q x^{\alpha_i+1}) \right] \left(\frac{D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q)}{(-qx; q)_{\infty}} \right) \\ &= \frac{q^{|\vec{n}|}}{q-1} \sum_{j=1}^r d_j q^{\alpha_j} \left[\prod_{i=1, i \neq j}^r (x^{-\alpha_i} D_q x^{\alpha_i+1}) \right] \left(\frac{\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(qx; q)}{(-qx; q)_{\infty}} \right), \end{aligned} \tag{38}$$

where d_j are defined as in Theorem 3.

Proof. We apply the operator $\left[\prod_{i=1}^r (x^{-\alpha_i} D_q x^{\alpha_i+1}) \right] \frac{1}{(-qx; q)_{\infty}}$ to (35) then, the use of the raising operator (18) and the relation $D_q f(qx) = q(D_q f)(qx)$ in the right hand side leads to

$$\begin{aligned} & \left[\prod_{i=1}^r (x^{-\alpha_i} D_q x^{\alpha_i+1}) \right] \left(\frac{D_q \mathcal{L}_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(qx; q)}{(-qx; q)_{\infty}} \right) \\ &= \frac{q^{|\vec{n}|+\alpha_j}}{q-1} \left[\prod_{i=1, i \neq j}^r (x^{-\alpha_i} D_q x^{\alpha_i+1}) \right] \left(\frac{\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(qx; q)}{(-qx; q)_{\infty}} \right); \end{aligned}$$

since the operators $x^{-\alpha_i} D_q x^{\alpha_i+1}$, $i = 1, \dots, r$ are commuting. \square

COROLLARY 2. *The multiple q -Laguerre polynomials $\mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x; q)$ (for $r = 2$) satisfy the following third-order q -difference equation:*

$$a_4(x; q) D_q^3 y(x) + a_3(x; q) D_q^2 y(x) + a_2(x; q) D_q y(x) + a_1(x; q) D_q y(qx) + a_0(x; q) y(qx) = 0, \tag{39}$$

where

$$\begin{aligned} a_4(x; q) &= q^{\alpha+\beta+3} (q-1)^2 x^2 (1+qx)(1+q^2x); \\ a_3(x; q) &= (q-1)x(1+qx) \left[q^{\alpha+\beta+5} (1+q)x + q^{\alpha+\beta+2} (1+q) - q^{\beta+1} - q^{\alpha+1} \right]; \\ a_2(x; q) &= (1+qx) \left[q^{\alpha+\beta+2} (1+q^2x) - q^{\alpha+1} - q^{\beta+1} \right] - 1; \\ a_1(x; q) &= -q^{\alpha+\beta+m+n+1} (d_1 + d_2)x(1+qx); \\ a_0(x; q) &= -q^{m+n} \left[q^{\alpha+\beta+1} (d_1 + d_2)(1+qx) - d_1 q^{\alpha} - d_2 q^{\beta} \right]; \\ d_1 &= q^{1-m-n} \frac{[n]_q [m + \beta - \alpha]_q}{[\beta - \alpha]_q}; \quad d_2 = q^{1-m-n} \frac{[m]_q [n + \alpha - \beta]_q}{[\alpha - \beta]_q}. \end{aligned}$$

Proof. By Theorem 38, each polynomial $\mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x;q)$ satisfies a q -difference equation

$$\begin{aligned} & (x^{-\beta}D_q x^{\beta+1})(x^{-\alpha}D_q x^{\alpha+1}) \left(\frac{D_q \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x;q)}{(-qx;q)_\infty} \right) \\ & - \frac{d_1 q^{n+m+\alpha}}{q-1} x^{-\beta} \left(D_q \frac{x^{\beta+1}}{(-qx;q)_\infty} \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(qx;q) \right) \\ & - \frac{d_2 q^{n+m+\beta}}{q-1} x^{-\alpha} \left(D_q \frac{x^{\alpha+1}}{(-qx;q)_\infty} \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(qx;q) \right) = 0, \end{aligned}$$

from which the conclusion follows after some calculations. \square

Let us now clarify the relationship between the difference equation (38) and the one satisfy by the q -Chalier multiple polynomials in [6].

REMARK 7. Observe that the change of q to q^{-1} , x to $-x$ following by a second change of x to qx , transform the lowering operator (35) to

$$D_q \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(-x;q^{-1}) = \sum_{j=1}^r \lambda_j \mathcal{L}_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(-x;q^{-1}), \tag{40}$$

which is the lowering operator of the multiple Little q -Laguerre polynomials; where λ_j is obtained by changing q to q^{-1} in the expression of d_j .

Thus due to the substitutions mentioned in Remark 1 the q -Charlier multiple orthogonal polynomials satisfy the following q -difference equation

$$\begin{aligned} & \left[\prod_{i=1}^r \left(\mu_j^{-s} \nabla^{n_j} (q^{n_j} \mu_j)^s \right) \right] \left(\Delta C_{q,\vec{n}}^{\vec{\mu}}(s) \right) \\ & = - \sum_{j=1}^r q^{|\vec{n}|} \mu_j \lambda_{\vec{\mu},j} \left[\prod_{i=1, i \neq j}^r \left(\mu_i^{-s} \nabla^{n_i} (q^{n_i} \mu_i)^s \right) \right] C_{q,\vec{n}}^{\vec{\mu}}(s), \end{aligned} \tag{41}$$

where $\lambda_{\vec{\mu},j}$ is obtained by changing q^{α_j} to $(1-q)\mu_j$ in the expression of λ_j and $\Delta := \frac{\Delta}{\Delta x(s-1/2)}$.

5. Recurrence relations for the multiple q -Laguerre polynomials

Type II multiple orthogonal polynomials are known to satisfy nearest neighbor recurrence relations of the form [12, Theorem.23.1.11]

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \tag{42}$$

where $k = 1, \dots, r$ and

$$a_{\vec{n},j} = \frac{\int_{\mathbb{R}} x^{n_j} w_j(x) P_{\vec{n}}(x) dx}{\int_{\mathbb{R}} x^{n_j-1} w_j(x) P_{\vec{n}-\vec{e}_j}(x) dx}. \tag{43}$$

For multiple q -Laguerre polynomials, $a_{\vec{n},j}$ can be obtained in the same way as in the proof of the [12, Theorem 23.1.11], but with discrete measures.

THEOREM 5. *The nearest neighbor recurrence relations for multiple q -Laguerre polynomials are*

$$\begin{aligned} x \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) &= \mathcal{L}_{\vec{n}+\vec{e}_k}^{\vec{\alpha}}(x; q) - q^{-|\vec{n}|} \left(1 - q^{-1-\alpha_k-n_k} - \sum_{j=1}^r q^{-\alpha_j} (q^{-n_j} - 1) \right) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) \\ &+ \sum_{j=1}^r q^{1-2\alpha_j-2n_j-|\vec{n}|_j-|\vec{n}|} (q-1)^2 [n_j]_q [n_j + \alpha_j]_q \\ &\times \prod_{i=1, i \neq j}^r \frac{[n_j + \alpha_j - \alpha_i]_q}{[n_j - n_i + \alpha_j - \alpha_i]_q} \mathcal{L}_{\vec{n}-\vec{e}_j}^{\vec{\alpha}}(x; q), \end{aligned} \tag{44}$$

where $|\vec{n}|_j = \sum_{i=1}^j n_i$.

Proof. From (23) and after some calculations using the relations

$$\begin{aligned} \frac{(q^{-n}; q)_n}{(q; q)_n} &= (-1)^n q^{-\frac{n(n+1)}{2}}; \\ \frac{(q^{-n}; q)_{n-1}}{(q; q)_{n-1}} &= (-1)^{n-1} [n]_q q^{-\frac{(n-1)(n+2)}{2}}; \\ \sum_{i=1}^r n_i (\alpha_i + \sum_{j=1}^i n_j) &= \sum_{i=1}^r n_i \alpha_i + \sum_{i=1}^r n_i^2 + \sum_{i=1}^{r-1} \sum_{j=i+1}^r n_i n_j, \end{aligned}$$

we find that

$$\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(x; q) = x^{|\vec{n}|} + \delta_{q, \vec{n}} x^{|\vec{n}|-1} + \dots,$$

where

$$\delta_{q, \vec{n}} = q^{1-|\vec{n}|} [[\vec{n}]]_q - \sum_{j=1}^r [n_j]_q q^{1-\alpha_j-n_j-|\vec{n}|}.$$

If we identify the coefficients of $x^{|\vec{n}|}$ in (42), then $b_{\vec{n},k} = \delta_{q, \vec{n}} - \delta_{q, \vec{n}+\vec{e}_k}$, which gives

$$b_{\vec{n},k} = -q^{-|\vec{n}|} \left(1 - q^{-1-\alpha_k-n_k} - \sum_{j=1}^r q^{-\alpha_j} (q^{-n_j} - 1) \right). \tag{45}$$

For the recurrence coefficients $a_{\bar{n},j}$, we use (43) and the Rodrigues formula where we

$$\text{get } \tilde{K}_{q,\bar{n}} = (q-1)^{|\bar{n}|} q^{-\sum_{i=1}^r n_i} \binom{\alpha_i + \sum_{j=1}^i n_j}{i}.$$

$$\begin{aligned} & \int_0^\infty x^{n_j} w(x, \alpha_j; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= \tilde{K}_{q,\bar{n}} \int_0^\infty x^{n_j + \alpha_j - \alpha_1} D_q^{n_1} x^{n_1 + \alpha_1} \prod_{i=2}^r (x^{-\alpha_i} D_q^{n_i} x^{\alpha_i + n_i}) \frac{1}{(-x; q)_\infty} d_q x; \end{aligned}$$

and the integration by parts (n_1 times) leads to

$$\begin{aligned} & \int_0^\infty x^{n_j} w(x, \alpha_j; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= (-1)^{n_1} \tilde{K}_{q,\bar{n}} \left(\prod_{l=0}^{n_1-1} [n_j + \alpha_j - \alpha_1 - l]_q \right) q^{-n_1(n_j + \alpha_j - \alpha_1) + \frac{n_1(\alpha_1 - 1)}{2}} \\ & \times \int_0^\infty x^{n_j + \alpha_j} \prod_{i=2}^r (x^{-\alpha_i} D_q^{n_i} x^{\alpha_i + n_i}) \frac{1}{(-x; q)_\infty} d_q x. \end{aligned}$$

Repeating this process r times gives

$$\begin{aligned} & \int_0^\infty x^{n_j} w(x, \alpha_j; q) \mathcal{L}_{\bar{n}}^{\bar{\alpha}}(x; q) d_q x \\ &= (-1)^{|\bar{n}|} \tilde{K}_{q,\bar{n}} \left(\prod_{i=1}^r \prod_{l=0}^{n_i-1} [n_j + \alpha_j - \alpha_i - l]_q \right) q^{-\sum_{i=1}^r n_i(n_j + \alpha_j - \alpha_i) + \sum_{i=1}^r \frac{n_i(n_i-1)}{2}} \int_0^\infty \frac{x^{n_j + \alpha_j}}{(-x; q)_\infty} d_q x. \end{aligned}$$

Finally we use (43) with

$$\int_0^\infty \frac{x^{n_j + \alpha_j}}{(-x; q)_\infty} d_q x = (1-q) q^{-\alpha_j - n_j} [n_j + \alpha_j]_q \int_0^\infty \frac{x^{n_j + \alpha_j - 1}}{(-x; q)_\infty} d_q x,$$

to find that

$$a_{\bar{n},j} = q^{1-2\alpha_j-2n_j-|\bar{n}|j-|\bar{n}|} (q-1)^2 [n_j]_q [n_j + \alpha_j]_q \prod_{i=1, i \neq j}^r \frac{[n_j + \alpha_j - \alpha_i]_q}{[n_j - n_i + \alpha_j - \alpha_i]_q}. \quad (46)$$

□

COROLLARY 3. *The multiple q -Laguerre polynomials $\mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x; q)$ (for $r = 2$) satisfy the following recurrence relations:*

$$\begin{aligned} x \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x; q) &= \mathcal{L}_{(n+1,m)}^{(\alpha,\beta)}(x; q) \\ & - q^{-n-m} \left[1 - q^{-1-\alpha-n} - q^{-\alpha}(q^{-n} - 1) - q^{-\beta}(q^{-m} - 1) \right] \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x; q) \\ & + q^{1-2\alpha-4n-m} (q-1)^2 [n]_q [n + \alpha]_q \frac{[n + \alpha - \beta]_q}{[n - m + \alpha - \beta]_q} \mathcal{L}_{(n-1,m)}^{(\alpha,\beta)}(x; q) \\ & + q^{1-2\beta-2n-4m} (q-1)^2 [m]_q [m + \beta]_q \frac{[m + \beta - \alpha]_q}{[m - n + \beta - \alpha]_q} \mathcal{L}_{(n,m-1)}^{(\alpha,\beta)}(x; q). \end{aligned} \quad (47)$$

$$\begin{aligned}
 x\mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x;q) &= \mathcal{L}_{(n,m+1)}^{(\alpha,\beta)}(x;q) \\
 &\quad -q^{-n-m} \left[1 - q^{-1-\beta-m} - q^{-\beta}(q^{-m} - 1) - q^{-\alpha}(q^{-n} - 1) \right] \mathcal{L}_{(n,m)}^{(\alpha,\beta)}(x;q) \\
 &\quad + q^{1-2\alpha-4n-m}(q-1)^2 [n]_q [n+\alpha]_q \frac{[n+\alpha-\beta]_q}{[n-m+\alpha-\beta]_q} \mathcal{L}_{(n-1,m)}^{(\alpha,\beta)}(x;q) \\
 &\quad + q^{1-2\beta-2n-4m}(q-1)^2 [m]_q [m+\beta]_q \frac{[m+\beta-\alpha]_q}{[m-n+\beta-\alpha]_q} \mathcal{L}_{(n,m-1)}^{(\alpha,\beta)}(x;q).
 \end{aligned} \tag{48}$$

REMARK 8. If we take $m = 0$ in (47) or $n = 0$ in (48) then we obtain

$$\begin{aligned}
 x\mathcal{L}_{(n,0)}^{(\alpha,\beta)}(x;q) &= \mathcal{L}_{(n+1,0)}^{(\alpha,\beta)}(x;q) + q^{-2n-\alpha-1} \left[(1 - q^{n+1}) + q(1 - q^{\alpha+n}) \right] \mathcal{L}_{(n,0)}^{(\alpha,\beta)}(x;q) \\
 &\quad + q^{-4n-2\alpha+1} (1 - q^n) (1 - q^{n+\alpha}) \mathcal{L}_{(n-1,0)}^{(\alpha,\beta)}(x;q).
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 x\mathcal{L}_{(0,m)}^{(\alpha,\beta)}(x;q) &= \mathcal{L}_{(0,m+1)}^{(\alpha,\beta)}(x;q) + q^{-2m-\beta-1} \left[(1 - q^{m+1}) + q(1 - q^{\beta+m}) \right] \mathcal{L}_{(0,m)}^{(\alpha,\beta)}(x;q) \\
 &\quad + q^{-4m-2\beta+1} (1 - q^m) (1 - q^{m+\beta}) \mathcal{L}_{(0,m-1)}^{(\alpha,\beta)}(x;q).
 \end{aligned} \tag{50}$$

which are both equivalent to the three terms recurrence relation for monic q -Laguerre polynomials [14, Equation 14.21.6].

In the same way as for the Rodrigues formula, we can obtain the nearest neighbor recurrence relations for multiple Little q -Laguerre polynomials by changing x to $-x$ and q to q^{-1} :

REMARK 9. Notice that:

$$\begin{aligned}
 -x\mathcal{L}_{\vec{n}}^{\vec{\alpha}}(-x;q^{-1}) &= \mathcal{L}_{\vec{n}+\vec{e}_k}^{\vec{\alpha}}(-x;q^{-1}) - q^{|\vec{n}|} \left(1 - q^{1+\alpha_k+n_k} - \sum_{j=1}^r q^{\alpha_j} (q^{n_j} - 1) \right) \mathcal{L}_{\vec{n}}^{\vec{\alpha}}(-x;q^{-1}) \\
 &\quad + \sum_{j=1}^r q^{\alpha_j+|\vec{n}|_j+|\vec{n}|-1} (q-1)^2 [n_j]_q [n_j+\alpha_j]_q \\
 &\quad \times \prod_{i=1, i \neq j}^r \frac{q^{-n_i} [n_j+\alpha_j-\alpha_i]_q}{[n_j-n_i+\alpha_j-\alpha_i]_q} \mathcal{L}_{\vec{n}-\vec{e}_j}^{\vec{\alpha}}(-x;q^{-1}).
 \end{aligned}$$

Hence the multiple Little q -Laguerre polynomials satisfy the following nearest neighbor recurrence relations

$$\begin{aligned}
 xp_{\vec{n}}(x, \vec{\alpha}; q) &= p_{\vec{n}+\vec{e}_k}(x, \vec{\alpha}; q) + q^{|\vec{n}|} \left(1 - q^{1+\alpha_k+n_k} - \sum_{j=1}^r q^{\alpha_j} (q^{n_j} - 1) \right) p_{\vec{n}}(x, \vec{\alpha}; q) \\
 &\quad + \sum_{j=1}^r q^{\alpha_j+|\vec{n}|_j-1} (q-1)^2 [n_j]_q [n_j+\alpha_j]_q \\
 &\quad \times \prod_{i=1, i \neq j}^r \frac{[n_j+\alpha_j-\alpha_i]_q}{[n_j-n_i+\alpha_j-\alpha_i]_q} p_{\vec{n}-\vec{e}_j}(x, \vec{\alpha}; q);
 \end{aligned} \tag{51}$$

and taking $q^{\alpha_j} = (1-q)\mu_j$ and $x = q^s$ one gets the nearest neighbor recurrence relations for q -Charlier multiple orthogonal polynomials $C_{q,\bar{n}}^{\bar{\mu}}(s; q)$:

$$\begin{aligned} q^s C_{q,\bar{n}}^{\bar{\mu}}(s; q) &= C_{\bar{n}+\bar{e}_k}^{\bar{\mu}}(s; q) + q^{|\bar{n}|} \left[1 - (1-q) \left(\mu_k q^{n_k+1} + \sum_{j=1}^r \mu_j (q^{n_j} - 1) \right) \right] C_{q,\bar{n}}^{\bar{\mu}}(s; q) \\ &+ \sum_{j=1}^r (1-q) \mu_j q^{|\bar{n}|j-1} (1-q^{n_j}) (\mu_j q^{n_j} (1-q) - 1) \\ &\times \prod_{i=1, i \neq j}^r \frac{q^{n_j+n_i} \mu_j - q^{n_i} \mu_i}{q^{n_j} \mu_j - q^{n_i} \mu_i} C_{\bar{n}-\bar{e}_j}^{\bar{\mu}}(s; q). \end{aligned} \quad (52)$$

Conclusion

We have studied multiple q -Laguerre orthogonal polynomials starting from a vector of q -Laguerre weight functions and working out their Rodrigues formulas and their explicit expressions. We showed that they satisfy an $(r+1)$ -order q -difference equation and an $(r+2)$ -term recurrence relation. An interesting fact is that, the explicit expression of the coefficients of this relation can be used to study some type of asymptotic behavior of these polynomials. For instance, in [21] the author used the explicit expression of the multiple Little q -Jacobi polynomials to study their asymptotic behavior. The recurrence relations are used to study the asymptotic behavior of multiple Meixner polynomials of the first and second kind in [3] and to study the asymptotic behavior of the ratio of two multiple Charlier polynomials in [17]. The results in this paper can also be useful to the study of asymptotic distribution of zeros of multiple q -Laguerre polynomials.

REFERENCES

- [1] R. ÁLVAREZ-NODARSE AND J. ARVESÚ, *On the q -polynomials in the exponential lattice*, Integr. Transf. Spec. F. **8**, (1999), 299–324.
- [2] A. I. APTEKAREV, *Multiple orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), 423–447.
- [3] A. I. APTEKAREV, J. ARVESÚ, *Asymptotics for multiple Meixner polynomials*, J. Math. Anal. Appl. **411** (2014), 485–505, [arXv:1207.0463](https://arxiv.org/abs/1207.0463).
- [4] J. ARVESÚ, J. COUSSEMENT, W. VAN ASSCHE, *Some discrete multiple orthogonal polynomials*, J. Comput. Appl. Math. **153** (2003), 19–45.
- [5] J. ARVESÚ, C. ESPOSITO, *A high-order q -difference equation for q -Hahn multiple orthogonal polynomials*, J. Difference Equ. Appl. **18** (2012), 833–847, <http://dx.doi.org/10.1080/10236198.2010.524211>.
- [6] J. ARVESÚ, A. M. RAMÍREZ-ABERASTURIS, *On q -Charlier multiple orthogonal polynomials*, Symmetry, Integrability and Geometry: Methods and Applications **11** (2015), Paper 026, 14 pp.
- [7] B. BECKERMANN, J. COUSSEMENT, W. VAN ASSCHE, *Multiple Wilson and Jacobi-Piñeiro polynomials*, J. Approx. Theory **132** (2005) 155–181.
- [8] M. BENDER, S. DELVAUX, A. B. J. KUIJLAARS, *Multiple Meixner-Pollaczek polynomials and the six-vertex model*, [arXv:1101.2982v2](https://arxiv.org/abs/1101.2982v2).
- [9] M. G. DE BRUIN, *Simultaneous Padé approximation and orthogonality*, in 'Polynomes Orthogonaux et Applications' (C. Brezinski et al. eds), Lecture Notes in Mathematics **1171**, Springer-Verlag, Berlin, 1985, 74–83.

- [10] M. G. DE BRUIN, *Some aspects of simultaneous rational approximation*, in 'Numerical Analysis and Mathematical Modeling', Banach Center Publications **24**, PWN-Polish Scientific Publishers, Warsaw, 1990, 51–83.
- [11] E. COUSSEMENT, W. VAN ASSCHE, *Some classical multiple orthogonal polynomials*, J. Comput. Appl. Math. **127** (2001), 317–347.
- [12] M.E.H. ISMAIL, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications **98**, Cambridge University Press, 2005 (paperback edition 2009).
- [13] V. KAC, P. CHEUNG, *Quantum calculus*, Springer, (2001).
- [14] R. KOEKOEK, P. A. LESKY, R. F. SWARTTOUW, *Hypergeometric Orthogonal Polynomials and their q -Analogues*, Springer, Berlin, 2010.
- [15] D. W. LEE, *Difference equations for discrete classical multiple orthogonal polynomials*, J. Approx Theory **150** (2008) 132–152.
- [16] K. MAHLER, *Perfect systems*, Composition math. **19** (1968), 95–166.
- [17] F. NDAYIRAGIJE, W. VAN ASSCHE, *Asymptotics for the ratio and zeros of multiple Charlier polynomials*, J. Approx Theory **164** (2012) 823–840, [arXv:1108.3918](https://arxiv.org/abs/1108.3918).
- [18] A. F. NIKISHIN, V. N. SOROKIN, *Rational Approximants and Orthogonality*, Translations of Mathematical Monographs **92**, Amer. Math. Soc., Providence, RI (1991).
- [19] P. NJIONOU SADJANG, S. MBOUTNGAM, *On fractional q -extensions of some q -orthogonals polynomials*, Axioms **9** (2020), Axioms 2020, 9 (3), 97.
- [20] J. P. NUWACU, W. VAN ASSCHE, *Multiple Askey-Wilson polynomials and related basic hypergeometric multiple orthogonal polynomials*, [arXv:1904.01252v1](https://arxiv.org/abs/1904.01252v1).
- [21] K. POSTELMANS, W. VAN ASSCHE, *Multiple q -Jacobi polynomials*, J. Comput. Appl. Math. **178** (2005) 361–375.

(Received February 19, 2023)

P. Njionou Sadjang
 Department of Common Core
 National Higher Polytechnic School Douala
 University of Douala
 Cameroon
 e-mail: pnjionou@yahoo.fr

J. C. Múnlúem Mouncharou
 Department of Mathematics
 Faculty of Sciences, University of Maroua
 Cameroon
 e-mail: munlchrist@gmail.com

Salifou Mboutngam
 Department of Mathematics
 Higher Teachers' Training College, University of Maroua
 Cameroon
 e-mail: mbsalif@gmail.com