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BOUNDED AND UNBOUNDED BERGMAN TYPE PROJECTIONS ON THE BLOCH SPACE

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Abstract. We prove that harmonic Bergman projection is unbounded on the Bloch space $\mathscr{B}$ over the unit ball in $\mathbb{R}^n$. Another family of Bergman type operators is found whose members continuously project the Bloch space of smooth functions $\mathscr{B}$ onto its harmonic subspace $h\mathscr{B}$. A generalization with more general indices is also given. Our method is mainly based on the techniques of a modified fractional integro-differentiation and two-sided estimates of the reproducing kernels and integrals.

1. Introduction and main result

Let B be the open unit ball in $\mathbb{R}^n$, $n \geqslant 2$, and $S = \partial B$ be its boundary, the unit sphere. The set of all (real) harmonic functions in the unit ball B is denoted by $h(B)$.

As is well known, Bergman projection operator T_β (see (2) below) continuously maps weighted Lebesgue space $L^p_\alpha(B) := L^p\big(B;(1-|x|)^\alpha\, dV(x)\big)$ onto its harmonic (Bergman) subspace $h^p_\alpha(B) := L^p_\alpha(B) \cap h(B)$ for suitable indices, $T_\beta : L^p_\alpha(B) \xrightarrow{\text{onto}} h^p_\alpha(B)$, e.g. [6, 8, 11, 12, 14]. It is also familiar that operator T_β continuously maps $L^\infty(B)$ onto the harmonic Bloch space $h\mathscr{B}$, that is, $T_\beta : L^\infty(B) \xrightarrow{\text{onto}} h\mathscr{B}$, see e.g. [7, 11, 15]. However, very little information is known about the action of the Bergman type projections on the Bloch space. In this paper, we prove (in Section 5) that Bergman projection T_β is unbounded on the Bloch space $\mathscr{B}$, that is, $T_\beta(\mathscr{B}) \not\subset \mathscr{B}$. That's why we construct another family of Bergman type operators Φ that continuously project the Bloch space of smooth functions $\mathscr{B}$ onto the harmonic Bloch space $h\mathscr{B}$ over the ball $B \subset \mathbb{R}^n$, that is, $\Phi : \mathscr{B} \xrightarrow{\text{onto}} h\mathscr{B}$, see Theorem 1 below.

We write $T : X \longrightarrow Y$ if T is a bounded operator mapping X to Y, i.e. $\|Tf\|_Y \leqslant C\|f\|_X$ for every $f \in X$. The letters $C(\alpha,\beta), C_\alpha$ etc. stand for various positive constants depending only on the parameters indicated. Sometimes we will not express the dependence on the parameters explicitly. For some real-valued quantities A and B, $A \approx B$ stands for the two-sided estimate $c_1|A| \leqslant |B| \leqslant c_2|A|$ with some inessential positive constants c_1 and c_2 independent of variable involved. For typical points in $\mathbb{R}^n$ we always write $x = r\zeta$, $y = \rho\eta$ or $x = rx'$, $y = \rho y'$ with $|x| = r$, $|y| = \rho$ and

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$\zeta, \eta, x', y' \in S$. The notation dV means the Lebesgue volume measure on B normalized to have total mass 1. In polar coordinates, we have $dV(x) = nr^{n-1}dr d\sigma(\zeta)$, where $d\sigma$ is the $(n-1)$-dimensional area-surface measure on S normalized so that $\sigma(S) = 1$.

DEFINITION 1. For functions f given in B, we define L^∞_α to be a Banach space under the norm

$$\|f\|_{L^\infty_\alpha} := \operatorname*{ess\,sup}_{x\in B} (1-|x|)^\alpha |f(x)|, \qquad \alpha \geqslant 0.$$

The subspace consisting of harmonic functions is denoted by h^∞_α, i.e. $h^\infty_\alpha := h(B) \cap L^\infty_\alpha$.

This norm may be expressed in polar coordinates:

$$\|f\|_{L^\infty_\alpha} = \sup_{0\leqslant r<1} (1-r)^\alpha M_\infty(f;r),$$

where

$$M_\infty(f;r) := \|f(r\cdot)\|_{L^\infty(S)}, \qquad 0 \leqslant r < 1.$$

DEFINITION 2. A function f smooth enough in B is said to be in the Lipschitz space Λ_α $(\alpha \geqslant 0)$, if

$$\|f\|_{\Lambda_\alpha} := \left\|\mathscr{D}^{[\alpha]+1} f\right\|_{L^\infty_{[\alpha]+1-\alpha}} = \operatorname*{ess\,sup}_{x\in B} (1-|x|)^{[\alpha]+1-\alpha} \left|\mathscr{D}^{[\alpha]+1} f(x)\right|$$

is finite. Here $[\alpha]$ stands for the largest integer less than or equal to α, while $\mathscr{D}^\gamma$ is the fractional differentiation operator of order $\gamma > 0$ which will be defined exactly in Section 2. Denote by $h\Lambda_\alpha$ the subspace consisting of harmonic functions, $h\Lambda_\alpha := h(B) \cap \Lambda_\alpha$.

Note that for $f \in h\Lambda_\alpha$, the index $[\alpha]+1$ can be replaced by any $\gamma > \alpha$, and equivalent norms appear: $\|f\|_{\Lambda_\alpha} \approx \|\mathscr{D}^\gamma_{n,\lambda} f\|_{L^\infty_{\gamma-\alpha}}$. For $\alpha = 0$, the limit space Λ_0 coincides with the well-known Bloch space of smooth or harmonic functions,

$$\mathscr{B} := \Lambda_0, \qquad h\mathscr{B} := h\Lambda_0, \qquad \|f\|_{\mathscr{B}} := \left\|\mathscr{D}^1 f\right\|_{L^\infty_1}.$$

The functions $u_1(x) := \log(1-|x|) \in \mathscr{B} \setminus h\mathscr{B}$ and $u_2(x) := \log\left[(1-x_1)^2 + x_2^2\right] \in h\mathscr{B}$ provide typical examples of unbounded non-harmonic and harmonic Bloch functions, respectively. Bloch spaces of holomorphic or harmonic functions are widely known ([8, 9, 11, 16, 18, 19, 20, 21]), while some information about smooth Bloch functions is contained in [17].

The following reproducing integral formula (1) is familiar for harmonic Bergman spaces $h^p_\alpha(B)$, $0 < p < \infty$, $\alpha > -1$, see, for example, [8, 11, 12]. For wider weighted spaces and a wide range of parameters, see also [4].

THEOREM A. *For $\beta > \alpha > 0$, every function $u \in h_\alpha^\infty$ or $u \in h_{\beta-1}^1(B)$ is representable in the form*

$$u(x) = \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B, \tag{1}$$

where P_β is the reproducing kernel of Poisson–Bergman type defined in Section 3.

Integral formula (1) induces a family of Bergman projection operators

$$T_\beta(u)(x) := \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B. \tag{2}$$

Theorem A asserts that operator T_β is the identity map on h_α^∞ or $h_{\beta-1}^1(B)$ or $h\mathscr{B}$. Holomorphic counterparts of operators T_β are studied, for example, in [6, 8, 20], while their harmonic analogues appear in [5, 6, 8, 11, 15, 16]. As already mentioned above, the operator T_β continuously projects weighted Lebesgue space $L_\alpha^p(B)$ onto its harmonic (Bergman) subspace $h_\alpha^p(B)$, i.e. $T_\beta : L_\alpha^p \xrightarrow{\text{onto}} h_\alpha^p$ for $\beta > (\alpha+1)/p > 0$, $1 \leqslant p < \infty$, but is unbounded on the Bloch space, $T_\beta(\mathscr{B}) \not\subset \mathscr{B}$, see Section 5. For that reason, we are interested in finding a similar projection result for the Bloch space $\mathscr{B}$. In the next theorem, a family of bounded harmonic projections on the Bloch space is found.

THEOREM 1. *For any $0 < \delta \leqslant 1$, $0 < \lambda \leqslant \beta - \delta$, $s \geqslant 1 - n/2$, the Bergman type operator*

$$\Phi_{\beta,\lambda,\delta}(f)(x) := \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\lambda(x,y)\, \mathscr{D}_{n,s}^\delta f(y)\, dV(y)$$

boundedly maps the Bloch space of smooth functions $\mathscr{B} = \Lambda_0$ onto the harmonic Lipschitz space $h\Lambda_{\beta-\delta-\lambda}$, that is, $\Phi_{\beta,\lambda,\delta} : \Lambda_0 \xrightarrow{\text{onto}} h\Lambda_{\beta-\delta-\lambda}$, with the norm inequality

$$\left\|\Phi_{\beta,\lambda,\delta}(f)\right\|_{h\Lambda_{\beta-\delta-\lambda}} \leqslant C(\beta,\delta,\lambda,n)\, \|f\|_{\Lambda_0}.$$

In particular, for $\beta = \delta + \lambda$ and $s = \lambda$, the operator $\Phi_{\beta,\lambda,\beta-\lambda}$ is a continuous projection of the Bloch space $\Lambda_0 = \mathscr{B}$ onto its harmonic subspace, $\Phi_{\beta,\lambda,\beta-\lambda} : \mathscr{B} \xrightarrow{\text{onto}} h\mathscr{B}$.

REMARK 1. Theorem 1 can be viewed as an extension of [3, Thm 1.3] and [1, Thm 5] but it cannot be obtained directly from [3] as a limit case $\alpha = 0$. On the other hand, in contrast to the recent papers [10], [9], our operators $\Phi_{\beta,\lambda,\beta-\lambda}$ are *true projections* of the Bloch space of smooth functions $\mathscr{B}$ onto its harmonic subspace $h\mathscr{B}$.

REMARK 2. It is well known that there are a large number of various norms for the Bloch space of harmonic or holomorphic functions, see e.g. [9, 11, 16, 17, 18, 20, 21]. For the Bloch space of smooth functions $\mathscr{B}$, our possibilities are restricted, however we still can replace the fractional derivative $\mathscr{D}^1$ in the Bloch norm $\|f\|_{\mathscr{B}} = \left\|\mathscr{D}^1 f\right\|_{L_1^\infty}$ by the gradient ∇ or $\frac{\partial}{\partial r}$ to obtain an equivalent definition of $\mathscr{B}$. In addition, the following inequalities hold:

$$C_1\left\|\mathscr{D}^1 f\right\|_{L_1^\infty} \leqslant |f(0)| + \left\|\nabla f\right\|_{L_1^\infty} \leqslant \sup_{|x|\leqslant 1/2} \left|\nabla f(x)\right| + C_2\left\|\mathscr{D}^1 f\right\|_{L_1^\infty}.$$

2. Fractional integro-differentiation

In this section we give definitions of the well-known Riemann–Liouville fractional integro-differentiation operator D^{α} and its modifications denoted by $\mathscr{D}^{\alpha}$ and $\mathscr{D}^{\alpha}_{n,\lambda}$ introduced in [4], [2]. We also review properties of operators $\mathscr{D}^{\alpha}$, $\mathscr{D}^{\alpha}_{n,\lambda}$ and state some new ones.

DEFINITION 3. (Classical Riemann–Liouville fractional integral and derivative) For a function $f(r)$ of a single variable $r \in [0,1)$, formally define the operator

$$D^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)\,dt = \frac{r^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)\,dt,$$

$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D^{\alpha} f(r) := D^{-(m-\alpha)} D^m f(r), \qquad D^0 f := f,$$

where $\alpha > 0$, $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}$, $\mathbb{N}$ is the set of all positive integers.

DEFINITION 4. (Fractional integral and derivative over $\mathbb{R}^n$, $n \geqslant 2$) Given a function f in the unit ball B and $\alpha > 0$, $\lambda \in \mathbb{R}$, let

$$\begin{aligned} \mathscr{D}^{-\alpha}_{n,\lambda} f(x) :=& r^{-(\alpha+\lambda+n/2-1)} D^{-\alpha} \left\{ r^{\lambda+n/2-1} f(x) \right\} \\ =& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{\lambda+n/2-1}\,dt, \\ \mathscr{D}^{\alpha}_{n,\lambda} f(x) :=& r^{-(\lambda+n/2-1)} D^{\alpha} \left\{ r^{\alpha+\lambda+n/2-1} f(x) \right\}, \qquad r = |x|. \end{aligned}$$

Actually the role of the subscript λ is not essential. For $\lambda = 0$, we will write simply $\mathscr{D}^{\beta} := \mathscr{D}^{\beta}_{n,0}$, $\beta \in \mathbb{R}$. The general formula for $\mathscr{D}^m_{n,\lambda} f$ implies an explicit form for fractional derivatives of lower orders

$$\mathscr{D}^1_{n,\lambda} f(x) = \left(\lambda + \frac{n}{2}\right) f + r\frac{\partial f}{\partial r} = \lambda f(x) + \mathscr{D}^1 f(x), \tag{3}$$

$$\begin{aligned} \mathscr{D}^2_{n,\lambda} f(x) &= \left(\lambda+1+\frac{n}{2}\right)\left(\lambda+\frac{n}{2}\right) f(x) + 2\left(\lambda+1+\frac{n}{2}\right) r\frac{\partial f}{\partial r} + r^2 \frac{\partial^2 f}{\partial r^2} \\ &= \lambda(\lambda+1) f(x) + 2\lambda\, \mathscr{D}^1 f(x) + \mathscr{D}^2 f(x). \end{aligned} \tag{4}$$

Definition 4 leads to the inversion formula for sufficiently smooth functions f

$$\mathscr{D}^{\alpha}_{n,\lambda} \mathscr{D}^{-\alpha}_{n,\lambda} f(x) = \mathscr{D}^{-\alpha}_{n,\lambda} \mathscr{D}^{\alpha}_{n,\lambda} f(x) = f(x), \quad x \in B, \quad \alpha > 0,\ \lambda \geqslant 1 - \frac{n}{2}. \tag{5}$$

Some other basic properties of the fractional operators such as semigroup and commutation relations are established in [4], [2] and also in the next two lemmas.

LEMMA 1. *For sufficiently smooth function f in the unit ball B, there hold the following semigroup and commutation relations* $(\lambda, s \geqslant 1-n/2)$*:*

$$\mathscr{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda} f(x) = \mathscr{D}^{-\beta}_{n,\lambda} \mathscr{D}^{\alpha}_{n,\lambda} f(x), \qquad \beta > \alpha > 0, \tag{6}$$

$$\mathscr{D}^{\beta}_{n,s} \mathscr{D}^{-\alpha}_{n,\lambda} f(x) = \mathscr{D}^{-\alpha}_{n,\lambda} \mathscr{D}^{\beta}_{n,s} f(x), \qquad \alpha, \beta > 0, \tag{7}$$

$$\mathscr{D}^{\alpha}_{n,\lambda} f(x) = \mathscr{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda} \mathscr{D}^{\beta}_{n,\lambda} f(x), \qquad \beta > \alpha > 0, \tag{8}$$

$$\mathscr{D}^{\alpha+\beta}_{n,\lambda} f(x) = \mathscr{D}^{\beta}_{n,\alpha+\lambda} \mathscr{D}^{\alpha}_{n,\lambda} f(x), \qquad \alpha, \beta > 0. \tag{9}$$

Proof. Definitions 3-4 together with the inversion formula (5) lead to

$$\begin{aligned}
\mathscr{D}^{-\beta}_{n,\lambda} \mathscr{D}^{\alpha}_{n,\lambda} f &= r^{-(\beta+\lambda+n/2-1)} D^{-(\beta-\alpha)} \Big[r^{\alpha+\lambda+n/2-1} r^{-(\alpha+\lambda+n/2-1)} \\
&\qquad \times D^{-\alpha} \big\{ r^{\lambda+n/2-1} \mathscr{D}^{\alpha}_{n,\lambda} f(x) \big\} \Big] \\
&= r^{-(\beta+\lambda+n/2-1)} D^{-(\beta-\alpha)} \Big[r^{\alpha+\lambda+n/2-1} \mathscr{D}^{-\alpha}_{n,\lambda} \mathscr{D}^{\alpha}_{n,\lambda} f(x) \Big] \\
&= r^{-\alpha} r^{-(\beta-\alpha+\lambda+n/2-1)} D^{-(\beta-\alpha)} \Big[r^{\lambda+n/2-1} r^{\alpha} f(x) \Big] \\
&= r^{-\alpha} \mathscr{D}^{-(\beta-\alpha)}_{n,\lambda} \big[r^{\alpha} f(x) \big] \\
&= r^{-\alpha} r^{\alpha} \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 (1-t)^{\beta-\alpha-1} f(tx) \, t^{\alpha+\lambda+n/2-1} \, dt \\
&= \mathscr{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda} f(x),
\end{aligned}$$

for $\beta > \alpha > 0$. This proves formula (6). Now, assume $\beta = m$ is an integer and show that

$$\mathscr{D}^{m}_{n,s} \mathscr{D}^{-\alpha}_{n,\lambda} f(x) = \mathscr{D}^{-\alpha}_{n,\lambda} \mathscr{D}^{m}_{n,s} f(x), \qquad m \in \mathbb{N}. \tag{10}$$

Indeed, first we show a simpler commutation formula

$$r^m D^m \mathscr{D}^{-\alpha}_{n,\lambda} f(x) = \mathscr{D}^{-\alpha}_{n,\lambda} \big\{ r^m D^m f(x) \big\}, \qquad x = r\zeta. \tag{11}$$

To this end, we expand it by using the obvious formula $\frac{\partial^m}{\partial r^m} f(tr\zeta) = t^m \frac{\partial^m f}{\partial (tr)^m}$,

$$\begin{aligned}
r^m D^m \mathscr{D}^{-\alpha}_{n,\lambda} f(x) &= r^m \frac{\partial^m}{\partial r^m} \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx) \, t^{\lambda+n/2-1} \, dt \right] \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \left[(tr)^m \frac{\partial^m}{\partial (tr)^m} f(tx) \right] t^{\lambda+n/2-1} \, dt \\
&= \mathscr{D}^{-\alpha}_{n,\lambda} \big\{ r^m D^m f(x) \big\},
\end{aligned}$$

which coincides with (11). For any $m \in \mathbb{N}$ there exist constants $c_k = c_k(m,n) > 0$ $(k = 0, 1, 2, \ldots, m-1)$ with $c_m = 1$ such that

$$\mathscr{D}^{m}_{n,s} f(x) = r^{-(s+n/2-1)} D^m \big\{ r^{m+s+n/2-1} f(x) \big\} = \sum_{k=0}^{m} c_k \, r^k D^k f(x). \tag{12}$$

On account of (11) and (12), we obtain (10):

$$\mathscr{D}^m_{n,s}\mathscr{D}^{-\alpha}_{n,\lambda} f(x)=\sum_{k=0}^{n} c_k r^k D^k \mathscr{D}^{-\alpha}_{n,\lambda} f(x)=\mathscr{D}^{-\alpha}_{n,\lambda}\left\{\sum_{k=0}^{m} c_k r^k D^k f(x)\right\}=\mathscr{D}^{-\alpha}_{n,\lambda}\mathscr{D}^m_{n,s} f(x).$$

Now by using (6), (10) and Fubini's theorem we have, with $m>\beta$,

$$\mathscr{D}^{-\alpha}_{n,\lambda}\,\mathscr{D}^{\beta}_{n,s} f(x)=\mathscr{D}^{-\alpha}_{n,\lambda}\,\mathscr{D}^{m}_{n,s}\,\mathscr{D}^{-(m-\beta)}_{n,\beta+s} f(x)=\mathscr{D}^{m}_{n,s}\,\mathscr{D}^{-(m-\beta)}_{n,\beta+s}\,\mathscr{D}^{-\alpha}_{n,\lambda} f(x)=\mathscr{D}^{\beta}_{n,s}\mathscr{D}^{-\alpha}_{n,\lambda} f(x).$$

This completes the proof of (7). Formulas (8) and (9) are combinations of (6) and (7) together with the inversion formula (5). □

Formulas (9) and (3) immediately imply the next recurrence relation for derivatives.

LEMMA 2. *For $m\geqslant 0, m\in\mathbb{Z}$, and all sufficiently smooth functions, there holds the recurrence relation*

$$\mathscr{D}^{m+1} f=\left(m+\mathscr{D}^1\right)\mathscr{D}^m f=\mathscr{D}^m\left(m+\mathscr{D}^1\right) f=\mathscr{D}^1_{n,m}\mathscr{D}^m f=\mathscr{D}^m\mathscr{D}^1_{n,m} f. \tag{13}$$

Notice formula (13) is a generalization of that with the Poisson kernel P_0 in place of the general function f, see [15, p. 91], [16, p. 238]. We next need some standard asymptotic relations ($\beta>\alpha>0$):

$$\int_S \frac{d\sigma(\xi)}{|\xi-x|^{\alpha+n-1}}\approx\frac{1}{(1-|x|)^{\alpha}},\qquad x\in B, \tag{14}$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt\approx\frac{1}{(1-r)^{\beta-\alpha}},\qquad 0\leqslant r<1, \tag{15}$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{|\eta-tx|^{\beta}}dt\approx\frac{1}{|\eta-x|^{\beta-\alpha}},\qquad \eta\in S,\ x\in B. \tag{16}$$

The estimates in (14)–(16) are well known and can be found, for example, in [5, 10, 11, 12, 15, 16].

3. Poisson-Bergman kernel P_α and upper estimates

The extended Poisson kernel $P\equiv P_0$ over the unit ball is given by

$$P(x,y)\equiv P_0(x,y):=\frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}=\frac{1-|x|^2|y|^2}{[x,y]^n},\qquad x\in B,\ y\in\overline{B}.$$

Here and afterward, we use the notation $[x,y]:=\sqrt{1-2x\cdot y+|x|^2|y|^2}$, where $x\cdot y$ stands for the inner product in $\mathbb{R}^n$.

DEFINITION 5. (Harmonic Poisson–Bergman kernel in B, [4])

$$P_\alpha(x,y):=\mathscr{D}^{\alpha}P(x,y),\quad x,y\in B,\quad \alpha\geqslant 0,$$

where the differentiation $\mathscr{D}^\alpha$ is understood with respect to either of x or y since $\mathscr{D}^\alpha_x P(x,y)=\mathscr{D}^\alpha_y P(x,y)$.

Equivalent or similar kernels mostly defined by means of series expansion in zonal and spherical harmonics can be found in [4, 5, 6, 8, 10, 11, 12, 15, 16]. Note that Poisson–Bergman type kernel $P_\alpha(x,y)$ is a harmonic function in B with respect to both x and y, $P_\alpha(x,y) = P_\alpha(y,x)$, and also continuously extendable to $\overline{B}$ in one variable whenever the second is fixed in B. A derivation of the first order Poisson–Bergman kernel $P_1 = \mathscr{D}^1 P_0$ by a direct differentiation via (3) gives us a closed explicit form of P_1,

$$P_1(x,y) = \mathscr{D}^1 P_0(x,y) = \frac{n}{2} P_0 + r\frac{\partial P_0}{\partial r} = \frac{1}{2\,[x,y]^n}\left[\frac{n\left(1-|x|^2|y|^2\right)^2}{[x,y]^2} - 4|x|^2|y|^2\right].$$

An application of formulas (8)–(9) in Lemma 1 to the extended Poisson kernel P_0 leads to the useful formulas for the kernels $P_\alpha = \mathscr{D}^\alpha P_0$:

$$P_\alpha(x,y) = \mathscr{D}_{n,\alpha}^{-(\beta-\alpha)} P_\beta(x,y), \qquad \beta > \alpha > 0, \tag{17}$$

$$P_{\alpha+\beta}(x,y) = \mathscr{D}_{n,\alpha}^{\beta} P_\alpha(x,y), \qquad \alpha,\beta > 0. \tag{18}$$

The growth rate of the Poisson-Bergman kernel and its derivatives/gradients are well known since 1980s, see [6, 8, 11, 12, 16],

$$|P_\alpha(x,y)| \leqslant \frac{C(\alpha,n)}{[x,y]^{\alpha+n-1}} \quad \text{and} \quad \left|\nabla_x P_\alpha(x,y)\right| \leqslant \frac{C(\alpha,n)}{[x,y]^{\alpha+n}}, \quad x,y \in B. \tag{19}$$

Although estimates (19) are well known, we give below a direct proof for them including higher-order gradients and fractional derivatives based on our Definitions 4 and 5.

LEMMA 3. *For* $k > 0$, $m \in \mathbb{N}$, *and for all* $x,y \in B$,

$$\left|\frac{\partial}{\partial r}[x,y]^{-k}\right| \leqslant \frac{k}{[x,y]^{k+1}} \quad \textit{and} \quad \left|\left(\frac{\partial}{\partial r}\right)^m [x,y]^{-k}\right| \leqslant \frac{C(k,m)}{[x,y]^{k+m}}. \tag{20}$$

Proof. Since $[x,y] = [y,x] = \left|y' - x|y|\right| = \left|x' - y|x|\right|$, $x,y \neq 0$, first we compute

$$\frac{\partial}{\partial r}[x,y]^2 = \frac{\partial}{\partial r}\left|x' - yr\right|^2 = -2y\cdot(x'-yr),$$
$$\frac{\partial}{\partial r}[x,y]^{-k} = -\frac{k}{2}\left(\left|x'-yr\right|^2\right)^{-k/2-1}\frac{\partial}{\partial r}\left|x'-yr\right|^2 = \frac{k\,y\cdot(x'-yr)}{\left|x'-yr\right|^{k+2}}.$$

From this, the first estimate in (20) follows immediately. Assuming the last inequality in (20) holds for all derivatives of order $1,2,\ldots,m-1$, the mth order case follows by induction. Indeed, by the Leibniz rule

$$\begin{aligned}\frac{\partial^m}{\partial r^m}[x,y]^{-k} &= \frac{\partial^{m-1}}{\partial r^{m-1}}\,\frac{k\,y\cdot(x'-yr)}{[x,y]^{k+2}}\\ &= k\,y\cdot(x'-yr)\frac{\partial^{m-1}}{\partial r^{m-1}}[x,y]^{-k-2} - (m-1)k|y|^2\frac{\partial^{m-2}}{\partial r^{m-2}}[x,y]^{-k-2}.\end{aligned}$$

So, by the induction assumption,

$$\left|\left(\frac{\partial}{\partial r}\right)^m [x,y]^{-k}\right| \leqslant k\left|x'-yr\right| \frac{C(k,m)}{[x,y]^{k+m+1}} + \frac{C(k,m)}{[x,y]^{k+m}} \leqslant \frac{C(k,m)}{[x,y]^{k+m}},$$

as desired. □

LEMMA 4. *Let U denote either of the following four differential operators $\frac{\partial}{\partial r}$, $r\frac{\partial}{\partial r}$, ∇, $\mathscr{D}^1$. Then*

$$\left|U^m P_0(x,y)\right| \leqslant \frac{C(m,n)}{[x,y]^{m+n-1}} \quad \textit{and} \quad \left|P_m(x,y)\right| \leqslant \frac{C(m,n)}{[x,y]^{m+n-1}}, \tag{21}$$

for $x,y\in B$, $m\in\mathbb{Z}$, $m\geqslant 0$. In the latter inequality (21), actually we have taken $U^m = \mathscr{D}^m$. Moreover, for general $\alpha\geqslant 0$, $\gamma>0$, $\lambda\geqslant 1-n/2$,

$$\left|P_\alpha(x,y)\right| \leqslant \frac{C(\alpha,n)}{[x,y]^{\alpha+n-1}} \quad \textit{and} \quad \left|\mathscr{D}^\gamma_{n,\lambda} P_\alpha(x,y)\right| \leqslant \frac{C(\alpha,\gamma,\lambda,n)}{[x,y]^{\alpha+\gamma+n-1}}, \tag{22}$$

$x,y\in B$. The differential operator $\mathscr{D}^\gamma_{n,\lambda}$ in (22) can be replaced by any higher-order gradient ∇^m.

Proof. By the Leibniz rule and Lemma 3, we get

$$\begin{aligned}\frac{\partial^m}{\partial r^m}P_0(x,y) &= (1-|x|^2|y|^2)\frac{\partial^m}{\partial r^m}[x,y]^{-n} - 2m|y|^2 r\,\frac{\partial^{m-1}}{\partial r^{m-1}}[x,y]^{-n}\\ &\quad - \frac{m(m-1)}{2}2m|y|^2\,\frac{\partial^{m-2}}{\partial r^{m-2}}[x,y]^{-n},\end{aligned}$$

$$\left|\left(\frac{\partial}{\partial r}\right)^m P_0(x,y)\right| \leqslant (1-|x|^2|y|^2)\frac{C(m,n)}{[x,y]^{m+n}} + \frac{C(m,n)}{[x,y]^{m+n-1}} + \frac{C(m,n)}{[x,y]^{m+n-2}} \leqslant \frac{C(m,n)}{[x,y]^{m+n-1}}.$$

Remaining three inequalities in (21) can be proved similarly taking into account the definition $P_m = \mathscr{D}^m P_0$.

For non-integer $\alpha>0$, $m-1<\alpha<m\,(m\in\mathbb{N})$, by (17), (21), (16) and Definition 4, we get

$$\begin{aligned}\left|P_\alpha(x,y)\right| = \left|\mathscr{D}^{-(m-\alpha)}_{n,\alpha}P_m(x,y)\right| &\leqslant \frac{1}{\Gamma(m-\alpha)}\int_0^1 t^{\alpha+n/2-1}(1-t)^{m-\alpha-1}\left|P_m(tx,y)\right|dt\\ &\leqslant C(\alpha,m)\int_0^1 \frac{t^{\alpha+n/2-1}(1-t)^{m-\alpha-1}}{\left|x'-y|x|\right|^{m+n-1}}\,dt \leqslant \frac{C(\alpha,n)}{\left|x'-y|x|\right|^{\alpha+n-1}},\end{aligned}$$

which is precisely the first inequality in (22). Further, the second inequality in (22) can be proved in much the same way by using the recurrence formula (13) or (18). We omit the details. □

4. Proof of Theorem 1

We need a modification of a fractional integration inequality in the weighted spaces L^∞_α, see e.g. [2],

$$\|\mathscr{D}^{-\beta}_{n,\lambda} f\|_{L^\infty_{\alpha-\beta}} \leqslant C(\alpha,\beta,\lambda)\|f\|_{L^\infty_\alpha}, \qquad \alpha>\beta>0, \quad \lambda>-\frac{n}{2}. \tag{23}$$

LEMMA 5. *For* $0<s=\lambda<\beta$, *any harmonic Bloch function* $u(x)\in h\mathscr{B}$ *can be represented in the form* ($x\in B$)

$$u(x)=\Phi_{\beta,\lambda,\beta-\lambda}(u)(x)=\frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1} P_\lambda(x,y)\,\mathscr{D}^{\beta-\lambda}_{n,\lambda} u(y)\,dV(y),$$

that is, $\Phi_{\beta,\lambda,\beta-\lambda}$ *is the identity operator on* $h\mathscr{B}$.

Proof. Any harmonic Bloch function $u(x)\in h\mathscr{B}$ can be characterized by $\mathscr{D}^{\beta-\lambda}_{n,\lambda}u\in h^\infty_{\beta-\lambda}$, see, for example, [13], [11], [9]. Elementary embedding $h^\infty_{\beta-\lambda}\subset h^1_{\beta-1}$ and Theorem A enable us to provide the Bergman representation

$$\mathscr{D}^{\beta-\lambda}_{n,\lambda}u(x)=T_\beta\left(\mathscr{D}^{\beta-\lambda}_{n,\lambda}u\right)(x)=\frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1}\,\mathscr{D}^\beta P_0(x,y)\,\mathscr{D}^{\beta-\lambda}_{n,\lambda}u(y)\,dV(y).$$

Applying here the integral $\mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}$, we deduce that

$$\begin{aligned} u(x) &= \mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}\mathscr{D}^{\beta-\lambda}_{n,\lambda}u(x)\\ &= \frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1}\,\mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}\mathscr{D}^\beta P_0(x,y)\,\mathscr{D}^{\beta-\lambda}_{n,\lambda}u(y)\,dV(y). \end{aligned}$$

In view of the identity (8) or (17), $\mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}\mathscr{D}^\beta P_0=\mathscr{D}^\lambda P_0=P_\lambda$. Thus, $u(x)=\Phi_{\beta,\lambda,\beta-\lambda}(u)(x)$, and the lemma follows. $\square$

Proof of Theorem 1. It suffices to prove

$$\left\|\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)\right\|_{L^\infty_{\gamma-\beta+\delta+\lambda}}\leqslant C\left\|\mathscr{D}^1 f\right\|_{L^\infty_1}$$

for a given function $f\in\Lambda_0(B)$ and some $\gamma>\beta-\delta-\lambda\geqslant 0$. Since $f\in\Lambda_0$ and $\mathscr{D}^1 f\in L^\infty_1$, formula (8) in Lemma 1 in the form $\mathscr{D}^\delta_{n,s}f(x)=\mathscr{D}^{-(1-\delta)}_{n,\delta+s}\mathscr{D}^1_{n,s}f(x)$, $0<\delta\leqslant 1$, makes it possible to estimate the norm by using (23) and (3):

$$\left\|\mathscr{D}^\delta_{n,s}f\right\|_{L^\infty_\delta}=\left\|\mathscr{D}^{-(1-\delta)}_{n,\delta+s}\mathscr{D}^1_{n,s}f\right\|_{L^\infty_\delta}\leqslant C\left\|\mathscr{D}^1_{n,s}f\right\|_{L^\infty_1}\leqslant C\left\|\mathscr{D}^1 f\right\|_{L^\infty_1}.$$

Therefore $\mathscr{D}^\delta f(x)\in L^\infty_\delta$ and $(1-r)^\delta M_\infty(\mathscr{D}^\delta_{n,s}f;r)\leqslant C\|\mathscr{D}^1 f\|_{L^\infty_1}$.

Differentiation of $\Phi_{\beta,\lambda,\delta}(f)$ with $\mathscr{D}^\gamma$ together with the kernel estimates (22) yields

$$\left|\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)(x)\right| \leqslant C\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\gamma+\lambda+n-1}}\left|\mathscr{D}^\delta_{n,s}f(\rho\eta)\right|\rho^{n-1}d\rho\, d\sigma(\eta)$$

for some $\gamma>\beta-\delta-\lambda\geqslant 0$ and a constant $C=C(\beta,\gamma,\lambda,n)$. Replace here x by Qx, where Q is an arbitrary orthogonal linear transformation $Q:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, that is, $|Qx|=|x|$ for all $x\in\mathbb{R}^n$. Applying also the change $\eta\mapsto Q\eta$, we find that

$$\begin{aligned}\left|\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)(Qx)\right| &\leqslant C\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho Qx-Q\eta|^{\gamma+\lambda+n-1}}\left|\mathscr{D}^\delta_{n,s}f(\rho Q\eta)\right|\rho^{n-1}d\rho\, d\sigma(\eta)\\ &= C\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\gamma+\lambda+n-1}}\left|\mathscr{D}^\delta_{n,s}f(\rho Q\eta)\right|\rho^{n-1}d\rho\, d\sigma(\eta).\end{aligned}$$

Hence, for $\gamma>\beta-\delta-\lambda\geqslant 0$, we obtain

$$\begin{aligned}M_\infty\left(\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f);r\right) &\leqslant C(\beta,\gamma,\lambda,n)\int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\lambda+\gamma}}M_\infty(\mathscr{D}^\delta_{n,s}f;\rho)\,d\rho\\ &\leqslant C(\beta,\gamma,\lambda,n)\int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\lambda+\gamma}}\frac{\|\mathscr{D}^1 f\|_{L^\infty_1}}{(1-\rho)^\delta}\,d\rho\\ &\leqslant C(\beta,\gamma,\delta,\lambda,n)\frac{\|\mathscr{D}^1 f\|_{L^\infty_1}}{(1-r)^{\lambda+\gamma-\beta+\delta}}.\end{aligned}$$

Therefore, $(1-r)^{\gamma-\beta+\delta+\lambda}M_\infty\left(\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f);r\right)\leqslant C\left\|\mathscr{D}^1 f\right\|_{L^\infty_1}$, or

$$\left\|\mathscr{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)\right\|_{L^\infty_{\gamma-\beta+\delta+\lambda}}\leqslant C\left\|\mathscr{D}^1 f\right\|_{L^\infty_1}\quad\text{and}\quad \left\|\Phi_{\beta,\lambda,\delta}(f)\right\|_{\Lambda_{\beta-\delta-\lambda}}\leqslant C\|f\|_{\Lambda_0}.$$

Thus, the operator $\Phi_{\beta,\lambda,\delta}$ boundedly maps Λ_0 into $h\Lambda_{\beta-\delta-\lambda}$, that is, $\Phi_{\beta,\lambda,\delta}:\Lambda_0\xrightarrow{\text{into}} h\Lambda_{\beta-\delta-\lambda}$, $0<\lambda\leqslant\beta-\delta$. It remains to prove this map is onto as well. To this end, we take arbitrary harmonic function $g\in h\Lambda_{\beta-\delta-\lambda}$, so $\mathscr{D}^{\beta-\lambda}_{n,\lambda}g\in h^\infty_\delta\subset h^1_{\beta-1}$ since $\beta>\delta>0$. By Theorem A, $\mathscr{D}^{\beta-\lambda}_{n,\lambda}g(x)=T_\beta\left(\mathscr{D}^{\beta-\lambda}_{n,\lambda}g\right)(x)$, and then taking the inverse operator $\mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}$, we obtain

$$\begin{aligned}g(x) &= \frac{2}{n\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}\left[\mathscr{D}^{-(\beta-\lambda)}_{n,\lambda}\mathscr{D}^\beta P_0(x,y)\right]\mathscr{D}^{\beta-\lambda}_{n,\lambda}g(y)\,dV(y)\\ &= \frac{2}{n\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}P_\lambda(x,y)\,\mathscr{D}^\delta_{n,s}\left[\mathscr{D}^{-\delta}_{n,s}\,\mathscr{D}^{\beta-\lambda}_{n,\lambda}g(y)\right]dV(y)\\ &=: \Phi_{\beta,\lambda,\delta}(\psi)(x),\end{aligned}$$

where we have used Lemma 1 and the inversion formula (5). Now it suffices to show that the function $\psi:=\mathscr{D}^{-\delta}_{n,s}\mathscr{D}^{\beta-\lambda}_{n,\lambda}g$ is in $h\Lambda_0$. For $0<\delta\leqslant 1$, we apply Hardy–Littlewood type theorems on (fractional) integro-differentiation in harmonic spaces h^∞_α, see [13]. Also, by using the commutation formula (7), we consecutively obtain the following chain of implications:

$$\begin{aligned} g \in h\Lambda_{\beta-\delta-\lambda} \quad &\Longrightarrow \quad \mathscr{D}_{n,\lambda}^{\beta-\lambda} g \in h_\delta^\infty \\ &\Longrightarrow \quad \mathscr{D}^1 \mathscr{D}_{n,\lambda}^{\beta-\lambda} g \in h_{\delta+1}^\infty \\ &\Longrightarrow \quad \mathscr{D}^1 \mathscr{D}_{n,s}^{-\delta} \mathscr{D}_{n,\lambda}^{\beta-\lambda} g \in h_1^\infty \\ &\Longrightarrow \quad \mathscr{D}^1 \psi = \mathscr{D}^1 \big(\mathscr{D}_{n,s}^{-\delta} \mathscr{D}_{n,\lambda}^{\beta-\lambda} g \big) \in h_1^\infty \\ &\Longrightarrow \quad \psi \in h\Lambda_0 . \end{aligned}$$

In particular, for $\beta = \delta + \lambda$ and $s = \lambda$, by Lemma 5, the operator $\Phi := \Phi_{\beta,\lambda,\beta-\lambda}$ is the identity operator on the harmonic Bloch space $h\mathscr{B}$, so $\Phi^2 = \Phi$, and indeed a bounded projection from the Bloch space $\Lambda_0 = \mathscr{B}$ onto $h\Lambda_0 = h\mathscr{B}$. The same reasoning shows that more general mapping $\Phi_{\beta,\lambda,\delta}$ is still a bounded projection from $\mathscr{B}$ onto $h\Lambda_{\beta-\delta-\lambda}$.

This completes the proof of Theorem 1. □

5. Bergman projection is unbounded on the Bloch space

In this section we show there exists a Bloch function $u_0 \in \mathscr{B}$ whose Bergman projection is not Bloch, $T_\beta(u_0) \notin h\mathscr{B}$. To this end, we need some preliminary integral estimates.

LEMMA 6. *For $\alpha, \lambda > 0$ and $\gamma \in \mathbb{R}$, the following asymptotic relations hold,*

$$\int_r^1 t^{\lambda-1} (1-t)^{\alpha-1} \left(\log \frac{2}{1-t} \right)^{-\gamma} dt \sim \frac{1}{\alpha} (1-r)^\alpha \left(\log \frac{2}{1-r} \right)^{-\gamma}, \tag{24}$$

$$\int_0^r \frac{t^{\lambda-1}}{(1-t)^{1+\alpha}} \left(\log \frac{2}{1-t} \right)^{-\gamma} dt \sim \frac{1}{\alpha} \frac{1}{(1-r)^\alpha} \left(\log \frac{2}{1-r} \right)^{-\gamma}, \tag{25}$$

and

$$\int_0^r \frac{t^{\lambda-1}}{1-t} \left(\log \frac{2}{1-t} \right)^{-\gamma} dt \sim \frac{1}{1-\gamma} \left(\log \frac{2}{1-r} \right)^{1-\gamma}, \qquad \gamma < 1, \tag{26}$$

$$\int_0^r \frac{t^{\lambda-1}}{1-t} \left(\log \frac{2}{1-t} \right)^{-1} dt \sim \log \left(2 \log \frac{2}{1-r} \right), \qquad \gamma = 1, \tag{27}$$

$$\int_0^r \frac{t^{\lambda-1}}{1-t} \left(\log \frac{2}{1-t} \right)^{-\gamma} dt \sim C(\lambda, \gamma) > 0, \qquad \gamma > 1, \tag{28}$$

as $r \to 1^-$.

All the asymptotic relations except (28) can easily be proved by L'Hôpital's rule. Also, we need sharp estimates of an integral of hypergeometric type that generalizes (15).

LEMMA 7. *Suppose $\alpha,\lambda>0$ and $\beta,\gamma\in\mathbb{R}$. Then the integral*

$$J=J_{\alpha,\beta,\gamma,\lambda}(r):=\int_0^1\frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-rt)^{\beta}}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt$$

has the following two-sided estimates for all $0\leqslant r<1$:

$$J=J_{\alpha,\beta,\gamma,\lambda}(r)\approx\begin{cases}\dfrac{1}{(1-r)^{\beta-\alpha}}\left(\log\dfrac{2}{1-r}\right)^{-\gamma}, & \beta>\alpha,\\ 1, & \beta<\alpha,\end{cases}\tag{29}$$

and for $\alpha=\beta$,

$$J=J_{\alpha,\alpha,\gamma,\lambda}(r)\approx\begin{cases}\left(\log\dfrac{2}{1-r}\right)^{1-\gamma}, & \gamma<1,\\ 1, & \gamma>1,\\ \log\left(2\log\dfrac{2}{1-r}\right), & \gamma=1.\end{cases}\tag{30}$$

Everywhere here all the constants $C=C(\alpha,\beta,\gamma,\lambda)$ implicitly involved depend only on $\alpha,\beta,\gamma,\lambda$.

Proof. All the inequalities in (29)–(30) are trivial for $0\leqslant r\leqslant\frac{1}{2}$. So, it suffices to prove (29)–(30) only for $\frac{1}{2}\leqslant r<1$ or $r\to1^-$. First, assuming $\beta>\alpha>0$, we will estimate the integral (29) from above,

$$\begin{aligned}J=\int_0^r+\int_r^1&\leqslant\int_0^r\frac{t^{\lambda-1}}{(1-t)^{1+\beta-\alpha}}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt\\&\quad+\frac{1}{(1-r)^{\beta}}\int_r^1 t^{\lambda-1}(1-t)^{\alpha-1}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt\\&\leqslant C(\alpha,\beta,\gamma,\lambda)\frac{1}{(1-r)^{\beta-\alpha}}\left(\log\frac{2}{1-r}\right)^{-\gamma},\end{aligned}$$

by virtue of two asymptotic relations (24) and (25). Conversely, again by (24), we obtain

$$\begin{aligned}J\geqslant\int_r^1&\geqslant\frac{C_\lambda}{(1-r^2)^{\beta}}\int_r^1(1-t)^{\alpha-1}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt\\&\geqslant\frac{C(\alpha,\beta,\gamma,\lambda)}{(1-r)^{\beta-\alpha}}\left(\log\frac{2}{1-r}\right)^{-\gamma},\end{aligned}$$

so the case $\beta>\alpha>0$ is proved. Assume $0<\beta<\alpha$ to estimate

$$J\geqslant\int_0^1 t^{\lambda-1}(1-t)^{\alpha-1}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt=C_1(\alpha,\gamma,\lambda)>0,$$

$$J\leqslant\int_0^1\frac{t^{\lambda-1}(1-t)^{\alpha-1}}{(1-t)^{\beta}}\left(\log\frac{2}{1-t}\right)^{-\gamma}dt=C_2(\alpha,\beta,\gamma,\lambda).$$

For $\beta \leqslant 0 < \alpha$, the proof is similar. Thus, the proof of relation (29) is complete.

Proceeding to (30) notice that the integral J converges as $r \to 1^-$ when $\beta = \alpha$ and $\gamma > 1$.

For the case $\gamma < 1$, we split the integral into two parts and then apply asymptotic relations (24) and (26) to obtain

$$\begin{aligned} J &\leqslant \int_0^r \frac{t^{\lambda-1}}{1-t}\left(\log\frac{2}{1-t}\right)^{-\gamma} dt + \frac{1}{(1-r)^\alpha}\int_r^1 t^{\lambda-1}(1-t)^{\alpha-1}\left(\log\frac{2}{1-t}\right)^{-\gamma} dt \\ &\leqslant C_\gamma \left(\log\frac{2}{1-r}\right)^{1-\gamma} + C_\alpha \left(\log\frac{2}{1-r}\right)^{-\gamma} \\ &\leqslant C(\alpha,\gamma)\left(\log\frac{2}{1-r}\right)^{1-\gamma}. \end{aligned}$$

For the lower estimate, notice that the function $\frac{1-t}{1-rt}$ is decreasing in t and bounded away from zero on $(0,r)$,

$$\begin{aligned} J &\geqslant \int_0^r t^{\lambda-1}\left(\frac{1-t}{1-rt}\right)^\alpha \left(\log\frac{2}{1-t}\right)^{-\gamma}\frac{dt}{1-t} \\ &\geqslant \int_0^r t^{\lambda-1}\left(\frac{1-r}{1-r^2}\right)^\alpha \left(\log\frac{2}{1-t}\right)^{-\gamma}\frac{dt}{1-t} \\ &\geqslant \frac{1}{2^\alpha}\int_0^r \frac{t^{\lambda-1}}{1-t}\left(\log\frac{2}{1-t}\right)^{-\gamma} dt \\ &\geqslant C(\alpha,\gamma)\left(\log\frac{2}{1-r}\right)^{1-\gamma}, \end{aligned}$$

by asymptotic relation (26). This proves the estimates for $\gamma < 1$.

Finally, both inequalities in the case $\gamma = 1$ can be proved similarly by employing asymptotic relation (27). This completes the proof of Lemma 7. □

Actually in this section we need only lower estimates in (29). However, we have established two-sided estimates and covered all possible values of the indices in Lemma 7, for completeness.

One more lower estimate for the reproducing kernels P_α is borrowed from [3].

LEMMA 8. *For any $\alpha > 0, \gamma \geqslant 0$ and x, $\frac{1}{2} \leqslant |x| < 1$, there exist a small neighborhood $\mathscr{U}_\varepsilon(x') := \{y \in B : |y - x'| < \varepsilon\}$, $\varepsilon > 0$, and a constant $C(\alpha,\gamma,n) > 0$ such that*

$$\mathscr{D}^\gamma P_\alpha(x,y) \geqslant \frac{C(\alpha,\gamma,n)}{[x,y]^{\gamma+\alpha+n-1}}, \qquad y \in \mathscr{U}_\varepsilon(x'). \tag{31}$$

Although Lemma 8 is proved in [3, Corollary 4.3] only for $\gamma = 0$, the same method is applicable for the derivative $\mathscr{D}^\gamma P_\alpha$ $(\gamma > 0)$. Besides, the additive property (18), $\mathscr{D}^\gamma_{n,\alpha} P_\alpha = P_{\alpha+\gamma}$, allows us to reduce the proof of (31) to that with $\gamma = 0$.

THEOREM 2. *Bergman projection T_β is unbounded on the Bloch space.*

Proof. We have to find a Bloch function $u_0(x) \in \mathscr{B}$ such that $T_\beta(u_0) \notin h\mathscr{B}$. Introduce a smooth (and superharmonic) Bloch function

$$u_0(x) := \log \frac{1-|x|}{2} \in \mathscr{B} \setminus h\mathscr{B}.$$

It is easy to calculate $(1-|x|)|\nabla u_0(x)| = 1$, $x \in B$. On the other hand, by the triangle inequality and Lemma 8,

$$\begin{aligned}
\left|\mathscr{D}^1 T_\beta(u_0)(x)\right| &= C_\beta \left| \int_B (1-|y|^2)^{\beta-1}\, \mathscr{D}^1 P_\beta(x,y) \log \frac{2}{1-|y|}\, dV(y) \right| \\
&= C_\beta \left| \int_{\mathscr{U}_\varepsilon(x')} + \int_{B\setminus \mathscr{U}_\varepsilon(x')} \right| \geqslant C_\beta \left| \int_{\mathscr{U}_\varepsilon(x')} \right| - C_\beta \left| \int_{B\setminus \mathscr{U}_\varepsilon(x')} \right| \\
&\geqslant C(\beta,n) \int_{\mathscr{U}_\varepsilon(x')} \frac{(1-|y|^2)^{\beta-1}}{[x,y]^{\beta+n}} \log \frac{2}{1-|y|}\, dV(y) \\
&\quad - C_\beta \left| \int_{B\setminus \mathscr{U}_\varepsilon(x')} (1-|y|^2)^{\beta-1}\, \mathscr{D}^1 P_\beta(x,y) \log \frac{2}{1-|y|}\, dV(y) \right|.
\end{aligned}$$

It suffices to estimate the integral over the whole ball from below,

$$I_\beta(x) := \int_B \frac{(1-|y|^2)^{\beta-1}}{[x,y]^{\beta+n}} \log \frac{2}{1-|y|}\, dV(y).$$

By (14) and (29) in Lemma 7,

$$\begin{aligned}
I_\beta(x) &= \int_B \frac{(1-|y|^2)^{\beta-1}}{\left|y' - |y|x\right|^{\beta+n}} \log \frac{2}{1-|y|}\, dV(y) \\
&\approx \int_0^1 \frac{\rho^{n-1}(1-\rho)^{\beta-1}}{(1-r\rho)^{\beta+1}} \log \frac{2}{1-\rho}\, d\rho \\
&\approx \frac{1}{1-r} \log \frac{2}{1-r}
\end{aligned}$$

as $r \to 1^-$. Hence,

$$(1-|x|)\left|\mathscr{D}^1 T_\beta(u_0)(x)\right| \geqslant C \log \frac{2}{1-|x|} \qquad \text{as} \quad |x| \to 1^-,$$

thus $\left\|T_\beta(u_0)\right\|_{\mathscr{B}} = +\infty$ and $T_\beta(u_0) \notin h\mathscr{B}$. $\square$

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