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ON THE CONVERGENT AND NULL GENERALIZED MOTZKIN SEQUENCE SPACES

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Abstract. The BK sequence spaces $c(\mathscr{G})$ and $c_0(\mathscr{G})$ generated by generalized Motzkin (GM) matrix $\mathscr{G}$ are constructing in this study. The Schauder bases of these spaces are obtained and the inclusion relations are presented. Additionally, the α-, β- and γ-duals of $c(\mathscr{G})$ and $c_0(\mathscr{G})$ are determined and finally, the necessary and sufficient conditions for a matrix to be in the class of matrices containing new spaces are explored.

1. Introduction

The Generalized Motzkin (GM) numbers, recently introduced and explored by Sun [29], provide a unifying framework that extends several fundamental integer sequences, including Motzkin numbers, Catalan numbers, Super-Catalan numbers, and restricted hexagonal numbers. This perspective offers new insights into the combinatorial and algebraic properties of these classical sequences. Given the k^{th} Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$, the k^{th} GM number $M_k(a,b)$ is defined as follows for any integers a and b:

$$M_k(a,b) := \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2l} C_l a^{k-2l} b^l, \quad (k \in \mathbb{N}) \tag{1}$$

where $\lfloor . \rfloor$ is the floor function and $\mathbb{N} = \{0,1,2,3,\dots\}$. Sun [29] further derived the following recursion

$$(k+3)M_{k+1}(a,b) = a(2k+3)M_k(a,b) - (a^2-4b)kM_{k-1}(a,b),$$

and the generating function

$$M(a,b;s) := \sum_{k\geqslant 0} M_k(a,b)s^k = \frac{1-as-\sqrt{(1-as)^2-4bs^2}}{2bs^2}.$$

The first seven GM numbers $M_k(a,b)$ are given by

$$1,\ a,\ a^2+b,\ a(a^2+3b),\ a^4+6a^2b+2b^2,\ a(a^4+10a^2b+10b^2)$$

and

$$a^6+15a^4b+30a^2b^2+5b^3.$$

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Upon examining the OEIS (The On-Line Encyclopedia of Integer Sequences) [27], it is observed that, in addition to the aforementioned number sequences, many other number sequences are special cases of GM numbers. Some of these are sequences A025235, A025237, A071356, and A122871 in [27]. Such sequences arise in the enumeration of paths of a given length, the counting of specific combinatorial objects using various transition functions and coefficients, and the analysis of transitions between nodes and edges [9, 10, 30].

The study of sequence spaces has been significantly influenced by the exploration of various number sequences characterized by recurrence relations. Among these, Fibonacci numbers stand out as the most well-known example, serving as a fundamental model in numerous mathematical and applied contexts.

The idea of utilizing Fibonacci numbers in the construction of sequence spaces was first introduced by Kara and Başarır [19] in 2012. This pioneering work demonstrated how recurrence relations could be systematically incorporated into the framework of sequence spaces, providing new avenues for mathematical analysis. Following this study, researchers began to explore other number sequences exhibiting similar recurrence properties, leading to the development of generalized sequence spaces based on alternative recurrence relations. These investigations have expanded the theoretical foundation of sequence space theory and opened new possibilities for applications in functional analysis, approximation theory, and discrete mathematics.

The space of all sequences with real entries is given by ω. The set $\aleph$ is entitled as sequence space if $\aleph \subset \omega$. The spaces of all bounded, convergent, null and p-absolutely summable sequences are denoted by ℓ_∞, c, c_0 and ℓ_p, respectively. If the maps $\varsigma : \aleph \to \mathbb{R}$, $\varsigma_k(s) = s_k$ are continuous, in that case the Banach space $\aleph$ is called as BK-space. Moreover; ℓ_∞, c, c_0 are BK-spaces with $\|s\|_{\ell_\infty} = \|s\|_c = \|s\|_{c_0} = \sup_{k\in\mathbb{N}} |s_k|$ and ℓ_p is BK-space with $\|s\|_{\ell_p} = (\sum_k |s_k|^p)^{\frac{1}{p}}$ for $s = (s_k) \in \omega$, $\sum_k |s_k| = \sum_{k=0}^{\infty} |s_k|$ and $1 \leqslant p < \infty$. Furthermore, the spaces of bounded and convergent series are given by bs and cs, respectively.

Consider an infinite matrix $F = (f_{kl})$ which has real entries and its k^{th} row F_k. Then, $Fs = (Fs)_k = \sum_l f_{kl} s_l$ is named as F-transform of $s \in \omega$ if the series is convergent for all $k \in \mathbb{N}$. The infinite matrix F is named as matrix mapping from $\aleph$ to $\mho$, if $Fs \in \mho$ for all $s \in \aleph$. The class of all matrix mappings from $\aleph$ to $\mho$ is indicated with $(\aleph : \mho)$. Furthermore, if $F \in (\aleph : \mho)$, then $F_k \in \aleph^\beta$ and $Fs \in \mho$ for all $s \in \aleph$. The set

$$\mho_F = \{s \in \omega : Fs \in \mho\} \tag{2}$$

is named as matrix domain of F in $\mho$.

Take into account that the multiplier set $(\aleph \star \mho)$ is defined by

$$(\aleph \star \mho) = \left\{\psi = (\psi_k) \in \omega : \psi s = (\psi_k s_k) \in \mho \text{ for all } s \in \aleph\right\}$$

for $\aleph, \mho \subset \omega$. Then, the α-, β- and γ-duals of $\aleph$ are described as

$$\aleph^\alpha = (\aleph \star \ell_1), \quad \aleph^\beta = (\aleph \star cs) \quad \text{and} \quad \aleph^\gamma = (\aleph \star bs).$$

It is important to say that, if $(f_{kl})_{l\in\mathbb{N}} \in \aleph^\beta$, then the F-transform $(Fs)_k = \sum_l f_{kl} s_l$ of any sequence $s = (s_l) \in \aleph$ is convergent for an infinite matrix F.

The spaces $\aleph$ and $\mho$ are called as linearly norm isomorphic when a norm-preserving linear bijection can be established between them, and this is denoted by $\aleph \cong \mho$.

Following the introduction of Fibonacci numbers-one of the most well known number sequences with a recurrence relation-into the construction of sequence spaces, the applications of such sequences in this field became rapidly widespread. Notable examples include the Schröder [5, 6], Mersenne [7], Catalan [17, 18, 21], Fibonacci [19, 20], Lucas [22, 23], Bell [24], Padovan [31], and Leonardo sequences [32]. In this context, fundamental concepts such as sequence spaces, summability theory, and matrix domain studies have been extensively examined in the literature [2, 3, 4, 11, 16, 26].

The aforementioned number sequences have found wide applications in the literature. In addition to these sequences, another important number sequence is the Motzkin numbers. This number sequence is named after Theodore Motzkin [25]. Motzkin numbers play a significant role in combinatorics, particularly in the enumeration of non-crossing partitions and polygon dissections, as explored in studies on hypersurface cross ratios and combinatorial partition formulas [1, 25, 30]. This sequence can be defined recursively by the relation

$$(k+3)M_{k+1} = (2k+3)M_k + 3kM_{k-1}.$$

The above recurrence formula provides a systematic method for generating Motzkin numbers.

The Motzkin numbers have a close relationship with the Catalan numbers, expressed by the following relations:

$$M_k = \sum_{l=0}^{k} \binom{k}{2l} C_l \quad \text{and} \quad C_{k+1} = \sum_{l=0}^{k} \binom{k}{l} M_l.$$

More detailed information on Motzkin numbers can be obtained from the studies [1, 9, 10, 30].

In [12], Erdem et al. introduced the Motzkin matrix $\mathscr{M} = (\mathfrak{m}_{kl})_{k,l\in\mathbb{N}}$ by using Motzkin numbers as follows:

$$\mathfrak{m}_{kl} := \begin{cases} \dfrac{M_l M_{k-l}}{M_{k+2} - M_{k+1}}, & \text{if } 0 \leqslant l \leqslant k, \\ 0, & \text{if } l > k, \end{cases} \tag{3}$$

and they described the BK-spaces $c(\mathscr{M})$ and $c_0(\mathscr{M})$ as the domain of $\mathscr{M}$ on the spaces c and c_0. In the continuation of this study, authors determined the duals, characterized matrix classes of the aforementioned spaces and presented the core theorems. After that, Erdem [13] exhibited the sufficient conditions for a matrix operator to be compact from the Motzkin sequence space $c_0(\mathscr{M})$ to the classical spaces. Additionally, Erdem [14] obtained two Motzkin sequence spaces by using the Motzkin matrix and the author investigated these spaces in terms of various topics such as completeness, duals, Schuder basis, isomorphism, inclusion relations, matrix transformations and compact operators.

Let $M_k(a,b)$ be the GM (generalized Motzkin) numbers described with (1). In that case, the GM numbers satisfy the recurrence relation

$$M_{k+1}(a,b) = aM_k(a,b) + b\sum_{l=0}^{k-1} M_l(a,b)M_{k-1-l}(a,b)$$

for $k \geqslant 1$.

Demiriz et al. [8] exposed the limit of the ratio of consecutive terms as

$$\lim_{k\to\infty} \frac{M_k(a,b)}{M_{k-1}(a,b)} = a + 2\sqrt{b},$$

and the authors described the infinite GM matrix $\mathscr{G} = (\mathfrak{g}_{kl})_{k,l\in\mathbb{N}}$ as:

$$\mathfrak{g}_{kl} := \begin{cases} \dfrac{bM_l(a,b)M_{k-l}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)}, & \text{if } 0 \leqslant l \leqslant k, \\ 0, & \text{if } l > k. \end{cases} \tag{4}$$

The matrix $\mathscr{G}$ can be expressed as follows:

$$\mathscr{G} := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{a^2+b}{3a^2+2b} & \frac{a^2}{3a^2+2b} & \frac{a^2+b}{3a^2+2b} & 0 & \cdots \\ \frac{a^2+3b}{4a^2+8b} & \frac{a^2+b}{4a^2+8b} & \frac{a^2+b}{4a^2+8b} & \frac{a^2+3b}{4a^2+8b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Based on its definition, it can be inferred that $\mathscr{G}$ is a triangular matrix and the $\mathscr{G}$-transform $\mathscr{G}s$ of any $s = (s_l) \in \omega$ is given by

$$t_k := (\mathscr{G}s)_k = \sum_{l=0}^{k} \frac{bM_l(a,b)M_{k-l}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} s_l, \quad (k \in \mathbb{N}). \tag{5}$$

Furthermore, it is given the inverse $\mathscr{G}^{-1} = (\mathfrak{g}_{kl}^{-1})$ of the GM matrix $\mathscr{G}$ as

$$\mathfrak{g}_{kl}^{-1} := \begin{cases} (-1)^{k-l}\dfrac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}, & \text{if } 0 \leqslant l \leqslant k, \\ 0, & \text{if } l > k, \end{cases} \tag{6}$$

where $\lambda_0 = 1$ and

$$\lambda_k = \begin{vmatrix} M_1(a,b) & M_0(a,b) & 0 & 0 & \cdots & 0 \\ M_2(a,b) & M_1(a,b) & M_0(a,b) & 0 & \cdots & 0 \\ M_3(a,b) & M_2(a,b) & M_1(a,b) & M_0(a,b) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_k(a,b) & M_{k-1}(a,b) & M_{k-2}(a,b) & M_{k-3}(a,b) & \cdots & M_1(a,b) \end{vmatrix}$$

for all $k \in \mathbb{N}\backslash\{0\}$.

LEMMA 1. [28] *An infinite matrix F is conservative, that is $F \in (c:c)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_{l} |f_{kl}| < \infty, \tag{7}$$

$$\textit{there exists } \rho_l \in \mathbb{R} \textit{ such that } \lim_{k\to\infty} f_{kl} = \rho_l \textit{ for each } l \in \mathbb{N}, \tag{8}$$

$$\textit{there exists } \rho \in \mathbb{R} \textit{ such that } \lim_{k\to\infty} \sum_{l} f_{kl} = \rho. \tag{9}$$

Additionally, $B \in (c_0 : c_0)$ if and only if the conditions (7) and

$$\lim_{k\to\infty} f_{kl} = 0 \tag{10}$$

hold.

Again, we know from the study [8] that the matrix $\mathscr{G}$ is conservative, that is $\mathscr{G} \in (c:c)$.

In this study, we focus on the construction of sequence spaces and the investigation of their fundamental properties using the generalized Motzkin sequence, which serves as a unifying framework for various number sequences and can be regarded as a polynomial sequence. The motivation behind this work arises from the observation that many important results obtained via Motzkin numbers can be extended within a more general setting, allowing for broader theoretical insights and enhanced applicability. In this direction, two novel sequence spaces by the aid of GM matrix operator in the spaces c and c_0 are introduced. Additionally, their algebraic and topological characteristics, inclusion relationships, bases and dual spaces are subsequently explored. The final section intends on some matrix mappings of the new sequence spaces.

2. The novel GM sequence spaces $c(\mathscr{G})$ and $c_0(\mathscr{G})$

In this section, the BK sequence spaces $c(\mathscr{G})$ and $c_0(\mathscr{G})$ are introduced using the GM matrix, which are linearly isomorphic to c and c_0, respectively. Next, Schauder bases and inclusion relations are discussed.

We are now ready to define the sequence spaces $c(\mathscr{G})$ and $c_0(\mathscr{G})$ as follows:

$$c(\mathscr{G}) = \{s \in \omega : \mathscr{G}s \in c\}$$

and

$$c_0(\mathscr{G}) = \{s \in \omega : \mathscr{G}s \in c_0\}.$$

Then, $c(\mathscr{G})$ and $c_0(\mathscr{G})$ can be rewritten as $c(\mathscr{G}) = (c)_{\mathscr{G}}$ and $c_0(\mathscr{G}) = (c_0)_{\mathscr{G}}$ by using notation (2). For a normed space $\aleph \in \omega$, $\aleph_{\mathscr{G}}$ is referred as the GM sequence space.

It is worth noting that BK-spaces have a significant role in summability theory because the matrix operators defined on BK-spaces are continuous.

THEOREM 1. *$c(\mathscr{G})$ and $c_0(\mathscr{G})$ are BK-spaces with the norm*

$$\|s\|_{c(\mathscr{G})} = \|s\|_{c_0(\mathscr{G})} = \|\mathscr{G}s\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}\left|\sum_{l=0}^{k}\frac{bM_l(a,b)M_{k-l}(a,b)}{M_{k+2}(a,b)-aM_{k+1}(a,b)}s_l\right|.$$

Proof. We see in [33] that, $\aleph_F$ is a BK-space for $\|s\|_{\aleph_F} = \|Fs\|_{\aleph}$ whenever F is triangle and $\aleph$ is BK-space. Consequently, it is seen that $c(\mathscr{G})$ and $c_0(\mathscr{G})$ are BK-spaces. □

THEOREM 2. *$c(\mathscr{G}) \cong c$ and $c_0(\mathscr{G}) \cong c_0$.*

Proof. It is necessary to ensure that there exists a linear bijection between $c_0(\mathscr{G})$ and c_0 for the first part. The operator $\wp : c_0(\mathscr{G}) \to c_0$, $\wp(s) = \mathscr{G}s$ is linear. Since, $\wp(s) = 0 \Rightarrow s = 0$, $\wp$ is injective. Let us take $t = (t_l) \in c_0$ and $s = (s_l) \in \omega$ with

$$s_l = \sum_{j=0}^{l}(-1)^{l-j}\frac{M_{j+2}(a,b)-aM_{j+1}(a,b)}{bM_l(a,b)}\lambda_{l-j}t_j \quad (l \in \mathbb{N}).$$

It is obtained that $\wp$ is surjective from the equation

$$\begin{aligned}(\mathscr{G}s)_k &= \sum_{l=0}^{k}\frac{bM_l(a,b)M_{k-l}(a,b)}{M_{k+2}(a,b)-aM_{k+1}(a,b)}s_l\\ &= \sum_{l=0}^{k}\frac{bM_l(a,b)M_{k-l}(a,b)}{M_{k+2}(a,b)-aM_{k+1}(a,b)}\sum_{j=0}^{l}(-1)^{l-j}\frac{M_{j+2}(a,b)-aM_{j+1}(a,b)}{bM_l(a,b)}\lambda_{l-j}t_j\\ &= t_k.\end{aligned}$$

From the equality $\|s\|_{c_0(\mathscr{G})} = \|\mathscr{G}s\|_{c_0}$, we see that $\wp$ keeps the norm.

The other part is similar. □

If there is a unique scalars' sequence $(\overline{h}_k)$ as

$$\left\|s - \sum_{l=0}^{k}\overline{h}_l\tau_l\right\| \longrightarrow 0 \quad (k \to \infty)$$

for each $s \in \aleph$, normed sequence space $(\aleph, \|.\|)$ and $(\tau_k) \in \aleph$, in that case (τ_k) is Schauder basis for $\aleph$, and this is stated as $s = \sum_l \overline{h}_l\tau_l$.

It is noticed that the inverse image of the basis $(e^{(l)})_{l\in\mathbb{N}}$ of c and c_0 becomes the bases of $c(\mathscr{G})$ and $c_0(\mathscr{G})$ since $\wp$ is an isomorphism given in Theorem 2, so it will be given following result without proof.

THEOREM 3. *Consider the sequence $\tau^{(l)} = \left(\tau_k^{(l)}\right) \in c_0(\mathscr{G})$ expressed by*

$$\tau_k^{(l)} := \begin{cases}(-1)^{k-l}\dfrac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}\,, & \text{if } 0 \leqslant l \leqslant k,\\ 0\,, & \text{if } l > k.\end{cases}$$

Thus;

(i) *The set* $\{\tau^{(0)},\tau^{(1)},\tau^{(2)},\dots\}$ *is basis of* $c_0(\mathscr{G})$ *and each* $s\in c_0(\mathscr{G})$ *is stated as* $s=\sum_l \overline{h}_l\tau^{(l)}$ *uniquely for* $\overline{h}_l=(\mathscr{G}s)_l$.

(ii) *The set* $\{e,\tau^{(0)},\tau^{(1)},\tau^{(2)},\dots\}$ *is basis of* $c(\mathscr{G})$ *and each* $s\in c(\mathscr{G})$ *is stated as* $s=\sum_l(\overline{h}_l-\delta)\tau^{(l)}$ *uniquely for* $\delta=\lim_{l\to\infty}\overline{h}_l=\lim_{l\to\infty}(\mathscr{G}s)_l$.

THEOREM 4. *The inclusion* $c\subset c(\mathscr{G})$ *holds.*

Proof. Since the matrix $\mathscr{G}$ is conservative, the inclusion is seen. □

REMARK 1. From the expression

$$\lim_{k\to\infty}\mathfrak{g}_{kl}=bM_l(a,b)\lim_{k\to\infty}\frac{M_{k-l}(a,b)}{M_{k+2}(a,b)-aM_{k+1}(a,b)}=\frac{\sqrt{b}M_l(a,b)}{2(a+2\sqrt{b})^{l+1}}\neq 0,$$

it is explored that $\mathscr{G}\notin(c_0,c_0)$. Therefore, $c_0\nsubseteq c_0(\mathscr{G})$.

3. Dual spaces

The α-, β- and γ-duals of the newly defined spaces will be discussed in this part of the article.

LEMMA 2. [28] $F=(f_{kl})\in(c_0:\ell_1)=(c:\ell_1)$ *iff*

$$\sup_{A,B\in\mathscr{E}}\left|\sum_{k\in A}\sum_{l\in B}f_{kl}\right|<\infty,$$

where $\mathscr{E}$ *represents the family of all finite subsets of* $\mathbb{N}$.

THEOREM 5. $(c(\mathscr{G}))^\alpha=(c_0(\mathscr{G}))^\alpha=\mathfrak{I}_1$, *where*

$$\mathfrak{I}_1=\left\{\psi=(\psi_k)\in\omega:\sup_{A,B\in\mathscr{E}}\left|\sum_{k\in A}\sum_{l\in B}(-1)^{k-l}\frac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}\psi_k\right|<\infty\right\}.$$

Proof. By (5), it is reached that

$$\begin{aligned}\psi_k s_k&=\psi_k\left(\sum_{l=0}^{k}(-1)^{k-l}\frac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}t_l\right)\\&=\sum_{l=0}^{k}(-1)^{k-l}\frac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}\psi_k t_l=(Dt)_k\end{aligned}\tag{11}$$

for all $k\in\mathbb{N}$, where the infinite matrix $D=(d_{kl})$ described as

$$d_{kl}:=\begin{cases}(-1)^{k-l}\dfrac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_k(a,b)}\lambda_{k-l}\psi_k\,, & \text{if } 0\leqslant l\leqslant k,\\ 0\,, & \text{if } l>k.\end{cases}$$

Then, by (11), it is concluded that $\psi s = (\psi_k s_k) \in \ell_1$ while $s \in c(\mathscr{G})$ (or $s \in c_0(\mathscr{G})$) iff $Dt \in \ell_1$ while $t \in c$ (or $t \in c_0$). Thus, $\psi \in (c(\mathscr{G}))^{\alpha}$ (or $\psi \in (c_0(\mathscr{G}))^{\alpha}$) iff $D \in (c : \ell_1)$ (or $D \in (c_0 : \ell_1)$). By Lemma 2, $(c(\mathscr{G}))^{\alpha} = (c_0(\mathscr{G}))^{\alpha} = \Im_1$. $\square$

LEMMA 3. [28] *The following statements hold:*

(i) $F = (f_{kl}) \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ *iff the condition (7) holds.*

(ii) $F = (f_{kl}) \in (c_0 : c)$ *if and only if the conditions (7) and (8) hold.*

Let us describe the sets $\Im_2 - \Im_4$ which will be used to compute the β- and γ-duals:

$$\Im_2 = \left\{ \psi = (\psi_k) \in \omega : \sum_{k=l}^{\infty} (-1)^{k-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_k(a,b)} \lambda_{k-l} \psi_k \text{ exists for each } l \in \mathbb{N} \right\},$$

$$\Im_3 = \left\{ \psi = (\psi_k) \in \omega : \sup_{k\in\mathbb{N}} \sum_{l=0}^{k} \left| \sum_{j=l}^{k} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} \psi_j \right| < \infty \right\},$$

$$\Im_4 = \left\{ \psi = (\psi_k) \in \omega : \lim_{k\to\infty} \sum_{l=0}^{k} \sum_{j=l}^{k} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} \psi_j \text{ exists} \right\}.$$

THEOREM 6. *The following statements hold:*

(i) $\{c_0(\mathscr{G})\}^{\beta} = \Im_2 \cap \Im_3$,

(ii) $\{c(\mathscr{G})\}^{\beta} = \Im_2 \cap \Im_3 \cap \Im_4$.

Proof.

(i) Consider that $\psi = (\psi_l) \in \omega$ and $s \in c_0(\mathscr{G})$ with $t \in c_0$ as (5). By bearing in mind the equation (11) we reach that

$$\begin{aligned} \xi_k = \sum_{l=0}^{k} \psi_l s_l &= \sum_{l=0}^{k} \psi_l \left(\sum_{j=0}^{l} (-1)^{l-j} \frac{M_{j+2}(a,b) - aM_{j+1}(a,b)}{bM_l(a,b)} \lambda_{l-j} t_j \right) \\ &= \sum_{l=0}^{k} \left(\sum_{j=l}^{k} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} \psi_j t_l \right) \\ &= (Ot)_k \end{aligned} \tag{12}$$

for $O = (o_{kl})$ defined by

$$o_{kl} := \begin{cases} \sum_{j=l}^{k} (-1)^{j-l} \dfrac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} \psi_j \ , & 0 \leqslant l \leqslant k, \\ 0 & , \ l > k \end{cases} \tag{13}$$

for every $k, l \in \mathbb{N}$. Then, from (12), $\psi s \in cs$ while $s = (s_l) \in c_0(\mathscr{G})$ iff $\xi = (\xi_k) \in c$ while $t \in c_0$. Then, $\psi \in (c_0(\mathscr{G}))^{\beta}$ iff $O \in (c_0 : c)$. Thus, it is reached the expected result by conditions of $(c_0 : c)$.

The desired result for the other part can be achieved with the same thought. □

THEOREM 7. $\{c_0(\mathscr{G})\}^\gamma = \{c(\mathscr{G})\}^\gamma = \mathfrak{I}_3$.

Proof. This can be seen as similar to Theorem 6. □

4. Matrix mappings

This section aims to present some classes of matrix mappings from new sequence spaces to other known classical sequence spaces and vice versa. Now let us give the fundamental theorem of this section and some important related results.

THEOREM 8. *Consider the space* $\mho \subset \omega$ *and infinite matrices* $Z^{(k)} = \left(z_{il}^{(k)}\right)$ *and* $Z = (z_{kl})$ *expressed as*

$$z_{il}^{(k)} := \begin{cases} \sum_{j=l}^{i} (-1)^{j-l} \dfrac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \ , & 0 \leqslant l \leqslant i, \\ 0 & , \ l > i \end{cases} \tag{14}$$

and

$$z_{kl} = \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \tag{15}$$

for all $k,l \in \mathbb{N}$. *In that case,*

(i) $F = (f_{kl}) \in (c_0(\mathscr{G}) : \mho)$ *if and only if* $Z^{(k)} \in (c_0 : c)$ *and* $Z \in (c_0 : \mho)$,

(ii) $F = (f_{kl}) \in (c(\mathscr{G}) : \mho)$ *if and only if* $Z^{(k)} \in (c : c)$ *and* $Z \in (c : \mho)$.

Proof. (i) Assume that $F = (f_{kl}) \in (c_0(\mathscr{G}) : \mho)$ and $s \in c_0(\mathscr{G})$. Then,

$$\begin{aligned} \sum_{l=0}^{i} f_{kl} s_l &= \sum_{l=0}^{i} f_{kl} \left(\sum_{j=0}^{l} (-1)^{l-j} \frac{M_{j+2}(a,b) - aM_{j+1}(a,b)}{bM_l(a,b)} \lambda_{l-j} t_j \right) \\ &= \sum_{l=0}^{i} \left(\sum_{j=l}^{i} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \right) t_l \\ &= \sum_{l=0}^{i} z_{il}^{(k)} t_l \end{aligned} \tag{16}$$

for all $i,k \in \mathbb{N}$. Since Fs exists, it is expected that $Z^{(k)} \in (c_0 : c)$. By passing limit for $i \to \infty$ in the relation (16), it is reached that $Fs = Zt$. Since $Fs \in \mho$, in that case $Zt \in \mho$ and so $Z \in (c_0 : \mho)$.

Conversely, let us suppose that $Z^{(k)} \in (c_0 : c)$ and $Z \in (c_0 : \mho)$. Then, it is seen that $z_{kl} \in c_0^{\beta}$ which gives $(f_{kl})_{l\in\mathbb{N}} \in (c_0(\mathscr{G}))^{\beta}$ for all $k \in \mathbb{N}$. Hence, Fs exists. Therefore, it is reached from the relation (16) for $i \to \infty$ that $Fs = Zt$ and this implies that $F \in (c_0(\mathscr{G}) : \mho)$, which is desired result.

(ii) Since the proof of this part can be done similarly to the first part, we skip the proof to avoid repetition. □

LEMMA 4. [28] *$F = (f_{kl}) \in (c : c_0)$ if and only if the conditions (7), (10) and (9) hold with $\rho = 0$.*

Let us state the followings which will be required to give some matrix classes related with the new described sequence spaces.

$$\sup_{i\in\mathbb{N}} \sum_{l=0}^{i} \left| \sum_{j=l}^{i} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \right| < \infty \ (\forall k \in \mathbb{N}), \tag{17}$$

$$\lim_{i\to\infty} \sum_{j=l}^{i} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \text{ exists } (\forall k, l \in \mathbb{N}), \tag{18}$$

$$\lim_{i\to\infty} \sum_{l=0}^{i} \sum_{j=l}^{i} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \text{ exists } (\forall k \in \mathbb{N}), \tag{19}$$

$$\sup_{A,B\in\mathscr{E}} \left| \sum_{k\in A} \sum_{l\in B} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \right| < \infty, \tag{20}$$

$$\sup_{k\in\mathbb{N}} \sum_{l} \left| \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \right| < \infty, \tag{21}$$

$$\lim_{k\to\infty} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} = 0 \text{ for each } l \in \mathbb{N}, \tag{22}$$

$$\lim_{k\to\infty} \sum_{l} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} = 0, \tag{23}$$

$$\lim_{k\to\infty} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \text{ exists for each } l \in \mathbb{N}, \tag{24}$$

$$\lim_{k\to\infty} \sum_{l} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{kj} \text{ exists}, \tag{25}$$

$$\sup_{k\in\mathbb{N}} \sum_{l} \left| \sum_{s=0}^{k} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{sj} \right| < \infty, \tag{26}$$

$$\lim_{k\to\infty} \sum_{s=0}^{k} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{sj} = 0 \text{ for each } l \in \mathbb{N}, \tag{27}$$

$$\lim_{k\to\infty} \sum_{l} \sum_{s=0}^{k} \sum_{j=l}^{\infty} (-1)^{j-l} \frac{M_{l+2}(a,b) - aM_{l+1}(a,b)}{bM_j(a,b)} \lambda_{j-l} f_{sj} = 0, \tag{28}$$

$$\lim_{k\to\infty}\sum_{s=0}^{k}\sum_{j=l}^{\infty}(-1)^{j-l}\frac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_j(a,b)}\lambda_{j-l}f_{sj} \text{ exists for each } l\in\mathbb{N}, \tag{29}$$

$$\lim_{k\to\infty}\sum_{l}\sum_{s=0}^{k}\sum_{j=l}^{\infty}(-1)^{j-l}\frac{M_{l+2}(a,b)-aM_{l+1}(a,b)}{bM_j(a,b)}\lambda_{j-l}f_{sj} \text{ exists for each } l\in\mathbb{N}. \tag{30}$$

THEOREM 9. *The necessary and sufficient conditions for the classes* $(\aleph:\mho)$ *can be read from the Table 1, where* $\aleph\in\{c_0(\mathscr{G}),c(\mathscr{G})\}$ *and* $\mho\in\{\ell_1,c_0,c,\ell_\infty\}$.

Table 1: *Characterizations of* $(\aleph:\mho)$, *where* $\aleph\in\{c_0(\mathscr{G}),c(\mathscr{G})\}$ *and* $\mho\in\{\ell_1,c_0,c,\ell_\infty\}$.

$(\aleph\downarrow:\mho\rightarrow)$	ℓ_1	c_0	c	ℓ_∞
$c_0(\mathscr{G})$	(17), (18) (20)	(17), (18) (21), (22)	(17), (18) (21), (24)	(17), (18) (21)
$c(\mathscr{G})$	(17), (18) (19), (20)	(17), (18), (19) (21), (22), (23)	(17), (18), (19) (21), (24), (25)	(17), (18) (19), (21)

COROLLARY 1. *The necessary and sufficient conditions for the classes* $(\aleph:\mho)$ *can be read from the Table 2, where* $\aleph\in\{c_0(\mathscr{G}),c(\mathscr{G})\}$ *and* $\mho\in\{cs_0,cs,bs\}$.

Table 2: *Characterizations of* $(\aleph:\mho)$, *where* $\aleph\in\{c_0(\mathscr{G}),c(\mathscr{G})\}$ *and* $\mho\in\{cs_0,cs,bs\}$.

$(\aleph\downarrow:\mho\rightarrow)$	cs_0	cs	bs
$c_0(\mathscr{G})$	(17), (18) (26), (27)	(17), (18) (26), (29)	(17), (18) (26)
$c(\mathscr{G})$	(17), (18), (19) (26), (27), (28)	(17), (18), (19) (26), (29), (30)	(17), (18) (19), (26)

Now, we may give the following conditions collected from the study [28] to characterize some matrix classes in Lemma 5:

$$\sup_{k,l\in\mathbb{N}}|f_{kl}|<\infty, \tag{31}$$

$$\lim_{k\to\infty}\sum_{l}|f_{kl}|=0, \tag{32}$$

$$\lim_{k\to\infty}\sum_{l}|f_{kl}|=\sum_{l}\left|\lim_{k\to\infty}f_{kl}\right|. \tag{33}$$

LEMMA 5. [28] *For the infinite matrix* $F=(f_{kl})$, *the following statements hold:*

(i) $F\in(\ell_1:c_0)$ *if and only if the conditions* (10) *and* (31) *hold,*

(ii) $F\in(\ell_\infty:c_0)$ *if and only if the conditions* (10) *and* (32) *hold,*

(iii) $F\in(\ell_1:c)$ *if and only if the conditions* (8) *and* (31) *hold,*

(iv) *$F \in (\ell_\infty : c)$ if and only if the conditions (8) and (33) hold.*

THEOREM 10. *Consider the infinite matrices $U = (u_{kl})$ and $F = (f_{kl})$ whose elements are connected with the relation*

$$u_{kl} = \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \tag{34}$$

for all $k,l \in \mathbb{N}$. In that case, $F \in (\aleph : \mho(\mathscr{G}))$ if and only if $U \in (\aleph : \mho)$ for $\mho \in \{c_0, c\}$ and $\aleph \subset \omega$.

Proof. From the relation (34), it is achieved that

$$\sum_{l=0}^{\infty} u_{kl} s_l = \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} \sum_{l=0}^{\infty} f_{jl} s_l$$

for an arbitrary $s = (s_l) \in \aleph$. The last equation implies us that $U_k(s) = \mathscr{G}_k(Fs)$ for all $k \in \mathbb{N}$. In that case, it is reached the biconditional statement "$Fs \in \mho(\mathscr{G})$ if and only if $U_k \in \mho$ for $s \in \aleph$", which is desired result. □

Now, by keeping in mind the Theorem just mentioned above, we can give some conditions and the final result of the last section after these:

$$\lim_{k\to\infty} \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} = 0 \text{ for each } l \in \mathbb{N}, \tag{35}$$

$$\sup_{k,l\in\mathbb{N}} \left| \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \right| < \infty, \tag{36}$$

$$\lim_{k\to\infty} \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \text{ exists for each } l \in \mathbb{N}, \tag{37}$$

$$\sup_{k\in\mathbb{N}} \sum_l \left| \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \right| < \infty, \tag{38}$$

$$\lim_{k\to\infty} \sum_l \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} = 0, \tag{39}$$

$$\lim_{k\to\infty} \sum_l \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \text{ exists}, \tag{40}$$

$$\lim_{k\to\infty} \sum_{j=0}^{k} \left| \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \right| = 0, \tag{41}$$

$$\begin{aligned} &\lim_{k\to\infty} \sum_l \left| \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \right| \\ &= \sum_l \left| \lim_{k\to\infty} \sum_{j=0}^{k} \frac{bM_j(a,b)M_{k-j}(a,b)}{M_{k+2}(a,b) - aM_{k+1}(a,b)} f_{jl} \right|. \end{aligned} \tag{42}$$

THEOREM 11. *The necessary and sufficient conditions for the classes* $(\aleph : \mho)$ *can be read from the Table 3, where* $\aleph \in \{\ell_1, c_0, c, \ell_\infty\}$ *and* $\mho \in \{c_0(\mathscr{G}), c(\mathscr{G})\}$.

Table 3: *Characterizations of* $(\aleph : \mho)$, *where* $\aleph \in \{\ell_1, c_0, c, \ell_\infty\}$ *and* $\mho \in \{c_0(\mathscr{G}), c(\mathscr{G})\}$.

$(\aleph \downarrow : \mho \rightarrow)$	$c_0(\mathscr{G})$	$c(\mathscr{G})$
ℓ_1	(35), (36)	(36), (37)
c_0	(35), (38)	(37), (38)
c	(35), (38) (39)	(37), (38) (40)
ℓ_∞	(35), (41)	(37), (42)

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