Journal of
Classical
Analysis
Volume 27, Number 1 (2026), 35–43 doi:10.7153/jca-2026-27-03

GENERAL ONE–SIDED MAXIMAL OPERATORS ON WEIGHTED ONE–SIDED HERZ SPACES

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Abstract. This paper establishes the mapping properties of the general one-sided maximal operators from the weak weighted one-sided Herz spaces to the weighted one-sided Herz spaces. These mapping properties are inherited from the corresponding mapping properties from the Lebesgue spaces.

1. Introduction

This main result of this paper establishes the mapping properties of the general one-sided maximal function on the weighted one-sided Herz spaces.

The studies of the Hardy-Littlewood maximal function and the singular integral operators are the core results of harmonic analysis [31, 32]. The studies of the the singular integral operators and the Hardy-Littlewood maximal functions had been extended to the one-sided versions in [2, 20, 21, 22, 30]. They give us the mapping properties of the the Riemann-Liouville and Weyl fractional integral operators, the one-sided singular integral operators and the one-sided Hardy-Littlewood maximal functions on the weighted Lebesgue spaces [1, 16, 24]. The mapping properties of the one-sided operators also give applications on the ergodic theory [23]. In particular, the general one-sided maximal operators were introduced in [26]. The results in [26] provide a unified approach on for the studies of the one-sided operators on the weighted Lebesgue spaces.

The Herz spaces were introduced by Herz in [6] for the studies of the Fourier series. It is an extension of the Lebesgue spaces and it becomes one of the most important function spaces in harmonic analysis. It provides a number of applications on the partial differential equations and Fourier analysis [5, 19, 27, 28, 34]. Moreover, recently, there are several extensions of Herz spaces [13, 15, 17, 18, 33, 35]. The one-sided Herz spaces with variable exponent are introduced in [9] to study the boundedness of the one-sided maximal operators on the Herz type spaces. The one-sided Herz type spaces introduced in [9] are different form the classical Herz spaces.

In this paper, we introduce the weighted one-sided Herz spaces and establish the mapping properties of the general one-sided maximal operators on the weighted one-sided Herz spaces.

Mathematics subject classification (2020): 26D10, 42B25, 42B35, 46E40.
Keywords and phrases: One-sided maximal functions, one-sided fractional maximal operators, Herz spaces.

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Paper JCA-27-03

This paper is organized as follows. The definitions of the general one-sided maximal operators and the weighted one-sided Herz spaces are presented in Section 2. The mapping properties of the general one-sided maximal operators on the weighted Lebesgue spaces are also presented in this section. The main result of this paper is established in Section 3.

2. Preliminaries and definitions

Let $\mathscr{M}$ be the class consisting of all Lebesgue measurable functions $f:\mathbb{R}\to[-\infty,\infty]$. Let L^1_{loc} denote the class of locally integrable functions on $\mathbb{R}$.

For any Lebesgue measurable set E on $\mathbb{R}$ and non-negative Lebesgue measurable function v, we write $v(E)=\int_{\mathbb{E}} v(x)dx$.

For any $p\in(0,\infty)$ and non-negative Lebesgue measurable function v, the weighted Lebesgue space $L^p(v)$ consists of all $f\in\mathscr{M}$ satisfying

$$\|f\|_{L^p(v)}=\left(\int_{\mathbb{R}}|f(x)|^p v(x)dx\right)^{\frac{1}{p}}<\infty.$$

The weighted weak Lebesgue space $L^{p,\infty}(v)$ consists of all $f\in\mathscr{M}$ satisfying

$$\|f\|_{L^{p,\infty}(v)}=\sup_{\lambda>0}\lambda\,(v(\{x\in\mathbb{R}:|f(x)|>\lambda\}))^{\frac{1}{p}}<\infty.$$

We recall the general one-sided maximal operator from [26, Definition 1.1].

DEFINITION 1. Let $h:\{(x,y)\in\mathbb{R}^2: x<y\}\to(0,\infty)$ and $k:\{(x,y,z)\in\mathbb{R}^3: x<y<z\}\to(0,\infty)$ be Lebesgue measurable functions. Suppose that for any $(s,t)\in\{(x,y)\in\mathbb{R}: x<y\}$, $k(s,\cdot,t)$ is locally integrable. For any locally integrable function f on $\mathbb{R}$, define

$$M^+_{h,k}f(x)=\sup_{c\in(x,\infty)}h(x,c)\int_x^c|f(s)|k(x,s,c)ds.$$

When $h(x,y)=(y-x)^{-\beta}$ and $k(x,y,z)=(z-y)^{\alpha-1}$, $0\leqslant\beta\leqslant\alpha\leqslant1$, we write $M^+_{h,k}$ by $M^+_{\alpha,\beta}$. When $\alpha=\beta$, $M^+_{\alpha,\beta}$ is the operator introduced and studied in [14]. These operators also have applications on the Kakeya maximal operator for radial functions [4]. Moreover, when $\alpha=\beta\neq1$, $M^+_{\alpha,\beta}$ is the maximal operator associated to the Cesáro averages. When $\alpha=1$ and $\beta\in(0,1)$, $M^+_{\alpha,\beta}$ is the fractional one-sided maximal function [2, 24, 25].

We have the following result from [26, Theorem 2.1].

THEOREM 1. *Let $1\leqslant p<q<\infty$ and u,v be non-negative Lebesgue measurable functions on $\mathbb{R}$. If for any $x,y\in\mathbb{R}$ $h(x,\cdot)$ and $k(x,y,\cdot)$ are decreasing and there is a* *constant* $C>0$ *such that for any* $a<b<c$, u,v *satisfy*

$$h(a,c)\left(\int_a^b u(t)dt\right)^{\frac{1}{q}}\left(\int_a^c v^{1-p'}(s)k^{p'}(a,s,c)ds\right)^{\frac{1}{p'}}<C, \quad \text{if } p>1, \tag{1}$$

$$h(a,c)\left(\int_a^b u(t)dt\right)^{\frac{1}{q}}<C\operatorname*{ess\,inf}_{s\in(b,c)} v(s)k^{-1}(a,s,c), \quad \text{if } p=1, \tag{2}$$

then there is a constant C_0 *such that for any* $f\in L^p(v)$

$$\|M^+_{h,k}f\|_{L^{q,\infty}(u)}\leqslant C_0\|f\|_{L^p(v)}.$$

The following is the mapping properties of $M^+_{\alpha,\beta}$ on the weighted Lebesgue spaces from [26, Theorem 3.5].

THEOREM 2. *Let* $0<\beta\leqslant\alpha\leqslant 1$, $p\in(1,\infty)$ *and* u,v *be positive Lebesgue measurable functions on* $\mathbb{R}$. *If* $v^q=u^p$, u,v *satisfy* (1) *and*

$$\frac{1}{p}-\frac{1}{q}=\alpha-\beta,$$

then there exists a constant $C>0$ *such that for any* $f\in L^p(v)$

$$\|M^+_{\alpha,\beta}f\|_{L^q(u)}\leqslant C\|f\|_{L^p(v)}.$$

For further results on the weighted norm inequalities of the one-sided operators, see [1, 2, 16, 21, 22, 23, 24, 26, 30].

We now give the definition of the weighted one-sided Herz spaces on $(0,\infty)$. For any $k\in\mathbb{N}\backslash\{0\}$, define

$$I_0=(-1,1), \quad I_k=[2^{k-1},2^k), \quad I_{-k}=(-2^k,-2^{k-1}].$$

Define $\chi_k=\chi_{I_k}$.

DEFINITION 2. Let $\alpha\in\mathbb{R}$, $\theta,p\in(0,\infty)$ and v be a non-negative Lebesgue measurable function. The weighted one-sided Herz space $\mathring{K}^{\alpha,\theta}_p(v)$ consists of all $f\in\mathscr{M}$ satisfying

$$\|f\|_{\mathring{K}^{\alpha,\theta}_p(v)}=\left(\sum_{k\in\mathbb{Z}}2^{k\alpha\theta}\|f\chi_k\|^\theta_{L^p(v)}\right)^{\frac{1}{\theta}}<\infty.$$

The weighted weak one-sided Herz space $\mathring{K}^{\alpha,\theta}_{p,\infty}(v)$ consists of all $f\in\mathscr{M}$ satisfying

$$\|f\|_{\mathring{K}^{\alpha,\theta}_{p,\infty}(v)}=\left(\sum_{k\in\mathbb{Z}}2^{k\alpha\theta}\|f\chi_k\|^\theta_{L^{p,\infty}(v)}\right)^{\frac{1}{\theta}}<\infty.$$

When $v \equiv 1$, we write $\mathring{K}_p^{\alpha,\theta}(v)$ as $\mathring{K}_p^{\alpha,\theta}$. When $p = \theta$ and $\alpha = 0$, the weighted Herz space $\mathring{K}_p^{\alpha,\theta}(v)$ becomes the weighted Lebesgue space $L^p(v)$, therefore, the weighted one-sided Herz spaces are extensions of the weighted Lebesgue spaces. The boundedness of the one-sided geometric maximal operator on the weighted one-sided Herz spaces was obtained in [12]. The one-sided Herz spaces had also been extended to the one-sided Herz spaces with variable exponents in [9, 10].

Recall that the classical weighted Herz space $K_{p,\theta}^{\alpha}(v)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{K_{p,\theta}^{\alpha}(v)} = \left(\sum_{k=0}^{\infty} 2^{k\alpha\theta} \|f\chi_{I_k \cup I_{-k}}\|_{L^p(v)}^{\theta} \right)^{\frac{1}{\theta}} < \infty.$$

When $v \equiv 1$, we write $K_{p,\theta}^{\alpha}(v)$ as $K_{p,\theta}^{\alpha}$.

When $\alpha > 0$, it is easy to see that $K_{p,\theta}^{\alpha}(v) \hookrightarrow \mathring{K}_p^{\alpha,\theta}(v)$. Moreover, we have $K_{p,\theta}^{\alpha}(v) \subsetneq \mathring{K}_p^{\alpha,\theta}(v)$.

Define

$$f(x) = \sum_{k=-\infty}^{-1} \frac{1}{v(I_k)^{\frac{1}{p}}} \chi_{I_k}.$$

As $\alpha > 0$, we see that $\|f\|_{K_{p,\theta}^{\alpha}} = \infty$ while

$$\|f\|_{\mathring{K}_p^{\alpha,\theta}(v)} = \left(\sum_{k=-\infty}^{-1} 2^{k\alpha\theta} \right)^{\frac{1}{\theta}} < \infty.$$

Thus, $\mathring{K}_p^{\alpha,\theta}(v) \backslash K_{p,\theta}^{\alpha}(v) \neq \emptyset$. Similarly, whenever $\alpha < 0$, we find that $\mathring{K}_p^{\alpha,\theta}(v) \hookrightarrow K_{p,\theta}^{\alpha}(v)$ and $K_{p,\theta}^{\alpha}(v) \backslash \mathring{K}_p^{\alpha,\theta}(v) \neq \emptyset$.

Thus, the weighted one-sided Herz spaces are different from the classical weighted Herz spaces. In [9], by taking the exponent function to be a constant function $p(\cdot) = p$ with $p \in (1,\infty)$, we find that when $\alpha > 0$ the one-sided Hardy-Littlewood maximal function is bounded on the one-sided Herz type spaces, while the boundedness of the Hardy-Littlewood maximal function on the classical Herz spaces requires that the α satisfies $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$ [19, Theorem 5.1.2 and Remark 5.1.3].

For the studies of the classical Herz spaces and its applications in harmonic analysis and partial differential equations, see [5, 6, 7, 8, 11, 13, 15, 19, 27, 28, 34, 35].

3. Main result

We establish the mapping properties of the general one-sided maximal operators on the weighted one-sided Herz spaces in this section. The followings are the main results of this paper, THEOREM 3. *Let* $\alpha \in (0,\infty)$, $\theta \in (0,\infty)$, $1 \leqslant p < q < \infty$ *and* u,v *be non-negative Lebesgue measurable functions on* $\mathbb{R}$. *If for any* $x,y \in \mathbb{R}$ $h(x,\cdot)$ *and* $k(x,y,\cdot)$ *is decreasing and* u,v *satisfy* (1)–(2), *then there exists a constant* $C > 0$ *such that for any* $f \in \mathring{K}_p^{\alpha,\theta}(v)$

$$\|M_{h,k}^+ f\|_{\mathring{K}_{q,\infty}^{\alpha,\theta}(u)} \leqslant C\|f\|_{\mathring{K}_p^{\alpha,\theta}(v)}.$$

Proof. Let $f \in \mathring{K}_p^{\alpha,\theta}(v)$ and $i \in \mathbb{N}$. Whenever $x \in I_i$, for any $c > x$, it follows that $(x,c) \subset (x,\infty) \subset \cup_{j=i}^{\infty} I_j$. Therefore,

$$\begin{aligned} &h(x,c)\int_x^c |f(s)|k(x,s,c)ds \\ &= h(x,c)\int_x^c \chi_{\cup_{j=i}^{\infty} I_j}(s)|f(s)|k(x,s,c)ds \\ &\leqslant M_{h,k}^+(\chi_{\cup_{j=i}^{\infty} I_j} f)(x). \end{aligned} \tag{3}$$

By taking the supremum over $c \in (x,\infty)$ on (3), we find that for any $x \in I_i$

$$\chi_{I_i}(x)M_{h,k}^+ f(x) \leqslant M_{h,k}^+(\chi_{\cup_{j=i}^{\infty} I_j} f)(x). \tag{4}$$

When $x \notin I_i$, we have

$$\chi_{I_i}(x)M_{h,k}^+ f(x) = 0 \leqslant M_{h,k}^+(\chi_{\cup_{j=i}^{\infty} I_j} f)(x). \tag{5}$$

Thus, (4) and (5) guarantee that for any $x \in \mathbb{R}$,

$$\chi_{I_i}(x)M_{h,k}^+ f(x) \leqslant M_{h,k}^+(\chi_{\cup_{j=i}^{\infty} I_j} f)(x). \tag{6}$$

We apply the norm $\|\cdot\|_{L^{q,\infty}(u)}$ on both sides of (6), we find that

$$\|\chi_{I_i} M_{h,k}^+ f\|_{L^{q,\infty}(u)} \leqslant \|M_{h,k}^+(\chi_{\cup_{j=i}^{\infty} I_j} f)\|_{L^{q,\infty}(u)}.$$

Theorem 1 yields

$$\|\chi_{I_i} M_{h,k}^+ f\|_{L^{q,\infty}(u)} \leqslant C\|\chi_{\cup_{j=i}^{\infty} I_j} f\|_{L^p(v)} \leqslant C\sum_{j=i}^{\infty} \|\chi_{I_j} f\|_{L^p(v)} \tag{7}$$

for some $C > 0$.

We now split the estimate into two cases, $\theta \in (1,\infty)$ and $\theta \in (0,1]$.

We first consider the case $\theta \in (0,1]$. The θ-inequality and (7) yield

$$\begin{aligned} \sum_{i\in\mathbb{Z}} (2^{i\alpha}\|\chi_{I_i} M_{h,k}^+ f\|_{L^{q,\infty}(u)})^\theta &\leqslant C\sum_{i\in\mathbb{Z}} \left(\sum_{j=i}^{\infty} 2^{i\alpha}\|\chi_{I_j} f\|_{L^p(v)}\right)^\theta \\ &= C\sum_{i\in\mathbb{Z}} \left(\sum_{j=i}^{\infty} 2^{(i-j)\alpha} 2^{j\alpha}\|\chi_{I_j} f\|_{L^p(v)}\right)^\theta \\ &\leqslant C\sum_{i\in\mathbb{Z}} \sum_{j=i}^{\infty} 2^{\theta(i-j)\alpha}(2^{j\alpha}\|\chi_{I_j} f\|_{L^p(v)})^\theta. \end{aligned}$$

We interchange the summations and obtain

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+_{h,k}f\|_{L^{q,\infty}(u)})^\theta &\leqslant C\sum_{j\in\mathbb{Z}}\sum_{i=-\infty}^{j}2^{\theta(i-j)\alpha}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\\ &= C\sum_{j\in\mathbb{Z}}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\sum_{i=-\infty}^{j}2^{\theta(i-j)\alpha}.\end{aligned}$$

As $\alpha>0$, we have $\sum_{i=-\infty}^{j}2^{\theta(i-j)\alpha}<C$ for some $C>0$ independent of j. Hence,

$$\begin{aligned}\|M^+_{h,k}f\|_{\dot{K}^{\alpha,\theta}_{q,\infty}(u)} &= \left(\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+f\|_{L^{q,\infty}(u)})^\theta\right)^{\frac{1}{\theta}}\\ &\leqslant C\left(\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}f\|_{L^p(v)})^\theta\right)^{\frac{1}{\theta}} = C\|f\|_{\dot{K}^{\alpha,\theta}_{p}(v)}\end{aligned}\tag{8}$$

for some $C>0$.

When $\theta\in(1,\infty)$, (7) gives

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+_{h,k}f\|_{L^{q,\infty}(u)})^\theta &\leqslant C\sum_{i\in\mathbb{Z}}\left(\sum_{j=i}^{\infty}2^{i\alpha}\|\chi_{I_j}f\|_{L^p(v)}\right)^\theta\\ &= C\sum_{i\in\mathbb{Z}}\left(\sum_{j=i}^{\infty}2^{(i-j)\alpha}2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)}\right)^\theta.\end{aligned}$$

The Hölder inequality and [29, Proposition 1.2] give

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+_{h,k}f\|_{L^{q,\infty}(u)})^\theta\\ &\leqslant C\sum_{i\in\mathbb{Z}}\left(\sum_{j=i}^{\infty}2^{\frac{\theta}{2}(i-j)\alpha}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\right)\left(\sum_{j=i}^{\infty}2^{\frac{\theta'}{2}(i-j)\alpha}\right)^{\frac{\theta}{\theta'}}.\end{aligned}$$

As $\alpha>0$, we find that $\sum_{j=i}^{\infty}2^{\frac{\theta'}{2}(i-j)\alpha}<C$ for some $C>0$ independent of j. Therefore, by interchanging the summations, we obtain

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+_{h,k}f\|_{L^{q,\infty}(u)})^\theta &\leqslant C\sum_{i\in\mathbb{Z}}\sum_{j=i}^{\infty}2^{\frac{\theta}{2}(i-j)\alpha}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\\ &\leqslant C\sum_{j\in\mathbb{Z}}\sum_{i=-\infty}^{j}2^{\frac{\theta}{2}(i-j)\alpha}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\\ &= C\sum_{j\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\sum_{i=-\infty}^{j}2^{\frac{\theta}{2}(i-j)\alpha}\\ &\leqslant C\sum_{j\in\mathbb{Z}}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\end{aligned}$$

where we have the last inequality because $\alpha > 0$. Hence,

$$\|M^+_{h,k}f\|_{\mathring{K}^{\alpha,\theta}_{q,\infty}(u)} = \left(\sum_{i\in\mathbb{Z}}(2^{i\alpha}\|\chi_{I_i}M^+_{h,k}f\|_{L^{q,\infty}(u)})^\theta\right)^{\frac{1}{\theta}}$$
$$\leqslant C\left(\sum_{j\in\mathbb{Z}}(2^{j\alpha}\|\chi_{I_j}f\|_{L^p(v)})^\theta\right)^{\frac{1}{\theta}} = C\|f\|_{\mathring{K}^{\alpha,\theta}_{p}(v)}. \tag{9}$$

Therefore, (8) and (9) yield the boundedness of $M^+_{h,k} : \mathring{K}^{\alpha,\theta}_{p}(v) \to \mathring{K}^{\alpha,\theta}_{q,\infty}(u)$. $\square$

We also have the mapping properties of $M^+_{\alpha,\beta}$ on the weighted one-sided Herz spaces.

THEOREM 4. *Let* $\gamma,\theta \in (0,\infty)$, $0<\beta\leqslant\alpha\leqslant 1$, $p\in(1,\infty)$, $q\in(p,\infty)$ *and* u,v *be positive Lebesgue measurable functions on* $\mathbb{R}$. *If* $v^q = u^p$, u,v *satisfy* (1) *and*

$$\frac{1}{p}-\frac{1}{q} = \alpha-\beta,$$

then there exists a constant $C>0$ *such that for any* $f\in\mathring{K}^{\gamma,\theta}_{p}(v)$

$$\|M^+_{\alpha,\beta}f\|_{\mathring{K}^{\gamma,\theta}_{q}(u)} \leqslant C\|f\|_{\mathring{K}^{\gamma,\theta}_{p}(v)}.$$

The proof of the above theorem is similar to the proof of Theorem 3, therefore, for brevity, we leave the proof to the reader.

When $\beta\in(0,1)$, $M^+_{1,\beta}$ is the one-sided fractional maximal operator. Theorem 4 gives the mapping properties of the one-sided fractional maximal operators on the weighted one-sided Herz spaces.

COROLLARY 1. *Let* $\gamma,\theta \in (0,\infty)$, $0<\beta<1$, $p\in(1,\infty)$, $q\in(p,\infty)$ *and* u,v *be positive Lebesgue measurable functions on* $\mathbb{R}$. *If* $v^q = u^p$, u,v *satisfy* (1) *and*

$$\frac{1}{p}-\frac{1}{q} = 1-\beta,$$

then there exists a constant $C>0$ *such that for any* $f\in\mathring{K}^{\gamma,\theta}_{p}(v)$

$$\|M^+_{1,\beta}f\|_{\mathring{K}^{\gamma,\theta}_{q}(u)} \leqslant C\|f\|_{\mathring{K}^{\gamma,\theta}_{p}(v)}.$$

Declaration

Funding. Not applicable.

Conflicts of interest/Competing interests. Not applicable.

Availability of data and material. Not applicable.

Code availability. Not applicable.

Acknowledgements. The author thanks the reviewers for their valuable suggestions that improved the presentation of this paper.

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(Received May 1, 2025)

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