Journal of Classical Analysis
Volume 27, Number 1 (2026), 45–61
doi:10.7153/jca-2026-27-04

ALMOST CONVERGENCE AND QUINTET BAND MATRIX

KÜBRA TOPAL

Abstract. In this study, we define the sequence spaces $f(G(r,s,t,u,v))$, $f_0(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$. We demonstrate that these newly defined spaces are BK-spaces and are linearly isomorphic to the sequence spaces f, f_0 and fs, respectively. Additionally, we provide the Schauder basis and determine the β- and γ-duals of these spaces. Finally, we characterize certain matrix classes associated with these spaces.

1. Introduction

w is a set of all real (or complex) valued sequences any vektor subspace of w is called a sequence space where w is a vector space under pointwise addition and scalar multiplication. The symbols ℓ_∞, c, c_0 and ℓ_p symbolize all bounded, convergent, null and absolutely p-summable sequence spaces, respectively, where $1 \leqslant p < \infty$.

A Banach sequence space is defined as a BK-space each of the maps $p_n : Z \to \mathbb{C}$ defined by $p_n(z) = z_n$ which must be continuous for all $n \in \mathbb{N}$ [14]. Considering this definition, we can say that ℓ_∞, c and c_0 are BK-spaces with sup-norm defined by $\|z\|_\infty = \sup_{k\in\mathbb{N}} |z_k|$ and ℓ_p is a BK-space with its p-norm defined by

$$\|z\|_p = \left(\sum_{k=0}^{\infty} |z_k|^p \right)^{\frac{1}{p}}$$

where $1 \leqslant p < \infty$ [23].

Given an infinite matrix $A = (a_{nk})$, the A-transform of a sequence $z = (z_k) \in w$ is defined by:

$$(Az)_n = \sum_{k=0}^{\infty} a_{nk} z_k.$$

Here, it is assumed that this transformation is convergent for each $n \in \mathbb{N}$ [30].

Then, by $(Z : T)$, we denote the class of matrices from Z into T and it is define as follows:

$$(Z : T) = \{A = (a_{nk}) : Az \in T \text{ for all } z \in Z\}.$$

Mathematics subject classification (2020): 40C05, 40H05, 46B45.
Keywords and phrases: Matrix domain, Schauder basis, β- and γ-duals, Banach limits, almost convergence and matrix classes.
This work was supported by the TÜBİTAK BİDEB 2211.

Paper JCA-27-04

Furthermore, the matrix domain of A in a sequence space Z, denoted by Z_A, is a sequence space defined as:

$$Z_A = \{z = (z_k) \in w : Az \in Z\}. \tag{1}$$

Let Z and T be two any Banach spaces and suppose $L : Z \to T$ is a bounded linear operator. In this case, we denote by $D(L)$, $R(L)$ and $B(Z,T)$ the domain of L, the range of L and the set of all bounded linear operators from Z to T, respectively. Additionally, for simplicity, $B(Z)$ is written instead of $B(Z : Z)$.

Now, let us the denote the sets of all bounded and convergent series by $bs = (\ell_\infty)_s$ and $cs = c_s$, respectively, where $S = (s_{nk})$ denotes the summation matrix as follows:

$$s_{nk} = \begin{cases} 1, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Stefan Banach, using the Hahn-Banach theorem in the space ℓ_∞, introduced the idea of Banach Limits. Banach identified specific non-negative linear functionals on ℓ_∞ that are invariant under shift operators and serve as extensions of the functional l, defined on c as

$$\begin{array}{rcl} \ell : c & \longrightarrow & \mathbb{R} \\ z = (z_n) & \longmapsto & \ell(z) = \lim\limits_{n\to\infty} z_n \end{array}$$

which is a linear functional on c. These functionals later became known as Banach Limits [14].

A linear functional $L : \ell_\infty \to \mathbb{R}$ is classified as a Banach Limit if it fulfills the following properties:

1. $L(z_n) \geqslant 0$ if $z \geqslant 0$, $n = 0, 1, 2, \ldots$
2. $L(P_j(z_n)) = L(z_n)$, $P_j(z_n) = z_{n+j}$, $j = 1, 2, 3, \ldots$
3. $L(e) = 1$, where $e = (1, 1, 1, \ldots)$.

Lorentz further explored the concept of Banach Limits and introduced a new idea known as 'almost convergence'. A sequence $z = (z_n) \in \ell_\infty$ is said to be almost convergent and the value $Lim\, z_n = \lambda$ is referred to as its F-limit if $L(z_n) = \lambda$ is satisfied for every limit L [22]. For further details on fundamental theorems in functional analysis, summability theory, sequence spaces, and almost convergence, readers can refer to the recent textbooks [2] and [25].

By using the forward difference operator Δ the difference spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ from the classical sequence spaces c_0, c and ℓ_∞ are introduced by Kızmaz, in [19]. Later, in the year 2003 the difference space bv_p has been constructed as the domain of backward difference matrix ∇ in the classical space ℓ_p of absolutely p-summable sequences in the case $1 \leqslant p < \infty$ by Başar and Altay [3] and in the case $0 < p < 1$ by Altay and Başar [1]. The construction of a new sequence space was used through the domains of difference matrices $\Delta c_o(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$ in

[31], $c_0(u,\Delta,p)$, $c(u,\Delta,p)$ and $\ell_\infty(u,\Delta,p)$ in [9], $c_0(\Delta^2)$, $c(\Delta^2)$ and $\ell_\infty(\Delta^2)$ in [17], $c_0(u,\Delta^2)$, $c(u,\Delta^2)$ and $\ell_\infty(u,\Delta^2)$ in [24], $c_0(u,\Delta^2,p)$, $c(u,\Delta^2,p)$ and $\ell_\infty(u,\Delta^2,p)$ in [10], $c_0(\Delta^m)$, $c(\Delta^m)$ and $\ell_\infty(\Delta^m)$ in [16], $\hat{\ell_\infty}$, $\hat{c_o}$, $\hat{c}$ and $\hat{\ell_p}$ in [21], $\ell_\infty(B)$, $c_0(B)$, $c(B)$ and $\ell_p(B)$ in [28], $\hat{f_0}$ and $\hat{f}$ in [4], $f_0(B)$ and $f(B)$ in [29], $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u)$ and $f_s(Q(r,s,t,u)$ in [11].

2. Almost convergent quintet band matrix

In this part, we review some previous research and define new sequence spaces $f_0(G(r,s,t,u,v))$, $f(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$. Additionally, we show that they are BK-spaces. Later, we also show that these spaces are linearly isomorphic to the spaces f_0, f and fs, respectively.

Lorentz provided a characterization of almost convergent sequences through the following theorem.

THEOREM 1. [22] *For the F-limit, $Lim\, z_n = \lambda$, to exist for the sequence $z = (z_n)$, it is required and sufficient that the following condition is satisfied uniformly in n:*

$$\lim_{i\to\infty} \frac{z_n + z_{n+1} + \ldots + z_{n+i}}{i+1} = \lambda.$$

By relating the concept of almost convergence to Theorem 1, the spaces f_0, f and fs are defined as the sets of all almost null sequences, almost convergent sequences and almost convergent series are defined as:

$$f_0 = \left\{ z = (z_k) \in w : \lim_{i\to\infty} \sum_{k=0}^{i} \frac{z_{n+k}}{i+1} = 0 \text{ uniformly in } n \right\},$$

$$f = \left\{ z = (z_k) \in w : \exists\lambda \in \mathbb{C} \lim_{i\to\infty} \sum_{k=0}^{i} \frac{z_{n+k}}{i+1} = \lambda \text{ uniformly in } n \right\}$$

and

$$fs = \left\{ z = (z_k) \in w : \exists\lambda \in \mathbb{C} \lim_{i\to\infty} \sum_{k=0}^{i} \sum_{j=0}^{n+k} \frac{z_j}{i+1} = \lambda \text{ uniformly in } n \right\},$$

in turn. Taking into account (1), the sequence space fs can be expressed in a new form as follows:

$$fs = f_S.$$

THEOREM 2. [13] *The subsumptions $c \subset f \subset \ell_\infty$ strictly hold.*

THEOREM 3. [13] *The spaces f and f_0 are classified as BK-spaces, defined by the norm*

$$\|z\|_f = \sup_{i,n\in\mathbb{N}} \left| \sum_{k=0}^{i} \frac{z_{n+k}}{i+1} \right|$$

and fs is classified as BK-space, defined by the norm $\|z\|_{fs} = \|Sz\|_f$.

Kızmaz, as noted in [19], was the first to define sequence spaces using the forward difference matrix, introduced the spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ as follows:

$$c_0(\Delta) = \{z = (z_k) \in w : \lim_{k\to\infty}(z_k - z_{k+1}) = 0\},$$

$$c(\Delta) = \{z = (z_k) \in w : \lim_{k\to\infty}(z_k - z_{k+1}) \text{ exists}\}$$

and

$$\ell_\infty(\Delta) = \{z = (z_k) \in w : \sup_{k\in\mathbb{N}} |z_k - z_{k+1}| < \infty\}.$$

The difference matrix $\Delta = (e_{nk})$ is defined by,

$$e_{nk} = \begin{cases} 1, & k = n \\ -1, & k = n+1 \\ 0, & \text{o.w.} \end{cases}$$

for all $n,k \in \mathbb{N}$.

Thereafter, Başar and Kirişçi utilized the generalized difference matrix to define the generalized difference sequence spaces $\hat{f}_0$ and $\hat{f}$ in [4] as:

$$\hat{f}_0 = \left\{ z = (z_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rz_{n+j} + sz_{n+j-1}}{m+1} = 0 \text{ uniformly in } n \right\}$$

and

$$\hat{f} = \left\{ z = (z_k) \in w : \exists\lambda \in \mathbb{C} \ \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rz_{n+j} + sz_{n+j-1}}{m+1} = \lambda \text{ uniformly in } n \right\}.$$

Additionally, Sönmez utilized the triple band matrix in [29] to establish the spaces $f_0(B)$ and $f(B)$, which are defined by,

$$f_0(B) = \left\{ z = (z_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rz_{n+j} + sz_{n+j-1} + tz_{n+j-2}}{m+1} = 0 \text{ uniformly in } n \right\}$$

and

$$f(B) = \left\{ z = (z_k) \in w : \exists\lambda \in \mathbb{C} \ \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rz_{n+j} + sz_{n+j-1} + tz_{n+j-2}}{m+1} = \lambda \text{ uniformly in } n \right\}.$$

Furthermore, Bişgin employed the quadruple band matrix to define the almost convergent sequence spaces $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u)$ and $fs(Q(r,s,t,u))$ in [11], given as:

$$f_0(Q) = \left\{ z = (z_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rz_{n+j} + sz_{n+j-1} + tz_{n+j-2} + uz_{n+j-3}}{m+1} = 0 \right. \\ \left. \text{uniformly in } n \right\},$$

$$f(Q)=\left\{z=(z_k)\in w:\exists\lambda\in\mathbb{C}\ \lim_{m\to\infty}\sum_{j=0}^{m}\frac{rz_{n+j}+sz_{n+j-1}+tz_{n+j-2}+uz_{n+j-3}}{m+1}=\lambda \text{ uniformly in } n\right\}$$

and

$$fs(Q)=\left\{z=(z_k)\in w:\exists\lambda\in\mathbb{C}\ \lim_{m\to\infty}\sum_{j=0}^{m}\sum_{v=0}^{n+j}\frac{rz_{v}+sz_{v-1}+tz_{v-2}+uz_{v-3}}{m+1}=\lambda \text{ uniformly in } n\right\}.$$

Lately, Bişgin introduced the sequence spaces $c_0(G)$, $c(G)$, $\ell_\infty(G)$ and $\ell_p(G)$, defined as follows:

$$c_0(G)=\left\{z=(z_k)\in w:\lim_{k\to\infty}(rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4})=0\right\},$$

$$c(G)=\left\{z=(z_k)\in w:\lim_{k\to\infty}(rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4})\text{ exists}\right\},$$

$$\ell_\infty(G)=\left\{z=(z_k)\in w:\sup_{k\in\mathbb{N}}|rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4}|<\infty\right\}$$

and

$$\ell_p(G)=\left\{z=(z_k)\in w:\sum_k|rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4}|^p<\infty\right\},\quad(1\leqslant p<\infty).$$

Also, the quintet band matrix $G=G(r,s,t,u,v)=(g_{nk}(r,s,t,u,v))$, which was introduced by Bişgin [12], is defined as:

$$g_{nk}(r,s,t,u,v)=\begin{cases} r, & k=n\\ s, & k=n-1\\ t, & k=n-2\\ u, & k=n-3\\ v, & k=n-4\\ 0, & \text{o.w.}\end{cases}$$

for all $n,k\in\mathbb{N}$ and $r,s,t,u,v\in\mathbb{C}\backslash\{0\}$. Here, it is easily seen that the equations, $G(1,-4,6,-4,1)=\Delta^4$, $G(r,s,t,u,0)=Q(r,s,t,u)$, $G(1,-3,3,-1,0)=\Delta^3$, $G(1,-\frac{3}{2},1,-\frac{1}{4},0)=\Delta_i^3$, $G(r,s,t,0,0)=B(r,s,t)$, $G(1,-2,1,0,0)=\Delta^2$, $G(r,s,0,0,0)=B(r,s)$ and $G(1,-1,0,0,0)=\Delta$, where Δ^4, $Q(r,s,t,u)$, Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called fourth order difference, quadruple band, third order difference, triple band, second order difference, double band and difference matrix, respectively. Hence, our results obtained in this paper are more comprehensive than the corresponding results in the existing literature.

Now, we define the sequence spaces,

$$f_0(G)=\left\{z=(z_k)\in w:\lim_{i\to\infty}\sum_{j=0}^{i}\frac{rz_{n+j}+sz_{n+j-1}+tz_{n+j-2}+uz_{n+j-3}+vz_{n+j-4}}{i+1}=0\right.$$
$$\left.\text{uniformly in } n\right\},$$

$$f(G)=\left\{z=(z_k)\in w:\exists\lambda\in\mathbb{C}\ni\lim_{i\to\infty}\sum_{j=0}^{i}\frac{rz_{n+j}+sz_{n+j-1}+tz_{n+j-2}+uz_{n+j-3}+vz_{n+j-4}}{i+1}\right.$$
$$\left.=\lambda\text{ uniformly in } n\right\}$$

and

$$fs(G)=\left\{z=(z_k)\in w:\exists\lambda\in\mathbb{C}\ni\ \lim_{i\to\infty}\sum_{j=0}^{i}\sum_{l=0}^{n+j}\frac{rz_l+sz_{l-1}+tz_{l-2}+uz_{l-3}+vz_{l-4}}{i+1}=\lambda\right.$$
$$\left.\text{uniformly in } n\right\},$$

in turn.

From the expression (1), it is evident that the transformation $G(r,s,t,u,v)$ of an arbitrary sequence $z=(z_k)\in w$ can be defined as follows:

$$y_k=(G(r,s,t,u,v)z)_k=rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4}$$

for $\forall k\in\mathbb{N}$. Also, the spaces $f_0(G)$, $f(G)$ and $fs(G)$ can be redefined with the domain of the quintet band matrix $G=G(r,s,t,u,v)$ by:

$$f_0(G)=(f_0)_G,\ \ f(G)=(f)_G\ \text{ and }\ fs(G)=(fs)_G. \tag{2}$$

THEOREM 4. *The spaces $f_0(G)$ and $f(G)$ are BK-spaces with the norm*

$$\|x\|_{f(G)}=\|x\|_{f_0(G)}=\|G(r,s,t,u,v)\|_f$$

and the space $fs(G)$ is BK-space with the norm

$$\|x\|_{fs(G)}=\|G(r,s,t,u,v)x\|_{fs}.$$

Proof. We know that f, f_0 and fs are BK-spaces. If we consider that the quintet band matrix is a triangle matrix and condition (2) is satisfied, we obtain that $f(G(r,s,t,u,v))$, $f_0(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$ are BK-spaces by Wilansky's Theorem 4.3.12 [30]. Thus, the proof is complete. □

Now, for $r,s,t,u,v\in\mathbb{C}\backslash\{0\}$, we consider the equation

$$rz^4+sz^3+tz^2+uz+v=0.$$

The roots of this equation are as follows:

$$\begin{aligned}
\sigma_1 &= -\frac{s}{4r}-\frac{1}{2}c-\frac{1}{2}d,\\
\sigma_2 &= -\frac{s}{4r}-\frac{1}{2}c+\frac{1}{2}d,\\
\sigma_3 &= -\frac{s}{4r}+\frac{1}{2}c-\frac{1}{2}p,\\
\sigma_4 &= -\frac{s}{4r}+\frac{1}{2}c+\frac{1}{2}p,
\end{aligned}$$

where

$$\begin{aligned}
a &= t^2-3su+12vr,\\
b &= -2t^3+9stu+72rtv-27ru^2-27s^2v,\\
c &= \sqrt{\frac{s^2}{4r^2}-\frac{2t}{3r}-\frac{1}{3\sqrt[3]{2}r}\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}-\frac{\sqrt[3]{2}a}{3r\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}}},\\
d &= \sqrt{\frac{s^2}{2r^2}-\frac{4t}{3r}+\frac{1}{3\sqrt[3]{2}r}\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}+\frac{\sqrt[3]{2}a}{3r\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}}-\frac{-\frac{s^3}{r^3}+\frac{4st}{r^2}-\frac{8u}{r}}{4c}}\\
\rho &= \sqrt{\frac{s^2}{2r^2}-\frac{4t}{3r}+\frac{1}{3\sqrt[3]{2}r}\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}+\frac{\sqrt[3]{2}a}{3r\left(b-\sqrt{b^2-4a^3}\right)^{\frac{1}{3}}}+\frac{-\frac{s^3}{r^3}+\frac{4st}{r^2}-\frac{8u}{r}}{4c}}.
\end{aligned}$$

Also, after some calculations we obtain:

$$\sigma_1+\sigma_2+\sigma_3+\sigma_4=-\frac{s}{r}$$

$$\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4=\frac{t}{r}$$

$$\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4=-\frac{u}{r}$$

$$\sigma_1\sigma_2\sigma_3\sigma_4=\frac{v}{r}$$

$$\sigma_1^4+\frac{s}{r}\sigma_1^3+\frac{t}{r}\sigma_1^2+\frac{u}{r}\sigma_1+\frac{v}{r}=0 \tag{3}$$

$$\begin{aligned}
\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4&\\
+\frac{s}{r}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+\frac{t}{r}=0&
\end{aligned} \tag{4}$$

$$\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3+\frac{s}{r}(\sigma_1+\sigma_2+\sigma_3)+\frac{t}{r}=0 \tag{5}$$

$$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\frac{s}{r}=0 \tag{6}$$

$$\sigma_1^3+\sigma_2^3+\sigma_1\sigma_2^2+\sigma_2\sigma_1^2+\frac{s}{r}(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)+\frac{t}{r}(\sigma_1+\sigma_2)+\frac{u}{r}=0 \tag{7}$$

$$\begin{aligned}&\sigma_1^3+\sigma_2^3+\sigma_3^3+\sigma_1\sigma_2^2+\sigma_1\sigma_3^2+\sigma_2\sigma_1^2+\sigma_2\sigma_3^2+\sigma_3\sigma_1^2+\sigma_3\sigma_2^2+\sigma_1\sigma_2\sigma_3\\&\quad+\frac{s}{r}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)+\frac{t}{r}(\sigma_1+\sigma_2+\sigma_3)+\frac{u}{r}=0\end{aligned} \tag{8}$$

$$\begin{aligned}&\sigma_1^3+\sigma_2^3+\sigma_3^3+\sigma_4^3+\sigma_1\sigma_2^2+\sigma_1\sigma_3^2+\sigma_1\sigma_4^2+\sigma_2\sigma_1^2+\sigma_2\sigma_3^2+\sigma_2\sigma_4^2+\sigma_3\sigma_1^2\\&\quad+\sigma_3\sigma_2^2+\sigma_3\sigma_4^2+\sigma_4\sigma_1^2+\sigma_4\sigma_2^2+\sigma_4\sigma_3^2+\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4\\&\quad+\frac{s}{r}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)\\&\quad+\frac{t}{r}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+\frac{u}{r}=0.\end{aligned} \tag{9}$$

THEOREM 5. *The spaces $f(G(r,s,t,u,v))$, $f_0(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$ are linearly isomorphic to the spaces f, f_0 and fs, in turn.*

Proof. We give the proof only for $f(G(r,s,t,u,v))$. Others can be proved by similar methods. Accordingly, it is shown that there exists a linear bijection between the sequence spaces $f(G(r,s,t,u,v))$ and f. Define the transformation $M: f(G(r,s,t,u,v)) \to f$ by $M(z)=G(r,s,t,u,v)z$. Clearly, if $z=(z_k)\in f(G(r,s,t,u,v))$, then $M(z)=G(r,s,t,u,v)z\in f$. Moreover, M is obviously a linear transformation and $M(z)=0$ implies $z=0$. Thus, M is injective. Let us define the sequence $z=(z_k)$ for all $k\in\mathbb{N}$ as follows:

$$z_k=\frac{1}{r}\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j,$$

where the sequence $y=(y_k)\in f$. Considering this equation and the expressions (3)–(9), we get

$$\begin{aligned}(Gz)_k&=rz_k+sz_{k-1}+tz_{k-2}+uz_{k-3}+vz_{k-4}\\&=\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\\&\quad+\frac{s}{r}\sum_{j=0}^{k-1}\sum_{\alpha=0}^{k-j-1}\sum_{i=0}^{k-j-\alpha-1}\sum_{l=0}^{k-j-\alpha-i-1}\sigma_1^{k-j-\alpha-i-l-1}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\\&\quad+\frac{t}{r}\sum_{j=0}^{k-2}\sum_{\alpha=0}^{k-j-2}\sum_{i=0}^{k-j-\alpha-2}\sum_{l=0}^{k-j-\alpha-i-2}\sigma_1^{k-j-\alpha-i-l-2}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\\&\quad+\frac{u}{r}\sum_{j=0}^{k-3}\sum_{\alpha=0}^{k-j-3}\sum_{i=0}^{k-j-\alpha-3}\sum_{l=0}^{k-j-\alpha-i-3}\sigma_1^{k-j-\alpha-i-l-3}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\\&\quad+\frac{v}{r}\sum_{j=0}^{k-4}\sum_{\alpha=0}^{k-j-4}\sum_{i=0}^{k-j-\alpha-4}\sum_{l=0}^{k-j-\alpha-i-4}\sigma_1^{k-j-\alpha-i-l-4}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-4}\left[\sum_{\alpha=0}^{k-j-4}\left[\sum_{i=0}^{k-j-\alpha-4}\left[\sum_{l=0}^{k-j-\alpha-i-4}\sigma_1^{k-j-\alpha-i-l-4}\sigma_2^l\sigma_3^i\sigma_4^\alpha\right.\right.\right.\\
&\quad\times\left(\sigma_1^4+\frac{s}{r}\sigma_1^3+\frac{t}{r}\sigma_1^2+\frac{u}{r}\sigma_1+\frac{v}{r}\right)+\sigma_2^{k-j-\alpha-i-3}\sigma_3^i\sigma_4^\alpha\\
&\quad\left.\left.\times\left[\sigma_1^3+\sigma_2^3+\sigma_1\sigma_2^2+\sigma_2\sigma_1^2+\frac{s}{r}(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)+\frac{t}{r}(\sigma_1+\sigma_2)+\frac{u}{r}=0\right]\right]\right.\\
&\quad+\sigma_3^{k-j-\alpha-3}\sigma_4^\alpha\left[\sigma_1^3+\sigma_2^3+\sigma_3^3+\sigma_1\sigma_2^2+\sigma_1\sigma_3^2+\sigma_2\sigma_1^2+\sigma_2\sigma_3^2+\sigma_3\sigma_1^2+\sigma_3\sigma_2^2\right.\\
&\quad\left.\left.+\sigma_1\sigma_2\sigma_3+\frac{s}{r}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)+\frac{t}{r}(\sigma_1+\sigma_2+\sigma_3)+\frac{u}{r}\right]\right]\\
&\quad+\sigma_4^{k-j-3}\left[\sigma_1^3+\sigma_2^3+\sigma_3^3+\sigma_4^3+\sigma_1\sigma_2^2+\sigma_1\sigma_3^2+\sigma_1\sigma_4^2+\sigma_2\sigma_1^2+\sigma_2\sigma_3^2+\sigma_2\sigma_4^2\right.\\
&\quad+\sigma_3\sigma_1^2+\sigma_3\sigma_2^2+\sigma_3\sigma_4^2+\sigma_4\sigma_1^2+\sigma_4\sigma_2^2+\sigma_4\sigma_3^2+\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4\\
&\quad+\sigma_2\sigma_3\sigma_4+\frac{s}{r}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3\\
&\quad\left.\left.+\sigma_2\sigma_4+\sigma_3\sigma_4)+\frac{t}{r}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+\frac{u}{r}\right]\right]y_j\\
&\quad+\left[\sigma_1^3+\sigma_2^3+\sigma_3^3+\sigma_4^3+\sigma_1\sigma_2^2+\sigma_1\sigma_3^2+\sigma_1\sigma_4^2+\sigma_2\sigma_1^2+\sigma_2\sigma_3^2+\sigma_2\sigma_4^2+\sigma_3\sigma_1^2\right.\\
&\quad+\sigma_3\sigma_2^2+\sigma_3\sigma_4^2+\sigma_4\sigma_1^2+\sigma_4\sigma_2^2+\sigma_4\sigma_3^2+\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4\\
&\quad+\frac{s}{r}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)\\
&\quad\left.+\frac{t}{r}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+\frac{u}{r}\right]y_{k-3}\\
&\quad+\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3\right.\\
&\quad\left.+\sigma_2\sigma_4+\sigma_3\sigma_4+\frac{s}{r}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)+\frac{t}{r}\right]y_{k-2}\\
&\quad+\left[\sigma_1+\sigma_2+\sigma_3+\sigma_4+\frac{s}{r}\right]y_{k-1}+y_k\\
&= y_k
\end{aligned}
$$

for all $k \in \mathbb{N}$. So we obtain,

$$\lim_{i\to\infty}\sum_{j=0}^{i}\frac{rz_{n+j}+sz_{n+j-1}+tz_{n+j-2}+uz_{n+j-3}+vz_{n+j-4}}{i+1}=\lim_{i\to\infty}\sum_{j=0}^{i}\frac{y_{n+j}}{i+1}=F-\lim y_n.$$

Namely, $z=(z_k)\in f(G(r,s,t,u,v))$ and $M(z)=y$. Therefore, M is surjective. Additionally, for all $z=(z_k)\in f(G(r,s,t,u,v))$, we obtain

$$\|M(z)\|_f=\|G(r,s,t,u,v)z\|_f=\|z\|_{f(G(r,s,t,u,v))}.$$

Hence, M preserves the norm. Consequently, M is a linear bijection. As a result, we conclude that $f(G(r,s,t,u,v)) \cong f$. Thus, the proof is complete. $\square$

3. The Schauder basis and β,γ-duals of new sequence spaces

In this section, we talk about Schauder basis of the spaces $f(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$. Also, we identify the β- and γ-duals of this spaces.

Let X be a normed space and $b=(b_k)$ a sequence in X. The sequence $b=(b_k)$ is a Schauder basis for X if and only if for every $x=(x_k)\in X$, there exists a unique sequence of scalars (α_n) such that

$$\left\|x-\sum_{k=0}^{n}\alpha_k b_k\right\|_X \to 0$$

as $n\to\infty$.

COROLLARY 1. [4] *The space f has no Schauder basis, where f is almost convergent sequence space.*

REMARK 1. [18] Let X be an arbitrary sequence space and $A=(a_{nk})$ be a triangle matrix. Then X_A has a basis if and only if X has a basis.

COROLLARY 2. *The spaces $f(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$ has no Schauder basis.*

The set $M(T,U)$ is named as multiplier space of T and U. Also this set is defined by:

$$M(T,U)=\{a=(a_k)\in w : at=(a_kt_k)\in U \text{ for all } t=(t_k)\in T\},$$

where T and U are the any sequence spaces. Then, the β- and γ-duals of the space T are defined as:

$$T^{\beta}=M(T,cs) \quad\text{and}\quad T^{\gamma}=M(T,bs),$$

in turn.

Now, to use in the next lemma we write the following:

$$\sup_{n\in\mathbb{N}}\sum_k |t_{nk}|<\infty \tag{10}$$

$$\lim_{n\to\infty} t_{nk}=\mu_k \text{ for all } k\in\mathbb{N} \tag{11}$$

$$\lim_{n\to\infty}\sum_k t_{nk}=\mu \tag{12}$$

$$\lim_{n\to\infty}\sum_k |\Delta(t_{nk}-\mu_k)|=0 \tag{13}$$

$$\sup_{n\in\mathbb{N}}\sum_k |\Delta t_{nk}|<\infty \tag{14}$$

$$\lim_{k\to\infty} t_{nk}=0 \text{ for all } n\in\mathbb{N} \tag{15}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta^2 t_{nk}\right|=\lambda \tag{16}$$

where $\Delta t_{nk}=t_{nk}-t_{n,k+1}$ and $\Delta^2 t_{nk}=\Delta(\Delta t_{nk})$.

LEMMA 1. *Let us assume that $B=(b_{nk})$ be an infinite matrix. In this case the followings hold:*

(i) $B=(b_{nk})\in(f:c)\Leftrightarrow$ (10), (11), (12) *and* (13) *hold* [27],

(ii) $B=(b_{nk})\in(f:\ell_\infty)\Leftrightarrow$ (10) *holds* [27],

(iii) $B=(b_{nk})\in(fs:c)\Leftrightarrow$ (11), (14), (15) *and* (16) *hold* [26],

(iv) $B=(b_{nk})\in(fs:\ell_\infty)\Leftrightarrow$ (14) *and* (15) *hold* [4].

THEOREM 6. *Given the sets ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6 and ξ_7 as follows:*

$$\xi_1=\left\{b=(b_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_j\right|<\infty\right\}$$

$$\xi_2=\left\{b=(b_k)\in w:\lim_{n\to\infty}\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_j \text{ exists for each } k\in\mathbb{N}\right\}$$

$$\xi_3=\left\{b=(b_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}\right]b_k \text{ exists}\right\}$$

$$\xi_4=\left\{b=(b_k)\in w:\lim_{n\to\infty}\sum_k\left|\Delta\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_j-\mu_k\right]\right|=0\right\}$$

$$\xi_5=\left\{b=(b_k)\in w:\sup_{n\in\mathbb{N}}\sum_k\left|\Delta\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_j\right]\right|<\infty\right\}$$

$$\xi_6=\left\{b=(b_k)\in w:\lim_{k\to\infty}\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_j=0 \text{ for each } n\in\mathbb{N}\right\}$$

$$\xi_7=\left\{b=(b_k)\in w:\lim_{n\to\infty}\sum_k\left|\Delta^2\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha b_j\right]\right| \text{ exists}\right\}$$

where

$$\mu_k=\lim_{n\to\infty}\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha b_j\ ,\ \ \forall k\in\mathbb{N}.$$

In that case the followings hold:

(i) $\{f(G(r,s,t,u,v))\}^\beta=\xi_1\cap\xi_2\cap\xi_3\cap\xi_4$

(ii) $\{f(G(r,s,t,u,v))\}^\gamma=\xi_1$

(iii) $\{fs(G(r,s,t,u,v))\}^\beta=\xi_2\cap\xi_5\cap\xi_6\cap\xi_7$

(iv) $\{fs(G(r,s,t,u,v))\}^\gamma=\xi_5\cap\xi_6.$

Proof. It is sufficient to prove only (i), as the others can be done similarly. Let $b=(b_k)\in w$ and the sequence $x=(x_k)$ be as follows:

$$x_k=\frac{1}{r}\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j$$

for all $k\in\mathbb{N}$. In that case, we obtain

$$\begin{aligned}\sum_{k=0}^{n}b_kx_k&=\sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha y_j\right]b_k\\&=\sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha b_j\right]y_k\\&=(Qy)_n\end{aligned}$$

for all $n\in\mathbb{N}$. Also, $Q=(q_{nk})$ is defined as follows:

$$q_{nk}=\begin{cases}\frac{1}{r}\sum\limits_{j=k}^{n}\sum\limits_{\alpha=0}^{j-k}\sum\limits_{i=0}^{j-k-\alpha}\sum\limits_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^l\sigma_3^i\sigma_4^\alpha b_j, & 0\leqslant k\leqslant n\\ 0, & k>n\end{cases}$$

for all $n,k\in\mathbb{N}$. Therefore, $bx=(b_kx_k)\in cs$ whenever $x=(x_k)\in f(G(r,s,t,u,v))\ \Leftrightarrow$ $Qy\in c$ whenever $y=(y_k)\in f$.

This result gives the fact that $b=(b_k)\in\{f(G(r,s,t,u,v))\}^{\beta} \Leftrightarrow Q\in(f:c)$. If we consider this fact with (i) of Lemma 1, we have $b=(b_k)\in\{f(G(r,s,t,u,v))\}^{\beta} \Leftrightarrow$

$$\sup_{n\in\mathbb{N}}\sum_k |q_{nk}|<\infty,$$

$$\lim_{n\to\infty} q_{nk}=\mu_k \ \text{ for all } k\in\mathbb{N},$$

$$\lim_{n\to\infty}\sum_k q_{nk}=\mu$$

and

$$\lim_{n\to\infty}\sum_k |\Delta(q_{nk}-\mu_k)|=0.$$

Consequently, $\{f(G(r,s,t,u,v))\}^{\beta}=\xi_1\cap\xi_2\cap\xi_3\cap\xi_4$. Thus, the proof is complete. $\square$

4. Matrix classes

In this part, we provide a characterization of certain matrix classes that are associated with the spaces $f(G(r,s,t,u,v))$ and $fs(G(r,s,t,u,v))$. From this point onward, we work with the matrices $D=(d_{nk})$ and $H=(h_{nk})$ defined by

$$d_{nk}=\frac{1}{r}\sum_{j=k}^{\infty}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_{nj} \tag{17}$$

$$h_{nk}=rb_{nk}+sb_{n-1,k}+tb_{n-2,k}+ub_{n-3,k}+vb_{n-4,k} \tag{18}$$

for all $n,k\in\mathbb{N}$, in turn.

THEOREM 7. *Let X be any sequence space. If $B=(b_{nk})$ is an infinite matrix whose entries satisfy conditions (17) and (18) then the following statements hold:*

(i) $B\in(f(G(r,s,t,u,v)):X)\Leftrightarrow D\in(f:X)$ *and* $\{b_{nk}\}_{k\in\mathbb{N}}\in f(G(r,s,t,u,v))^{\beta}$ *for all* $n\in\mathbb{N}$,

(ii) $B\in(X:f(G(r,s,t,u,v)))\Leftrightarrow H\in(X:f)$.

Proof.

(i) Let $B\in(f(G(r,s,t,u,v)):X)$. Suppose that $f(G(r,s,t,u,v))\cong f$ and $y=G(r,s,t,u,v)x$, where $y=(y_k)\in f$ is any sequence. Then, $DG(r,s,t,u,v)$ exists and $\{b_{nk}\}_{k\in\mathbb{N}}\in\{f(G(r,s,t,u,v))\}^{\beta}$ for $\forall n\in\mathbb{N}$. This implies that $\{d_{nk}\}_{k\in\mathbb{N}}\in\ell_1$ for $\forall n\in\mathbb{N}$. Hence, Dy exists and $\sum_k d_{nk}y_k=\sum_k b_{nk}x_k$ for all $n\in\mathbb{N}$, that is $Dy=Bx$. So, $D\in(f:X)$.

Conversely, suppose that $D \in (f : X)$ and $\{b_{nk}\}_{k\in\mathbb{N}} \in \{f(G(r,s,t,u,v))\}^{\beta}$ for $\forall n \in \mathbb{N}$. Let $x = (x_k) \in f(G(r,s,t,u,v))$ be any sequence. It is clear that Bx exists. Furthermore, we obtain

$$\begin{aligned}\sum_{k=0}^{\gamma} b_{nk}x_k &= \sum_{k=0}^{\gamma}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{\alpha=0}^{k-j}\sum_{i=0}^{k-j-\alpha}\sum_{l=0}^{k-j-\alpha-i}\sigma_1^{k-j-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}y_j\right]b_{nk}\\ &= \sum_{k=0}^{\gamma}\left[\frac{1}{r}\sum_{j=k}^{\gamma}\sum_{\alpha=0}^{j-k}\sum_{i=0}^{j-k-\alpha}\sum_{l=0}^{j-k-\alpha-i}\sigma_1^{j-k-\alpha-i-l}\sigma_2^{l}\sigma_3^{i}\sigma_4^{\alpha}b_{nj}\right]y_k\end{aligned}$$

for all $n \in \mathbb{N}$. If we take the limit $\gamma \to \infty$, then we arrive with the conclusion $Bx = Dy$. Thus, we have that $B \in (f(G(r,s,t,u,v)) : X)$.

(ii) For an arbitrary sequence $x = (x_k) \in X$, take into account the following equality:

$$\begin{aligned}\{G(Bx)\}_n &= r(Bx)_n + s(Bx)_{n-1} + t(Bx)_{n-2} + u(Bx)_{n-3} + v(Bx)_{n-4}\\ &= \sum_k (rb_{nk} + sb_{n-1,k} + tb_{n-2,k} + ub_{n-3,k} + vb_{n-4,k})x_k\\ &= (Hx)_n\end{aligned}$$

for all $n \in \mathbb{N}$. By applying the generalized limit, we conclude that $Bx \in f(G(r,s,t,u,v)) \Leftrightarrow Hx \in f$. Thus, the proof is complete. $\square$

Now, let us consider an infinite matrix $B = (b_{nk})$. In this case, the following statements hold:

$$F - \lim_{n\to\infty} b_{nk} = \mu_k \text{ for all } k \in \mathbb{N} \tag{19}$$

$$F - \lim_{n\to\infty}\sum_k b_{nk} = \mu \tag{20}$$

$$F - \lim_{n\to\infty}\sum_{j=0}^{n} b_{jk} = \mu_k \tag{21}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\Delta\left(\sum_{j=0}^{n} b_{jk}\right)\right| < \infty \tag{22}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n} b_{jk}\right| < \infty \tag{23}$$

$$\sum_n b_{nk} = \mu_k \text{ for all } k \in \mathbb{N} \tag{24}$$

$$\sum_n\sum_k b_{nk} = \mu \tag{25}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta\left(\sum_{j=0}^{n} b_{jk} - \mu_k\right)\right| = 0 \tag{26}$$

$$\lim_{p\to\infty}\sum_k\left|\frac{1}{p+1}\sum_{j=0}^{p} b_{n+j,k}-\mu_k\right| = 0 \text{ uniformly in } n \tag{27}$$

$$\lim_{p\to\infty}\sum_k\left|\Delta\left(\frac{1}{p+1}\sum_{j=0}^{p} b_{n+j,k}-\mu_k\right)\right| = 0 \text{ uniformly in } n \tag{28}$$

$$\lim_{p\to\infty}\sum_k\frac{1}{p+1}\left|\sum_{i=0}^{p}\Delta\left(\sum_{j=0}^{n+i} b_{jk}-\mu_k\right)\right| = 0 \text{ uniformly in } n \tag{29}$$

$$\lim_{p\to\infty}\sum_k\frac{1}{p+1}\left|\sum_{i=0}^{p}\Delta^2\left(\sum_{j=0}^{n+i} b_{jk}-\mu_k\right)\right| = 0 \text{ uniformly in } n. \tag{30}$$

LEMMA 2. *Let us consider $B=(b_{nk})$ as an infinite matrix. Then, the followings hold:*

(i) $B=(b_{nk})\in(c:f)$ *if and only if* (10), (19) *and* (20) *hold* [20]

(ii) $B=(b_{nk})\in(\ell_\infty:f)$ *if and only if* (10), (19) *and* (27) *hold* [15]

(iii) $B=(b_{nk})\in(f:f)$ *if and only if* (10), (19), (20) *and* (28) *hold* [15]

(iv) $B=(b_{nk})\in(f:cs)$ *if and only if* (23), (24), (25) *and* (26) *hold* [7]

(v) $B=(b_{nk})\in(cs:f)$ *if and only if* (14) *and* (19) *hold* [5]

(vi) $B=(b_{nk})\in(cs:fs)$ *if and only if* (21) *and* (22) *hold* [5]

(vii) $B=(b_{nk})\in(bs:f)$ *if and only if* (14), (15), (19) *and* (29) *hold* [6]

(viii) $B=(b_{nk})\in(bs:fs)$ *if and only if* (15), (21), (22) *and* (29) *hold* [6]

(ix) $B=(b_{nk})\in(fs:f)$ *if and only if* (15), (19), (28) *and* (29) *hold* [8]

(x) $B=(b_{nk})\in(fs:fs)$ *if and only if* (21), (22), (29) *and* (30) *hold* [8].

By combining Lemma 1, conditions (17) and (18), Theorem 7 and Lemma 2 we can also present the following results.

COROLLARY 3. *Consider replacing the entries of the matrix $B=(b_{nk})$ in (10)–(16) and (19)–(30) with the entries of the matrix $D=(d_{nk})$. In this case, the followings hold:*

(i) $B=(b_{nk})\in(f(G(r,s,t,u,v)):c)$ *if and only if* $\{b_{nk}\}_{k\in\mathbb{N}}\in\{f(G(r,s,t,u,v))\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (10), (11), (12) *and* (16) *hold;*

(ii) $B=(b_{nk})\in(f(G(r,s,t,u,v)):\ell_\infty)$ *if and only if* $\{b_{nk}\}_{k\in\mathbb{N}}\in\{f(G(r,s,t,u,v))\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (10) *holds;*

(iii) $B=(b_{nk})\in(f(G(r,s,t,u,v)):cs)$ *if and only if* $\{b_{nk}\}_{k\in\mathbb{N}}\in\{f(G(r,s,t,u,v))\}^\beta$ *for all* $n\in\mathbb{N}$ *and* (23), (24), (25) *and* (26) *hold;*

(iv) $B=(b_{nk})\in(f(G(r,s,t,u,v)):bs)$ *if and only if* $\{b_{nk}\}_{k\in\mathbb{N}}\in\{f(G(r,s,t,u,v))\}^{\beta}$ *for all* $n\in\mathbb{N}$ *and* (24) *holds.*

COROLLARY 4. *Consider replacing the entries of the matrix* $B=(b_{nk})$ *in* (10)–(16) *and* (19)–(30) *with the entries of the matrix* $H=(h_{nk})$*. In this case, the followings hold:*

(i) $B=(b_{nk})\in(c:f(G(r,s,t,u,v)))$ *if and only if* (10), (19) *and* (20) *hold;*

(ii) $B=(b_{nk})\in(\ell_\infty:f(G(r,s,t,u,v)))$ *if and only if* (10), (19) *and* (27) *hold;*

(iii) $B=(b_{nk})\in(f:f(G(r,s,t,u,v)))$ *if and only if* (10), (19), (20) *and* (28) *hold;*

(iv) $B=(b_{nk})\in(cs:f(G(r,s,t,u,v)))$ *if and only if* (14) *and* (19) *hold;*

(v) $B=(b_{nk})\in(bs:f(G(r,s,t,u,v)))$ *if and only if* (14), (15) (19) *and* (29) *hold;*

(vi) $B=(b_{nk})\in(fs:f(G(r,s,t,u,v)))$ *if and only if* (15), (19) (28) *and* (29) *hold;*

(vii) $B=(b_{nk})\in(cs:fs(G(r,s,t,u,v)))$ *if and only if* (21) *and* (22) *hold;*

(viii) $B=(b_{nk})\in(bs:fs(G(r,s,t,u,v)))$ *if and only if* (15), (21) (22) *and* (29) *hold;*

(ix) $B=(b_{nk})\in(fs:fs(G(r,s,t,u,v)))$ *if and only if* (21), (22) (29) *and* (30) *hold.*

5. Conclusion

Upon examining the quintet band matrix operator $G(r,s,t,u,v)$ defined here, it is clear that $G(1,-4,6,-4,1)=\Delta^4$, $G(r,s,t,u,0)=Q(r,s,t,u)$, $G(1,-3,3,-1,0)=\Delta^3$, $G(r,0,0,u,0)=D(r,0,0,s)$, $G(r,s,t,0,0)=B(r,s,t)$, $G(1,-2,1,0,0)=\Delta^2$, $G(r,s,0,0,0)=B(r,s)$ and $G(1,-1,0,0,0)=\Delta$, where Δ^4, $Q(r,s,t,u)$, Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are named fourth order difference, quadruple band, third order difference, triple band, second order difference, double band and difference matrix, respectively. Accordingly, the conclusions derived from the quintet band matrix operator are more comprehensive compared to previous findings. As a result, this study addresses a gap in the existing literature.

REFERENCES

[1] B. ALTAY AND F. BAŞAR, *The fine spectrum and the matrix domain of the difference operator* Δ *on the sequence space* ℓ_p, $(0<p<1)$, Commun. Math. Anal. **2** (2) (2007), 1–11.

[2] F. BAŞAR, *Summability Theory and its Applications*, 2nd ed., CRC Press, Boca Raton · London · New York, 2022.

[3] F. BAŞAR AND B. ALTAY, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukr. Mat. Zh. **55** (1) (2003), 108–118; transl. in Ukrainian Math. J. **55** (1) (2003), 136–147.

[4] F. BAŞAR AND M. KIRIŞÇI, *Almost convergence and generalized difference matrix*, Comput. Math. Appl. **61** (3) (2011), 602–611.

[5] F. BAŞAR AND R. ÇOLAK, *Almost-conservative matrix transformations*, Doğa Mat. **13** (3) (1989), 91–100.

[6] F. BAŞAR AND İ. SOLAK, *Almost-coercive matrix transformations*, Rend. Mat. Appl. (7) **11** (2) (1991), 249–256.

[7] F. BAŞAR, *Strongly-conservative sequence-to-series matrix transformations*, Erciyes Univ. Fen Bil. Derg. **5** (12) (1989), 888–893.

[8] F. BAŞAR, *f-conservative matrix sequences*, Tamkang J. Math. **22** (2) (1991), 205–212.

[9] Ç. A. BEKTAŞ AND R. ÇOLAK, *On the Köthe-Toeplitz duals of some generalized sets of difference sequences*, Demonstratio Math. **33** (4) (2000), 797–803.

[10] Ç. A. BEKTAŞ, *On some new generalized sequence spaces*, J. Math. Anal. Appl. **277** (2) (2003), 681–688.

[11] M. C. BIŞGIN, *Almost convergent sequence spaces derived by the domain of quadruple band matrix*, Ann. Univ. Paedagog. Crac. Stud. Math. **19** (1) (2020), 155–170.

[12] M. C. BIŞGIN, *New sequence spaces defined by the domain of quintet band matrix*, under review.

[13] J. BOOS, *Classical and Modern Methods in Summability*, Oxford Math. Monogr., Oxford Univ. Press, New York, 2000.

[14] B. CHOUDHARY AND S. NANDA, *Functional Analysis with Applications*, Wiley, New Delhi, 1989.

[15] J. P. DURAN, *Infinite matrices and almost-convergence*, Math. Z. **128** (1972), 75–83.

[16] M. ET AND R. ÇOLAK, *On some generalized difference sequence spaces*, Soochow J. Math. **21** (4) (1995), 377–386.

[17] M. ET, *On some difference sequence spaces*, Turkish J. Math. **17** (1993), 18–24.

[18] A. M. JARRAH AND E. MALKOWSKY, *BK spaces, bases and linear operators*, Rend. Circ. Mat. Palermo (2) Suppl. **52** (1998), 177–191.

[19] H. KIZMAZ, *On certain sequence spaces*, Canad. Math. Bull. **24** (2) (1981), 169–176.

[20] J. P. KING, *Almost summable sequences*, Proc. Amer. Math. Soc. **17** (1966), 1219–1225.

[21] M. KIRIŞÇI AND F. BAŞAR, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl. **60** (5) (2010), 1299–1309.

[22] G. G. LORENTZ, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[23] I. J. MADDOX, *Elements of Functional Analysis*, Cambridge Univ. Press, Cambridge, 1988.

[24] M. MURSALEEN, *Applied Summability Methods*, Springer Briefs in Math., Springer, Heidelberg, 2014.

[25] M. MURSALEEN AND F. BAŞAR, *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Boca Raton, 2020.

[26] E. ÖZTÜRK, *On strongly regular dual summability methods*, Comm. Fac. Sci. Univ. Ankara Sér. A1 Math. **32** (1) (1983), 1–5.

[27] J. A. SIDDIQI, *Infinite matrices summing every almost periodic sequence*, Pacific J. Math. **39** (1971), 235–251.

[28] A. SÖNMEZ, *Some new sequence spaces derived by the domain of the triple band matrix*, Comput. Math. Appl. **62** (2) (2011), 641–650.

[29] A. SÖNMEZ, *Almost convergence and triple band matrix*, Math. Comput. Model. **57** (9–10) (2013), 2393–2402.

[30] A. WILANSKY, *Summability Through Functional Analysis*, North-Holland Math. Stud., vol. 85, North-Holland, Amsterdam, 1984.

[31] A. ZAHEER AND M. MURSALEEN, *Köthe-Toeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math. (Belgrad) (N.S.) **42** (56) (2002), 57–61.

(Received June 19, 2025)

Kübra Topal
Department of Mathematics
Recep Tayyip Erdoğan University
Rize, Türkiye
e-mail: kubra_topal22@erdogan.edu.tr