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QUATERNARY ELLIPTIC FUNCTIONS IN THE SPIRIT OF DU VAL

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Abstract. In 1964 Patrick Du Val introduced and studied his ‘ternary’ elliptic functions as cube roots of the suitably modified first derivative p' of an arbitrary Weierstrass $\wp$-function. We introduce and study ‘quaternary’ elliptic functions as fourth roots of its suitably modified second derivative p''.

Introduction

A standard construction of the Jacobian elliptic functions begins by subtracting from a Weierstrass $\wp$-function p its value at one of its halfperiods; as the resulting elliptic function has poles and zeros of second order, it has two square roots that are meromorphic and indeed elliptic; these square roots are generalizations of the Jacobian functions, to which they reduce when p is appropriately chosen. This construction was especially promoted by E. H. Neville in his masterly treatise [5].

In 1964, Patrick Du Val took this approach a step further. By subtracting from the first derivative of p a suitable affine expression in p, he obtained an elliptic function having poles and zeros of third order; the cube roots of this function are the *ternary* elliptic functions of Du Val. The introduction and study of these functions in [3] by Du Val was continued in a chapter of his noteworthy account [2] of elliptic function theory.

Our purpose here is to take this approach still further. Subtracting from the second derivative of p a suitable affine expression in p and its first derivative, we obtain an elliptic function whose zeros and poles all have order four; its meromorphic fourth roots are the quaternary elliptic functions of our title. Properties of the ternary elliptic functions have their quaternary counterparts, many of which we develop in detail.

Though it was not recorded by Du Val, his ternary elliptic functions are most intimately related to the elliptic functions sm and cm of ‘modulus’ α introduced and studied in [1] by Dixon, who presented them as providing an elliptic parametrization of the cubic curve

$$x^3 + y^3 = 1 + 3\alpha xy.$$

This intimate relationship was made quite explicit in [6]. In the present paper, we perform a like service in relating the quaternary elliptic functions to elliptic parametrizations of the quartic curves

$$x^4 - 2x^2y^2 + y^4 = 1 + 4\alpha xy.$$

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An Appendix characterizes thirdperiods and quarterperiods of the Weierstrass $\wp$-function p in terms of the values of p and its derivatives.

Throughout, we assume familiarity with the fundamentals of elliptic function theory, in particular those that pertain to $\wp$-functions, such as may be found in the monumental treatise [7] or in the more recent [2] and [4].

1. Quaternary elliptic functions

Let $p = \wp(-;\omega,\omega') = \wp(-;g_2,g_3)$ be an arbitrary Weierstrass elliptic $\wp$-function, identified either by choosing a fundamental pair $(2\omega, 2\omega')$ of periods or by specifying its quadrinvariant g_2 and cubinvariant g_3.

We shall assume without proof certain differential equations satisfied by p: it satisfies the first-order 'eliminant equation' $(p')^2 = 4p^3 - g_2 p - g_3$ and the 'derived equation' $p'' = 6p^2 - \frac{1}{2}g_2$; beyond these, we make free use of $p''' = 12pp'$ and the less familiar consequences

$$p'''' = 12(p')^2 + 12pp'' \text{ and } p^{\mathrm{v}} = 36p'p'' + 144p^2p'.$$

We begin our study by modifying p so as to produce an elliptic function having the same period lattice

$$\Lambda_p := \{2n\omega + 2n'\omega' : n, n' \in \mathbb{Z}\}$$

and having a fourth-order zero at the particular quarterperiod

$$a := 2\omega/4 = \omega/2.$$

THEOREM 1. *There exist unique constants $A, B, C \in \mathbb{C}$ such that the elliptic function*

$$\phi := \frac{1}{6}p'' + Ap' + Bp + C$$

has a fourth-order zero at a.

Proof. We require that each of the following conditions be satisfied

$$\phi(a) = \frac{1}{6}p''(a) + Ap'(a) + Bp(a) + C = 0 \tag{1}$$

$$\phi'(a) = \frac{1}{6}p'''(a) + Ap''(a) + Bp'(a) = 0 \tag{2}$$

$$\phi''(a) = \frac{1}{6}p''''(a) + Ap'''(a) + Bp''(a) = 0 \tag{3}$$

$$\phi'''(a) = \frac{1}{6}p^{\mathrm{v}}(a) + Ap''''(a) + Bp'''(a) = 0 \tag{4}$$

and that the fourth derivative $\phi''''(a)$ be nonzero.

Conditions (2) and (3) may be rewritten as the system

$$Ap''(a) + Bp'(a) = -2p(a)p'(a)$$

$$Ap'''(a) + Bp''(a) = -2p'(a)^2 - 2p(a)p''(a)$$

in which the determinant $p''(a)^2 - p'(a)p'''(a) = p''(a)^2 - 12p'(a)^2 p(a)$ is nonzero because a is not a third-period of p; see (27) in the Appendix for this. The constants A and B are then forced:

$$(p''^2 - p'p''')A = 2p'^3 \tag{5}$$

$$(p''^2 - p'p''')B = 24p^2p'^2 - 2p'^2p'' - 2pp''^2 \tag{6}$$

wherein evaluation at a of p and its derivatives has been suppressed for convenience. The unique constant C that now supports condition (1) is given by

$$6(p''^2 - p'p''')C = 12p^2p''^2 + 2p'p''p''' - pp'''^2 - 12p'^4 - p''^3 \tag{7}$$

where evaluation at a is again suppressed. To confirm that condition (4) is satisfied by A and B above involves first a free (but lengthy) use of the aforementioned formulas for the derivatives of p to deduce that

$$(p''^2 - p'p''')\,\phi'''(a) = 6p'(p''^3 - p'p''p''' + 4p'^4)$$

in which formula we still suppress a; and then an invocation of (27) and (29) in the Appendix to conclude that ϕ''' vanishes at the quarterperiod a.

Finally, the elliptic function ϕ has 2ω and $2\omega'$ as periods and in each period parallelogram has precisely one pole, of order four; it follows that its zero a has order at most four, hence exactly four: $\phi''''(a)$ is nonzero without further calculation. $\square$

As the elliptic function ϕ has both zeros of order four and poles of order four, it has four meromorphic fourth roots; further, the scaling of p'' by one-sixth secures $z \to 0 \Rightarrow z^4\phi(z) \to 1$. Our *quaternary elliptic function* ψ will be the meromorphic fourth root of ϕ that has unit residue at the origin: thus,

$$\psi^4 = \phi \ \text{ and } \ \mathrm{Res}_0\,\psi = 1. \tag{8}$$

The function ψ is indeed elliptic: from the identity

$$\psi(z + 2\omega)^4 = \phi(z + 2\omega) = \phi(z) = \psi(z)^4$$

it follows that identically

$$\psi(z + 2\omega) = \delta\,\psi(z) \tag{9}$$

for some constant fourth root of unity δ, while similarly

$$\psi(z + 2\omega') = \delta'\,\psi(z) \tag{10}$$

for some constant fourth root of unity δ'; this implies at once that ψ has 8ω and $8\omega'$ as periods. In fact, the fourth root of unity δ is unity itself, and $(2\omega, 8\omega')$ is a fundamental pair of periods. For these facts, see Theorem 4 and Theorem 5 below.

THEOREM 2.

$$\psi(z)\psi(-z) = p(a) - p(z).$$

Proof. Consider $\psi(z)\psi(-z)$ and $p(a)-p(z)$ as functions of z. Each function has a double pole at each point of Λ_p (and no other poles); each function has a simple zero at every point of $\Lambda_p \pm a$ (and no other zeros). As the two functions share the lattice $4\Lambda_p$ of periods, they differ by a multiplicative constant, this being

$$\lim_{z\to 0}\frac{\psi(z)\psi(-z)}{p(a)-p(z)} = \lim_{z\to 0}\frac{z\psi(z)\cdot(-z)\psi(-z)}{z^2 p(z) - z^2 p(a)} = \frac{1\cdot 1}{1-0} = 1. \quad \square$$

THEOREM 3.

$$\psi(a-z)\psi(z) = \psi(2a)\psi(-a) = \psi(3a)\psi(-2a) = -\psi'(a).$$

Proof. Consider the product $\psi(a-z)\psi(z)$ as a function of z. Where $z\in\Lambda_p$ the first factor has a simple zero while the second has a simple pole; where $z\in\Lambda_p + a$ the first factor has a simple pole while the second has a simple zero. The product is therefore an elliptic function without poles, hence constant. Its value is revealed by substituting $z=-a$ or $z=-2a$; alternatively, let $z\to 0$ and see that

$$\psi(a-z)\psi(z) = -\Big\{\frac{\psi(a-z)}{-z}\Big\} z\psi(z) \to -\psi'(a). \quad \square$$

THEOREM 4.

$$\psi(z)\psi(z+a)\psi(z+2a)\psi(z+3a) = \psi'(a)\psi(2a)\psi(3a).$$

Proof. Consider the fourfold product on the left as a function of z. The zeros and poles of the four factors cancel one with another, whence the product is constant. We may evaluate this constant by passage to the limit as $z\to 0$, whereupon

$$\psi(z)\psi(z+a) = \psi(z)z\Big\{\frac{\psi(z+a)-\psi(a)}{z}\Big\} \to \psi'(a). \quad \square$$

It follows at once from Theorem 4 that the function ψ has $4a = 2\omega$ as a period: simply compare the value of the constant fourfold product at z with its value at $z+a$; in other words, the fourth root of unity δ at (9) is unity itself.

THEOREM 5. *The elliptic function ψ has 2ω and $8\omega'$ as a fundamental pair of periods.*

Proof. Already, 2ω and $8\omega'$ are periods. It is immediately evident that $2\omega'$ cannot be a period, for a 'period' parallelogram corresponding to the pair $(2\omega, 2\omega')$ contains a lone (simple) pole of ψ. That $4\omega'$ cannot be a period lies a little deeper: in the 'period' parallelogram corresponding to the pair $(2\omega, 4\omega')$, ψ has (simple) zeros at a

and $a+2\omega'$ and has (simple) poles at 0 and $2\omega'$; the difference between zero sums and pole sums is $2a=\omega$, which is not a period of ψ (or indeed of ϕ). Thus the pair $(2\omega, 8\omega')$ is fundamental. $\square$

We now see that the order of ψ as an elliptic function is four. We also see that the fourth root of unity δ' at (10) is $\pm\mathrm{i}$ and then that ψ satisfies the identity

$$\psi(z+4\omega') = -\psi(z).$$

In fact we can identify δ' precisely, as follows.

THEOREM 6. *The fourth root of unity* δ' *is* $\pm\mathrm{i}$ *according as* $\pm\omega'/\omega$ *lies in the upper halfplane.*

Proof. We indicate the steps to be taken. Let $\sigma = \sigma(-;\omega,4\omega')$ be the Weierstrass sigma-function associated to the Weierstrass function $\wp(-;\omega,4\omega')$ and take congruence to be modulo $2\Omega = 2\omega$ and $2\Omega' = 8\omega'$. As a full set of incongruent zeros for ψ we choose

$$a_0 = a,\;\; a_1 = a+2\omega',\;\; a_2 = a+4\omega',\;\; a_3 = a+6\omega';$$

as a full set of incongruent poles for ψ we choose

$$b_0 = 2\omega,\;\; b_1 = 2\omega',\;\; b_2 = 4\omega',\;\; b_3 = 6\omega'.$$

These choices make the zero sum equal the pole sum, ensuring that the meromorphic function f defined by

$$f(z) = \prod_{j=0}^{3} \sigma(z-a_j) \Big/ \prod_{j=0}^{3} \sigma(z-b_j)$$

has 2Ω and $2\Omega'$ as periods. As f and ψ share precisely the same (simple) zeros and poles, it follows that

$$f = \lambda\,\psi$$

with constant multiplier $\lambda = \mathrm{Res}_0 f$. We need not compute the value of λ explicitly: all we need is its existence, which implies that

$$\delta' = \frac{\psi(z+2\omega')}{\psi(z)} = \frac{f(z+2\omega')}{f(z)}.$$

To calculate this last ratio we recall sigma-function quasiperiodicity, which here assumes the forms

$$\sigma(z+2\Omega) = -\exp\{2\eta(z+\Omega)\}\sigma(z)$$
$$\sigma(z+2\Omega') = -\exp\{2\eta'(z+\Omega')\}\sigma(z)$$

in which $\eta = \zeta(\Omega)$ and $\eta' = \zeta(\Omega')$ are the indicated values of the Weierstrass zeta-function $\zeta = \zeta(-;\Omega,\Omega')$. The result is that

$$\delta' = \exp\{\eta\Omega' - \eta'\Omega\}$$

and the proof is ended by the Legendre identity: if $\pm\Omega'/\Omega$ lies in the upper halfplane then

$$\eta\Omega' - \eta'\Omega = \pm\frac{1}{2}\pi\mathrm{i}. \quad \square$$

Further progress in our study of ψ is assisted by a closer inspection of the constants A, B, C.

THEOREM 7. *The constants A, B, C are given by the equations*

(i) $p''(a) + 2Ap'(a) = 0$, (ii) $2p(a) + B = 2A^2$, (iii) $Bp(a) + C = \frac{1}{3}\phi(-a)$.

Proof. (i) We again suppress evaluation at a for convenience. From (5) we deduce that

$$(p''^2 - p'p''')(p'' + 2Ap') = p''^3 - p'p''p''' + 4p'^4 = 0$$

by virtue of (29) in the Appendix; as a is not a third-period of p, the discussion after (27) in the Appendix then concludes the justification of (i).

(ii) We know from (2) in Theorem 1 that

$$Ap''(a) + Bp'(a) = -2p(a)p'(a)$$

whence

$$\Big\{2p(a) + B\Big\}p'(a) = -Ap''(a)$$

and now part (i) implies that

$$\Big\{2p(a) + B\Big\}p''(a) = 2A^2p''(a).$$

As noted after (12) below, $p''(a)$ is not zero, so we are done.

(iii) From (i) and (12) together we also know that

$$\frac{1}{6}p''(a) + Ap'(a) = -\frac{1}{3}p''(a) = -\frac{1}{3}\phi(-a)$$

so

$$Bp(a) + C = \phi(a) - \left\{\frac{1}{6}p''(a) + Ap'(a)\right\} = \frac{1}{3}\phi(-a). \quad \square$$

A further identification of A follows by evaluating the odd part of ϕ at a: thus,

$$\phi(z) - \phi(-z) = 2Ap'(z) \tag{11}$$

and so

$$-\phi(-a) = 2Ap'(a).$$

From this and Theorem 7(i) we see further that

$$\phi(-a) = p''(a). \tag{12}$$

Recall that the zeros of ϕ are all congruent to a modulo the period lattice Λ_p of p; consequently, p'' does not vanish at the quarterperiod a.

We state separately the following relation between the constants, in view of its crucial role in Theorem 10 below.

THEOREM 8.
$$8A^4 - 6A^2B = \phi(-a).$$

Proof. On the one hand, substitution of $p''(a) = -2Ap'(a)$ from Theorem 7(i) into (29) of the Appendix yields
$$p' = 2A(A^2 - 3p(a))$$
after cancellation of $4p'(a)^3$ and rearrangement; thus
$$4A^2(3p(a) - A^2) = p''(a) = \phi(-a)$$
in view of (12). On the other hand, $B = 2A^2 - 2p(a)$ from Theorem 7(ii) shows that
$$8A^4 - 6A^2B = 4A^2(3p(a) - A^2);$$
so we are done. □

The ternary elliptic functions of Du Val satisfy a cubic equation: see Sections 9 and 19 in [3]; see also equations (59.2) and (62.11) in [2]. Our quaternary elliptic functions satisfy a corresponding quartic equation.

THEOREM 9.
$$\psi^4(z) + \psi^4(-z) = 2\psi^2(z)\psi^2(-z) - 4A^2\psi(z)\psi(-z) + \psi^4(-a).$$

Proof. Here, the coefficient A is given by (5) and Theorem 7(i), while the constant term equals $\phi(-a) = p''(a)$ as noted at (12). From the definitions, twice the even part of $\psi^4 = \phi$ is given by
$$\psi^4(z) + \psi^4(-z) = 2p(z)^2 + 2Bp(z) + 2C - \frac{1}{6}g_2$$
in light of the fact that p satisfies the 'derived equation' $2p'' = 12p^2 - g_2$. Now, Theorem 2 permits us to replace $p(z)$ by $p(a) - \psi(z)\psi(-z)$: thus
$$\begin{aligned}\psi^4(z) + \psi^4(-z) = 2\psi^2(z)\psi^2(-z) - 2\{2p(a) + B\}\psi(z)\psi(-z) + 2p(a)^2 \\ -\frac{1}{6}g_2 + 2Bp(a) + 2C\end{aligned}$$
where
$$\begin{aligned}2p(a)^2 - \frac{1}{6}g_2 + 2Bp(a) + 2C &= \frac{1}{3}p''(a) + 2Bp(a) + 2C = \phi(a) + \phi(-a) \\ &= \phi(-a) = \psi^4(-a)\end{aligned}$$
by reuse of the 'derived equation'. Finally, Theorem 7(ii) replaces $2p(a) + B$ by $2A^2$. □

The ternary elliptic functions of Du Val satisfy a first-order differential equation: see Section 20 in [3]. Our quaternary elliptic functions satisfy a corresponding first-order differential equation.

THEOREM 10.

$$2A\,\psi'(z) = \Big\{A^2 - \psi(z)\psi(-z)\Big\}\psi(z) + \psi^3(-z).$$

Proof. Again, A is as in (5) and Theorem 7(i). Differentiation of $\psi^4 = \phi$ yields

$$4\psi^3\psi' = \phi' = \frac{1}{6}p''' + Ap'' + Bp' = 2pp' + Ap'' + Bp'$$

whence reusing the definition of ϕ yields

$$4\psi^3\psi' = 2pp' + 6A\{\psi^4 - Ap' - Bp - C\} + Bp'$$

so that

$$4\psi^3\psi' = 6A\psi^4 + \{2p + B - 6A^2\}p' - 6A(Bp + C).$$

Here, use (11) to replace $2Ap'(z)$ by $\psi^4(z) - \psi^4(-z)$ and use Theorem 2 to replace $p(z)$ by $p(a) - \psi(z)\psi(-z)$; then invoke Theorem 7. After simplification, this results in the following expression for $4A\psi^3(z)\psi'(z)$:

$$4A^2\psi^4(z) - \psi^5(z)\psi(-z) + \psi(z)\psi^5(-z) + 6A^2B\psi(z)\psi(-z) + 2A^2\{\psi^4(-z) - \psi^4(-a)\}.$$

At this point, summon the quartic equation from Theorem 9 in order to replace $\psi^4(-z)$. After further simplification, this produces the following expression for $4A\psi^3(z)\psi'(z)$:

$$2\{A^2 - \psi(z)\psi(-z)\}\psi^4(z) + 2\psi^3(z)\psi^3(-z) + \{6A^2B - 8A^4 + \psi^4(-a)\}\psi(z)\psi(-z).$$

Finally, Theorem 8 kills the last summand and cancellation of $2\psi^3(z)$ concludes the proof. $\square$

Here, we derived the differential equation of Theorem 10 from the quartic equation of Theorem 9. The direction can be reversed: the two sides of the quartic equation in Theorem 9 agree when $z = a$ since $\psi(a) = 0$; and it can be checked that when either side of the same quartic equation is differentiated with respect to z using Theorem 10 the result is

$$\frac{2}{A}\{A^2 - \psi(z)\psi(-z)\}\{\psi^4(z) - \psi^4(-z)\}.$$

2. Quartic Dixonians

For convenience, let us abbreviate $\psi(-a)$ to μ: note that

$$\mu^4 = \phi(-a) = p''(a)$$

according to (12); note also that when $2\omega'$ is added to a repeatedly, the values of μ cycle through the fourth roots of $p''(a)$ according to (10) and Theorem 6.

Now, with a constant $\lambda \in \mathbb{C}$ to be determined, we define elliptic functions s and c by

$$s(z) = \psi(a - \lambda z)/\mu \tag{13}$$

$$c(z) = \psi(\lambda z - a)/\mu. \tag{14}$$

Note that $s(0) = 0$ and $c(0) = 1$: the former since $\psi(a) = 0$; the latter by definition of μ.

THEOREM 11. *The elliptic functions s and c have derivatives*

$$s' = \Big[-\frac{\lambda\mu^2}{2A}\Big]c^3 - \Big[-\frac{\lambda\mu^2}{2A}\Big]s^2c - \Big[\frac{\lambda A}{2}\Big]s$$

$$c' = \Big[\frac{\lambda A}{2}\Big]c + \Big[-\frac{\lambda\mu^2}{2A}\Big]sc^2 - \Big[-\frac{\lambda\mu^2}{2A}\Big]s^3.$$

Proof. These of course derive from Theorem 10. For s we have at once

$$s'(z) = -\frac{\lambda}{\mu}\frac{1}{2A}\Big\{\big[A^2 - \psi(a-\lambda z)\psi(\lambda z - a)\big]\psi(a-\lambda z) + \psi^3(\lambda z - a)\Big\};$$

now simply replace $\psi(a-\lambda z)$ by $\mu s(z)$ and $\psi(\lambda z - a)$ by $\mu c(z)$, then rearrange. The calculation of $c'(z)$ proceeds similarly. $\square$

Here, recall that

$$A = -\frac{1}{2}\frac{p''(a)}{p'(a)} \text{ and } \mu = \psi(-a)$$

while λ remains at our disposal.

The *ternary* elliptic functions of Du Val bear an intimate relationship to the classical *Dixonian* elliptic functions of Dixon, as is explored at depth in [6]. Guided by this relationship, it is reasonable to demand that the derivatives s' and c' involve c^3 and $-s^3$ respectively: thus, to demand that $\lambda\mu^2 = -2A$ so that

$$\lambda = -\frac{2A}{\mu^2} \text{ or } \psi(-a)^2\lambda = \frac{p''(a)}{p'(a)}. \tag{15}$$

With this choice for λ, the derivatives of s and c assume the form

$$s' = c^3 - s^2c - \alpha s \tag{16}$$

$$c' = \alpha c + sc^2 - s^3 \tag{17}$$

where for convenience we introduce the abbreviation

$$\alpha = \frac{A\lambda}{2} = -\frac{A^2}{\mu^2} \text{ or } \psi(-a)^2\alpha = -\frac{1}{4}\frac{p''(a)^2}{p'(a)^2} \tag{18}$$

so that

$$16\alpha^2 = \frac{p''(a)^3}{p'(a)^4}. \tag{19}$$

Until further notice, the choice (15) for λ will be in force, as will the above meaning of α.

THEOREM 12. *The elliptic functions s and c satisfy the quartic identity*

$$s^4 - 2s^2c^2 + c^4 = 1 + 4\alpha sc.$$

Proof. Two routes present themselves. The simpler one is to substitute $z = a - \lambda w$ in the quartic of Theorem 9, so that $\psi(z) = \mu s(w)$ and $\psi(-z) = \mu c(w)$; then divide by μ^4 throughout, recalling that $A^2/\mu^2 = -\alpha$ and that $\mu = \psi(-a)$. Alternatively, differentiate each side of the proposed quartic identity with the aid of (16) and (17): mass cancellations reveal that the two sides have the same derivative; the two sides agree at the origin, where $s = 0$ and $c = 1$. $\square$

It is natural to ask whether the foregoing passage from the 'quaternary elliptic function' ψ to the 'quartic Dixonians' s and c is reversible; we now address this question.

Consider the initial value problem that augments the differential equations (16) and (17) with the initial conditions $s(0) = 0$ and $c(0) = 1$. This initial value problem is of considerable interest in itself; we shall discuss a two-parameter generalization in a separate publication. Here, we shall simply show how quaternary elliptic functions solve the initial value problem:

$$s' = c^3 - s^2c - \alpha s, \;\; s(0) = 0,$$

$$c' = \alpha c + sc^2 - s^3, \;\; c(0) = 1.$$

A solution to this initial value problem (on some disc about the origin) satisfies the quartic identity displayed in Theorem 12. In order that this quartic admit a parametrization by elliptic functions, we ask that the homogeneous quartic

$$f(x,y,z) = x^4 - 2x^2y^2 + y^4 - z^4 - 4\alpha xyz^2$$

have two double points, reducing the genus to unity. It is straightforward to check that this is brought about by the precise condition $\alpha \notin \{0, \pm 1\}$; this condition we assume from now on. The fact that α must be nonzero is not surprising, in light of (12) and (19).

Thus, let the complex number α_0 satisfy $\alpha_0 \neq 0$ and $\alpha_0^2 \neq 1$; and let

$$\kappa = \frac{2}{3}(2\alpha_0^2 - 1). \tag{20}$$

With

$$g_2 = 4(3\kappa^2 - 1) \text{ and } g_3 = 4\kappa(1 - 2\kappa^2) \tag{21}$$

we find that

$$g_2^3 - 27g_3^2 = 16(9\kappa^2 - 4) = 256\alpha_0^2(\alpha_0^2 - 1) \neq 0.$$

Consequently, the (new) Weierstrass $\wp$-function $p = \wp(-; g_2, g_3)$ is available for use. As an elliptic function, p assumes all complex numbers as values; choose $a \in \mathbb{C}$ so that $p(a) = 1 + \kappa$. By direct calculation,

$$p'(a)^2 = 4p(a)^3 - g_2 p(a) - g_3 = 8 + 12\kappa \tag{22}$$

and

$$p''(a) = 6p(a)^2 - \tfrac{1}{2}g_2 = 8 + 12\kappa \tag{23}$$

whence $\wp$-function duplication yields

$$p(2a) = \frac{1}{4}\frac{p''(a)^2}{p'(a)^2} - 2p(a) = \kappa$$

and therefore

$$p'(2a) = 4p(2a)^3 - g_2 p(2a) - g_3 = 0.$$

As the halfperiods of p coincide with the zeros of its derivative, it follows that a is a quarterperiod of p.

Now, from this pair (p, a) we fashion the quaternary elliptic function ψ as in Section 1 and we define the elliptic functions s and c as at (13) and (14) with $\mu = \psi(-a)$ as before. On taking (22) and (23) into account, (15) simplifies to

$$\lambda = \frac{1}{\psi(-a)^2}\frac{p''(a)}{p'(a)} = \frac{p'(a)}{\psi(-a)^2} \tag{24}$$

and (18) simplifies to

$$\alpha = -\frac{1}{4}\frac{1}{\psi(-a)^2}\frac{p''(a)^2}{p'(a)^2} = -\frac{1}{4}\frac{p'(a)^2}{\psi(-a)^2}. \tag{25}$$

Developing (22) a little further, (20) yields

$$p'(a)^2 = 8 + 12\kappa = 16\alpha_0^2;$$

by replacing a with $-a$ if necessary, we may arrange that $p'(a) = -4\alpha_0$. At this point, we invoke (12) to see that $\psi(-a)^4 = p'(a)^2$; by replacing a with $a + 2\omega'$ if need be, we may arrange that $\psi(-a)^2 = p'(a)$ thanks to Theorem 6. With these choices, (25) shows that the parameter α coincides with α_0 as given, while (24) shows that the parameter λ is here unity, so that s and c are coperiodic with ψ.

To summarize this discussion, the passage from 'quaternary elliptic functions' to 'quartic Dixonians' is reversible: each quartic Dixonian pair (s, c) arises from a suitable quaternary elliptic function ψ, as follows.

THEOREM 13. *If the complex parameter α is neither 0 nor ± 1 then the initial value problem*

$$s' = c^3 - s^2c - \alpha s, \;\; s(0) = 0$$
$$c' = \alpha c + sc^2 - s^3, \;\; c(0) = 1$$

has solutions given by

$$s(z) = \frac{\psi(a-z)}{\psi(-a)} \;\; \textit{and} \;\; c(z) = \frac{\psi(z-a)}{\psi(-a)}$$

where ψ is the quaternary elliptic function associated to a suitable pair (p, a).

Of course, the elliptic functions s and c of Theorem 13 provide an elliptic parametrization of the quartic displayed in Theorem 12 when an arbitrary $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$ is given.

We reserve for a future publication a detailed account of a two-parameter generalization of this 'quartic Dixonian' initial value problem; here, we simply exhibit (without proof) the solutions to the initial value problem of Theorem 13 in the degenerate cases $\alpha \in \{0, \pm 1\}$. If $\alpha = 0$ then the solutions are

$$s = \sinh \;\text{ and }\; c = \cosh;$$

if $\alpha = \pm 1$ then the solutions are given by

$$s(z) = \frac{\tan z}{1 - \alpha \tan z} \;\text{ and }\; c(z) = \frac{1}{1 - \alpha \tan z}.$$

As expected, in each of the degenerate cases the solutions are hyperbolic or trigonometric rather than elliptic.

Appendix

It is a familiar fact that the halfperiods of any Weierstrass $\wp$-function p coincide with the zeros of its derivative: that is, if Λ_p denotes the period lattice of p and if $z \in \mathbb{C} \setminus \Lambda_p$ then

$$2z \in \Lambda_p \Leftrightarrow p'(z) = 0.$$

We require characterizations of thirdperiods and quarterperiods in terms of the values of p and its derivatives; lacking a convenient reference, we establish such characterizations here. In each case, we make use of the familiar $\wp$-function duplication formula:

$$p(2z) + 2p(z) = \frac{1}{4} \frac{p''(z)^2}{p'(z)^2}. \tag{26}$$

In connexion with this formula, note that $z \in \mathbb{C} \setminus \Lambda_p$ cannot satisfy $p'(z) = 0$ and $p''(z) = 0$ simultaneously. For suppose it does: evaluation of the eliminant equation $(p')^2 = 4p^3 - g_2 p - g_3$ at z gives

$$g_3 = \{4p(z)^2 - g_2\} p(z)$$

while evaluation of the derived equation $p'' = 6p^2 - \frac{1}{2} g_2$ at z gives

$$p(z)^2 = \tfrac{1}{12} g_2;$$

elimination of $p(z)$ between these two conditions gives $g_2^3 = 27 g_3^2$, which violates the nature of p as a $\wp$-function.

Incidentally, $z \in \mathbb{C} \setminus \Lambda_p$ can satisfy $p''(z) = 0$ and $p(z) = 0$ simultaneously: on account of the derived equation and the eliminant equation respectively, this happens precisely when $g_2 = 0$ and $p'(z)^2 = -g_3$. Coincidentally, $z \in \mathbb{C} \setminus \Lambda_p$ can satisfy $p(z) =$

0 and $p'(z) = 0$ simultaneously: on account of the eliminant equation and the derived equation respectively, this happens precisely when $g_3 = 0$ and $p''(z) = -\frac{1}{2}g_2$. In each of these cases, the indicated points z exist, since p' and p'' are elliptic.

The characterization of thirdperiods is quite simple. Let $z \in \mathbb{C} \setminus \Lambda_p$ be a thirdperiod of p: thus, $3z$ is a period of p, so that $2z \equiv -z$ modulo Λ_p and therefore $p(2z) = p(-z) = p(z)$. The duplication formula (26) now simplifies to

$$p''(z)^2 = 12p'(z)^2 p(z) = p'(z)p'''(z) \tag{27}$$

by virtue of the identity $p''' = 12pp'$ satisfied by the $\wp$-function p.

The converse holds: as p' and p'' cannot vanish simultaneously, if $z \in \mathbb{C} \setminus \Lambda_p$ satisfies (27) then (26) shows that $p(2z) = p(z) = p(-z)$ from which follows $2z \equiv -z$ modulo Λ_p so that z is a thirdperiod of p.

The characterization of quarterperiods calls for just a little more work. Differentiate (26) and rearrange to obtain

$$4p'(z)^4 p'(2z) = p'(z)^2 p''(z)p'''(z) - p'(z)p''(z)^3 - 4p'(z)^5. \tag{28}$$

Now let $z \in \mathbb{C} \setminus \Lambda_p$ be a quarterperiod of p: as $2z$ is a halfperiod of p but z is not, $p'(2z)$ is zero but $p'(z)$ is not; consequently, $p'(z)$ may be cancelled in (28) to yield

$$p''(z)^3 + 4p'(z)^4 = p'(z)p''(z)p'''(z). \tag{29}$$

Here, note that the right side may be replaced by $12p(z)p'(z)^2 p''(z)$ by virtue of the identity $p''' = 12pp'$.

Conversely, let $z \in \mathbb{C} \setminus \Lambda_p$ satisfy (29); from (28) it follows that either $p'(2z) = 0$ or $p'(z) = 0$. The latter condition forces $p''(z) = 0$ and is therefore ruled out, so the former condition holds: thus $2z$ is a halfperiod of p and so z is a quarterperiod of p.

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