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DUALS OF PASCAL SEQUENCE SPACES VIA MODULUS FUNCTION

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Abstract. The main goal of this paper is to introduce the new concept of sequence spaces by using Pascal matrix via modulus function. We investigate various algebraic, topological properties and inclusion relation of the newly constructed sequence space. We also make and efforts to find duals of these spaces.

1. Introduction and preliminaries

Let us consider ω be the space of all real or complex valued sequences. Any vector subspace of ω is referred to be a sequence space. The spaces of all bounded, convergent and null sequences can be written as ℓ_∞, c and c_0 and we represent these as

$$\ell_\infty = \{\psi = (\psi_m) \in \omega : \sup_m |\psi_m| < \infty\},$$
$$c = \{\psi = (\psi_m) \in \omega : \lim_{m\to\infty} \psi_m \text{ exists}\},$$
$$c_0 = \{\psi = (\psi_m) \in \omega : \lim_{m\to\infty} \psi_m = 0\}.$$

Bounded, convergent, absolutely convergent, and p-absolutely convergent series are denoted by bs, cs, ℓ_1 and ℓ_p respectively. Sequence space maps $p_i : \varsigma \to \mathbb{C}$ defined by $q_i(\psi) = \psi_i$ are continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ and $\mathbb{N}$ indicates the sets of complex numbers and non-negative integers, respectively. A sequence ς with a linear topology is referred to as a K-space. An K-space ς is known to be FK-space provided ς is a complete linear metric space. An FK-space is a space with a normable topology, which is referred to as a BK-space [11].

Let X and Y be two sequence spaces and $A = (a_{lm})$ denote an infinite matrix of real entries, where $l, m \in \mathbb{N}$. Then, for a sequence ξ, we define the A-transform of ξ as $A\xi = \big((A\xi)_l\big)$, where

$$(A\xi)_l = \sum_m a_{lm}\xi_m,$$

provided that the series converges for each $l \in \mathbb{N}$. If $\xi \in X$ implies that $A\xi \in Y$, then we say that A determines a matrix transformation from X into Y and it is represented by $A : X \to Y$. The class of all such matrices is denoted by $(X : Y)$. For convenience,

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throughout this work, all summations without specified limits are understood to extend from 0 to ∞. Let F represent the family of all finite subsets of $\mathbb{N}$ and $K \subset F$. For infinite matrix A, we denote the matrix domain by X_A for sequence space X and is defined as follows:

$$X_A = \{\xi = (\xi_m) \in \omega : A\xi \in X\}. \tag{1}$$

Summability theory plays a fundamental role in functional analysis and has been extensively applied to the study of sequence spaces and matrix transformations (Başar [7]). Considerable effort has been devoted to constructing new sequence spaces as generalizations of the classical spaces ℓ_p, c_0, c, and ℓ_∞ and to examining their duals, basis and topological structures. For instance, Başar and Altay [8] investigated the space of sequences of p-bounded variation and related matrix mappings, while Altay and Başar [2] introduced paranormed sequence spaces of non-absolute type by means of weighted means. Further contributions include the introduction of Euler sequence spaces and their generalizations by various authors see ([1, 3, 18]), the study of generalized difference sequence spaces of non-absolute type defined by Sönmez and Başar [30]) and the analysis of matrix domains and their duals by Altay and Başar [4]). The reader can refer to the articles [9, 20, 26], and the recent textbook [24] in order to the reader is aware of some new results on the generated spaces of new sequence spaces and the spaces of almost convergent sequences, and fundamental theorems in functional analysis, summability theory, sequence spaces.

Building upon these developments, the present work is concerned with new sequence spaces generated by the Pascal mean and the associated Pascal matrix. We investigate their structural and topological properties and establish connections with existing results in summability theory.

A function $f : [0,\infty) \to [0,\infty)$ is referred to as a modulus function (or simply a modulus) if it satisfies the following conditions:

1. $f(u) = 0$ if and only if $u = 0$,
2. f is an increasing function,
3. $f(u+v) \leqslant f(u) + f(v)$ for every $u, v \in [0,\infty)$,
4. f is continuous from the right at 0.

Therefore, the function f must be continuous on the interval $[0,\infty)$. A modulus function can either be bounded or unbounded. Numerous authors have frequently utilized modulus functions in the construction of various difference sequence spaces, see ([15, 16, 21, 23, 27, 28, 32]).

Consider a linear metric space X. A mapping $g : X \to \mathbb{R}$ is known to be paranorm provided that

1. $g(\xi) \geqslant 0$ for $\xi \in X$;
2. $g(-\xi) = g(\xi)$ for $\xi \in X$;
3. $g(\xi + \chi) \leqslant g(\xi) + g(\chi)$ for $\xi, \chi \in X$;

4. If (α_l) be a sequence of scalars converging to α and $(\xi_l) \in X$ such that $g(\xi_l - \xi) \to 0$, then it follows that $g(\alpha_l \xi_l - \alpha\xi) \to 0$ as $l \to \infty$.

A paranorm g is referred to as a total paranorm if $g(\xi) = 0$ implies that $\xi = 0$. In this case, the space (X, g) is known to be total paranormed space. It is a well-known result that any linear metric space can be equipped with a metric derived from a total paranorm [33].

Let P denote the Pascal means, which are generated by the Pascal matrix $P = (p_{lm})$ given by [22], defined as

$$p_{lm} = \begin{cases} \binom{l}{l-m}, & 0 \leqslant m \leqslant l \\ 0, & \text{otherwise} \end{cases}, \quad (l, m \in \mathbb{N}).$$

The inverse $P^{-1} = (s_{lm})$ of Pascal matrix $P = (p_{lm})$, given by [13], as

$$s_{lm} = \begin{cases} (-1)^{l-m}\binom{l}{l-m}, & 0 \leqslant m \leqslant l \\ 0, & m > l \end{cases}, \quad (l, m \in \mathbb{N}). \tag{2}$$

Pascal matrix possesses several properties. For instance, for any number $l > 0$, we can form three different kinds of matrices namely symmetric, triangle and upper triangle matrices. The Pascal symmetric matrix $S_l = (s_{ij})$ of order l is defined as

$$s_{ij} = \binom{i+j-2}{j-1}, \quad i, j = 1, 2, \ldots, n. \tag{3}$$

The triangle Pascal matrix $L_l = (\ell_{ij})$ of order l can be defined by

$$\ell_{ij} = \begin{cases} \binom{i-1}{j-1}, & 0 \leqslant j \leqslant i \\ 0, & j > i \end{cases} \tag{4}$$

and the upper triangle Pascal matrix $U_l = (u_{ij})$ of order l is defined by

$$u_{ij} = \begin{cases} \binom{j-1}{i-1}, & 0 \leqslant i \leqslant j \\ 0, & j < i. \end{cases} \tag{5}$$

It can be observed that $U_l = (L_l)^T$, for every positive integer l.

1. If S_l is the Pascal symmetric matrix of order l as defined by (2), L_l be the triangle Pascal matrix of order l as defined by equation (4) and U_l is the upper triangle Pascal matrix of order l as defined by equation (5), then $S_l = L_l U_l$ and $\det(S_l) = 1$ [10].

2. If A and B are $l \times l$ matrices, then A is defined to be similar to B whenever there exists an invertible matrix P of order l such that $P^{-1}AP = B$ [14].

3. The Pascal symmetric matrix S_l of order l, defined by equation (3), is similar to its inverse S_l^{-1} [10].

4. The triangle Pascal matrix L_l of order l, defined by equation (4), has its inverse expressed as $L_l^{-1} = ((-1)^{i-j} l_{ij})$ [12].

Several authors have focused on extending the classical sequence spaces ℓ_p, c_0, c and ℓ_∞ to broader classes with richer structural properties. In this direction, Aydın and Başar [6] introduced certain sequence spaces including ℓ_p and ℓ_∞, while Aydın and Başar [5] considered spaces containing c_0 and c and Şengönül and Başar [29] examined Cesàro sequence spaces of non-absolute type. These works established the linear and topological structures of the newly defined spaces, investigated their duals, and studied associated matrix transformations.

Motivated by these contributions, we focus here on sequence spaces generated by the Pascal mean and its associated matrix. Our aim is to construct such spaces, analyze their structural properties, and determine their duals in connection with classical spaces.

The Pascal sequence spaces p_∞, p_c and p_0 to consist of all sequences such that their P-transforms belong to the classical spaces ℓ_∞, c and c_0, respectively, are defined by Polat [25] and are given by

$$p_\infty = \left\{ \xi = (\xi_m) \in \omega : \sup_l \left| \sum_{m=0}^{l} \binom{l}{l-m} \xi_m \right| < \infty \right\},$$

$$p_c = \left\{ \xi = (\xi_m) \in \omega : \lim_{l\to\infty} \sum_{m=0}^{l} \binom{l}{l-m} \xi_m \text{ exists} \right\},$$

$$p_0 = \left\{ \xi = (\xi_m) \in \omega : \lim_{l\to\infty} \sum_{m=0}^{l} \binom{l}{l-m} \xi_m = 0 \right\}.$$

Using the notation introduced in (1), the spaces p_∞, p_c and p_0 can be redefined as follows:

$$p_\infty = (\ell_\infty)_P, \quad p_c = (c)_P, \quad p_0 = (c_0)_P. \tag{6}$$

For a normed or paranormed sequence space ς, the associated matrix domain ς_P is called the Pascal sequence space. In this context, the sequence $y = (y_l)$ is defined as the P-transform of $\xi = (\xi_m)$, given by

$$y_l = \sum_{m=0}^{l} \binom{l}{l-m} \xi_m, \quad (l \in \mathbb{N}). \tag{7}$$

It is straightforward to verify that p_∞, p_c and p_0 are linear normed spaces under the norm defined by

$$\|\xi\|_{p_0} = \|\xi\|_{p_c} = \|\xi\|_{p_\infty} = \|P\xi\|_{\ell_\infty}. \tag{8}$$

In this paper, we introduce new Pascal sequence spaces p_∞, p_c, and p_0 defined via a modulus function and to establish several related results. In addition, we determine the α-, β-, and γ-duals of the spaces.

2. Main results

In this section, we introduce the Pascal sequence spaces over the modulus function. Moreover, we present topological and algebraic characterstics of these recently introduced sequence spaces. Let $q = (q_m)$ be a bounded sequence of positive real numbers. The sequence spaces below are then defined as follows

$$(P,f,q)_\infty = \left\{ \Psi = (\psi_m) \in \omega : \sup_l \left[\sum_{m=0}^{l} \left| f_m \binom{l}{l-m} \psi_m \right|^{q_m} \right] < \infty \right\},$$

where $0 < q_m \leqslant A < \infty$. In the case, $q_m = q$, for all $m \in \mathbb{N}$, the space reduces to classical space l_p, when $p \geqslant 1$.

$$(P,f,q)_c = \left\{ \Psi = (\psi_m) \in \omega : \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m \right|^{q_m} \text{ exists} \right\}.$$

$$(P,f,q)_0 = \left\{ \psi = (\psi_m) \in w : \lim_{l \to \infty} \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m \right|^{q_m} = 0 \right\}.$$

Through out the paper,the following inequality will be used. Let $q = (q_m)$ be the sequence of positive real numbers such that $0 < q_m \leqslant \sup_m q_m = A$ and define $F = \max\{1, 2^{A-1}\}$. Then for any sequences (ψ_m) and (σ_m) in $\mathbb{C}$ the following inequality holds

$$|\psi_m + \sigma_m|^{q_m} \leqslant F(|\psi_m|^{q_m} + |\sigma_m|^{q_m}).$$

THEOREM 1. *The spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. We prove the result for the space $(P,f,q)_\infty$. Let $\psi, \sigma \in (P,f,q)_\infty$. Then

$$\sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m \right|^{q_m} < \infty, \tag{9}$$

$$\sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \sigma_m \right|^{q_m} < \infty. \tag{10}$$

Suppose that $\alpha, \beta \in \mathbb{C}$. Then there exist integers R_α and Q_β such that

$$|\alpha| \leqslant R_\alpha \quad \text{and} \quad |\beta| \leqslant Q_\beta.$$

Thus, we have

$$\begin{aligned}
&\sup_l \left| \sum_{m=0}^{l} \alpha \left[f_m \binom{l}{l-m} \psi_m \right] + \sum_{m=0}^{l} \beta \left[f_m \binom{l}{l-m} \sigma_m \right|^{q_m} \right] \\
&\leqslant \sup_l \left(|\alpha| \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \varphi_m \right|^{q_m} + |\beta| \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \sigma_m \right|^{q_m} \right) \\
&\leqslant FR_\alpha^A \sup_l \left(\left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \varphi_m \right|^{q_m} \right) + FQ_\beta^A \sup_l \left(\left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \sigma_m \right|^{q_m} \right) \\
&< \infty.
\end{aligned}$$

Then, $\alpha\varphi + \beta\sigma \in (P,f,q)_\infty$. Thus, the space $(P,f,q)_\infty$ is linear. Similarly, we can prove for the spaces $(P,f,q)_c$ and $(P,f,q)_0$. This completes the proof of the theorem. □

THEOREM 2. *Let $f = (f_m)$ be a sequence of modulus functions. Then, the spaces $(P,f,q)_\infty$, $(P,f,q)_c$, $(P,f,q)_0$ are paranormed by g defined by*

$$g(\varphi) = \sup_l \left[\left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \varphi_m \right|^{q_m} \right]^{1/E},$$

where $0 < q_m \leqslant A < \infty$ for all $m \in \mathbb{N}$.

Proof. Consider the space $(P,f,q)_\infty$, clearly $g(\varphi) = 0$, whenever $\varphi = 0$. Since $q_m/E \leqslant 1$, by using the definition of the modulus function and the Minkowski inequality, we have

$$\begin{aligned}
g(\varphi + \sigma) &= \left[\sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} (\varphi_m + \sigma_m) + f_m \binom{l}{l-m} (\varphi_{m-1} + \sigma_{m-1}) \right|^{q_m} \right]^{1/E} \\
&= \left(\sup_l \left| \sum_{m=0}^{l} \left[f_m \binom{l}{l-m} \varphi_m + f_m \binom{l}{l-m} \varphi_{m-1} \right.\right.\right. \\
&\qquad \left.\left.\left. + f_m \binom{l}{l-m} \sigma_m + f_m \binom{l}{l-m} \sigma_{m-1} \right]^{q_m} \right|^{E} \right)^{1/E} \\
&\leqslant \left(\sup_l \left[\left| \sum_{m=0}^{l} \left(f_m \binom{l}{l-m} \varphi_m + f_m \binom{l}{l-m} \varphi_{m-1} \right) \right|^{q_m/E} \right.\right. \\
&\qquad \left.\left. + \left| \sum_{m=0}^{l} \left(f_m \binom{l}{l-m} \sigma_m + f_m \binom{l}{l-m} \sigma_{m-1} \right) \right|^{q_m/E} \right]^{E} \right)
\end{aligned}$$

$$\leqslant \left(\sup_{l}\sum_{m=0}^{l}\left(f_m\binom{l}{l-m}\varphi_m+f_m\binom{l}{l-m}\varphi_{m-1}\right)^{q_m}\right)^{1/E}$$
$$+\left(\sup_{l}\sum_{m=0}^{l}\left(f_m\binom{l}{l-m}\sigma_m+f_m\binom{l}{l-m}\sigma_{m-1}\right)^{q_m/E}\right)^{1/E}$$
$$= g(\varphi)+g(\sigma).$$

Furthermore, since the inequality

$$|\lambda|^{q_m} \leqslant \max\{1,|\lambda|^E\}$$

holds for all $\lambda \in \mathbb{R}$, it follows that

$$g(\lambda\varphi) = \left[\sup_{l}\sum_{m=0}^{l}\left(f_m\binom{l}{l-m}(\lambda\varphi_m)+f_m\binom{l}{l-m}(\lambda\varphi_{m-1})\right)^{q_m}\right]^{1/E}$$
$$= \sup_{l}\left[\sum_{m=0}^{l}|\lambda|^{q_m}f_m\binom{l}{l-m}\varphi_m+f_m\binom{l}{l-m}\varphi_{m-1}\right]^{1/E}$$
$$\leqslant \max\left[\{1,|\lambda|\}\cdot g(\varphi)\right].$$

Let the sequence of scalars λ_l converge to λ, i.e., $\lambda_l \to \lambda$ as $l \to \infty$ and let the sequence $\{\psi^l\} \subset (P,f,q)_\infty$ be such that $g(\psi^l-\psi) \to 0$ as $l \to \infty$. Then,

$$\begin{aligned}0 &\leqslant g(\lambda_l\psi^l-\lambda\psi)\\ &= g(\lambda_l\psi^l-\lambda\psi^l+\lambda\psi^l-\lambda\psi)\\ &= g((\lambda_l-\lambda)\psi^l+\lambda(\psi^l-\psi))\\ &\leqslant g((\lambda_l-\lambda)\psi^l)+g(\lambda(\psi^l-\psi))\\ &= |\lambda_l-\lambda|\cdot g(\psi^l)+\max\{1,|\lambda|\}]g(\psi^l-\psi).\end{aligned} \tag{11}$$

By combining the facts that $\psi^l \to \psi$ as $l \to \infty$ and $g(\psi^l-\psi) \to 0$ as $l \to \infty$, together with equality (11), we have

$$g(\lambda_l\psi^l-\lambda\psi) \to 0 \quad \text{as} \quad l \to \infty.$$

Thus, g is a paranorm on the space $(P,f,q)_\infty$. □

THEOREM 3. *The spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ are complete with the paranorm defined in Theorem 2.*

Proof. Now, we will prove our result for $(P,f,q)_\infty$. So let us consider that (ψ^l) be a Cauchy sequence in (P,f,q), where

$$\psi^l = \{\psi_1^l,\psi_2^l,\psi_3^l,\ldots\}.$$

For given $\varepsilon > 0$, there exists a positive integer $\eta_0(\varepsilon)$ such that

$$\left| f_m(\psi^l - \psi^t) \right|^E < \varepsilon^E \quad \forall\, l,t > \eta_0(\varepsilon).$$

For each $m \in \mathbb{N}$,

$$\begin{aligned}
&|((P,f,q)\psi^l)_m - ((P,f,q)\psi^t)_m|^{q_m} \\
\leqslant &\left| ((P,f,q)\psi^l)_m - ((P,f,q)\psi^t)_m \right|^{q_m} \\
= &\sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m^l - \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m^t \right|^{q_m} \\
&+ \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi^l \right| + \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi^t \right|.
\end{aligned}$$

Then,

$$\left| f_m(\psi^l - \psi^t) \right|^E < \varepsilon^E \quad \forall\, l,t > \eta_0(\varepsilon).$$

Thus, the sequence

$$\left\{ \left((P,f,q)\psi^0\right)_m, \left((P,f,q)\psi^1\right)_m, \left((P,f,q)\psi^2\right)_m, \dots \right\}$$

is a Cauchy sequence of real numbers for every $m \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it is convergent. Then

$$((P,f,q)\psi^l)_m \to ((P,f,q)\psi)_m \quad \text{as} \quad l \to \infty.$$

By employing the infinitely many limits $((P,f,q)\psi)_0, ((P,f,q_2)\psi)_1, \dots$, we have

$$\begin{aligned}
\left[f_m(\psi^l, \psi^t) \right]^{\mathbb{E}} &= \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \varphi_m \right| + \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \sigma_m \right|^{q_m} \\
&= \sup_l \left| \sum_{m=0}^{l} f_m \binom{l}{l-m} \psi_m^l - \sum_{m=0}^{l} f_m \binom{l}{l-m} \sigma_m^t \right|^{q_m} \\
&\leqslant \sup_l \left| \sum_{m=0}^{l} ((P,f,q)\psi^l)_m - ((P,f,q)\psi^t)_m \right|^{q_m} \\
&\leqslant \varepsilon.
\end{aligned}$$

Thus, $(\varphi^l - \varphi) \in (P,f,q)_\infty$. Since the space $(P,f,q)_\infty$ is linear, we have $\varphi \in (P,f,q)_\infty$ $\varphi^l \to \varphi$ as $l \to \infty$ in $(P,f,q)_\infty$. Hence the space (P,f,q) is complete. $\square$

THEOREM 4. *The spaces $(P,f,q)_\infty, (P,f,q)_c$ and $(P,f,q)_0$ are K-space.*

Proof. First we have to prove that $S_j(\varphi) = 1$ for $j \in \mathbb{N}$ is linear. For this, let $\varphi = \varphi_j, \sigma = \sigma_j \in (P,f,q)_\infty$ and $\lambda \in \mathbb{C}$, then $S_j(\varphi+\sigma) = (\varphi+\sigma)_j = \varphi_j + \sigma_j = S_j(\varphi) + S_j(\sigma)$

$$S_j(\lambda\psi) = (\lambda\psi)_j = \lambda\psi = \lambda S_j(\psi),$$

for all $j \in \mathbb{N}$. Therefore, S_j is linear. Now, we have to prove that S_j is continuous. It is sufficient to establish that it is bounded. For this, let $\psi = (\psi_j) \in (P,f_i,q_i)_\infty$, then $|S_j(\psi)| = |\psi_j|$, $\forall j \in \mathbb{N}$, we have

$$\|S_j\| = \sup_{\psi \neq 0} \frac{|S_j(\psi)|}{\|\psi\|_{(P,f,q)_\infty}} = \sup_{\psi \neq 0} \frac{|\psi|}{\|\psi\|_{(P,f,q)_\infty}} \leqslant \sup_{\psi \neq 0} \frac{\|\psi\|_{(P,f_i,q_i)_\infty}}{\|\psi\|_{(P,f,q)_\infty}} = 1.$$

Thus, S_j is bounded implies S_j is linear and continuous. Hence, $(P,f,q)_\infty$ is a K-space. □

THEOREM 5. *The spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ are FK-space.*

Proof. Consider a space $(P,f,q)_\infty$ is complete space. Then the space $(P,f,q)_\infty$ is an FK-space. □

THEOREM 6. *The spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ are a Frechet space.*

Proof. Let us consider that $\varphi = (\varphi_l)$ and $\sigma = (\sigma_l)$ be two sequences in $(P,f,q)_\infty$ and (λ_l) be a sequence of scalars such that

$$\widetilde{\delta}(\psi_l,\psi) \to 0, \quad \widetilde{\delta}(\sigma_l,\sigma) \to 0, \quad \widetilde{\delta}(\sigma_l+\psi_l,\sigma+\psi) \to 0 \quad \text{and} \quad \lambda_l \to \lambda \text{ as } l \to \infty.$$

Then, we have

$$\begin{aligned} 0 &\leqslant \lim_{l\to\infty} \widetilde{\delta}(\psi_l+\sigma_l,\psi+\sigma) \\ &= \lim_{l\to\infty} \|(\psi_l+\sigma_l)-(\psi+\sigma)\| \\ &\leqslant \lim_{l\to\infty} (\|\psi_l-\psi\|+\|\sigma_l-\sigma\|) \\ &= \lim_{l\to\infty} \widetilde{\delta}(\psi_l,\psi) + \lim_{l\to\infty} \widetilde{\delta}(\sigma_l,\sigma) = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} 0 &\leqslant \lim_{l\to\infty} \widetilde{\delta}(\lambda_l\psi_l,\lambda\psi) \\ &= \lim_{l\to\infty} \|\lambda_l\psi_l-\lambda\psi\| \\ &= \lim_{l\to\infty} \|(\lambda_l-\lambda)\psi_l+\lambda(\psi_l-\psi)\| \\ &\leqslant \lim_{l\to\infty} (|\lambda_l-\lambda|\cdot\|\psi_l\|+|\lambda|\cdot\|\psi_l-\psi\|) \\ &= \lim_{l\to\infty} (|\lambda_l-\lambda|\cdot\|\psi_l\|+|\lambda|\cdot\|\psi_l-\psi\|) = 0 \end{aligned} \tag{13}$$

from the inequalities (12) and (13) that the algebraic operations are continous on space $(P,f,q)_\infty$. Therefore, $(P,f,q)_\infty$ is Frechet space. □

3. The α, β, γ-duals of the spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$

The α,β, and γ-duals of the sequence spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ are defined by the propositions and their proofs in this section. We define the multiplier space $S(X,Y)$ for the sequence spaces X, Y as follows. For the sequence spaces X, Y, we define the multiplier space $S(X,Y)$ as follows

$$S(X,Y) = \{z = (z_m) \in \omega : \xi z = (\xi_m z_m) \in Y \text{ for all } \xi = (\xi_m) \in X\}.$$

Garling [17] defined the α, β, and γ-duals of a sequence spaces X as follows: X^α, X^β and X^γ, respectively.

$$X^\alpha = S(X,\ell_1), \quad X^\beta = S(X,cs), \quad X^\gamma = S(X,bs).$$

We proceed by utilizing certain lemmas established by Stieglitz and Tietz [31], which are essential in the proof of the Theorems.

LEMMA 1. *$A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_l \sum_{m\in F} \left| \sum_{r\in K} a_{lr} \right| < \infty. \tag{14}$$

LEMMA 2. *$A \in (c_0 : c)$ if and only if*

$$\sup_l \sum_r |a_{lr}| < \infty, \tag{15}$$

$$\lim_{l\to\infty} a_{lm} = \alpha_m \quad (\alpha_m \in \mathbb{R}). \tag{16}$$

LEMMA 3. *$A \in (c_0 : \ell_\infty)$ if and only if (15) holds.*

THEOREM 7. *The α-dual of the sequence spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ is the set*

$$D = \left\{ a = (a_m) \in \omega : \sup_{m\in F} \sum_l \left| \sum_{m\in K} f_m (-1)^{l-m} \binom{l}{l-m} a_l \right| < \infty \right\}.$$

Proof. Let $a = (a_l) \in \omega$ and let us consider the matrix B whose rows of the matrix B are the products of the rows of the matrix P^{-1} and sequence $a = (a_l)$. The relation

$$L_l = (\ell_{ij}) = \begin{cases} \binom{i-1}{j-1}, & 0 \leqslant j \leqslant i, \\ 0, & j > i. \end{cases} \tag{17}$$

We can derive that

$$\begin{aligned} a_l\varphi_l &= \sup_l \left| \sum_{m=0}^{l} f_m(-1)^{l-m} \binom{l}{l-m} a_l \sigma_m \right| \\ &= \sum_{m=0}^{l} b_{lm}\sigma_m \\ &= (B\sigma)_l \quad (l \in \mathbb{N}). \end{aligned} \tag{18}$$

By equality (18), we analyze that $a\varphi = (a_l\varphi_l) \in z$, whenever $\varphi \in (P,f,q)_\infty, (P,f,q)_c$ and $(P,f,q)_0$, $B_\sigma \in z_1$, whenever $\sigma \in l_\infty$, c and c_0 then, we derive by Lemma 1 that

$$\sup_{m\in F} \sum_l \left| \sum_{m\in K} f_m(-1)^{l-m} \binom{l}{l-m} a_l \right| < \infty,$$

which yields the consequence that

$$((P,f,q)_\infty)^\alpha = ((P,f,q)_c)^\alpha \quad \text{and} \quad ((P,f,q)_0)^\alpha = D. \quad \square$$

THEOREM 8. *Let us consider the sets D_1, D_2, D_3 defined as follows*

$$\begin{aligned} D_1 &= \left\{ a = (a_m) \in \omega : \sup_{l\in\mathbb{N}} \sum_{m=0}^{l} \left| \sum_{m=0}^{l} f_m(-1)^{i-m} \binom{i}{i-m} \right| \right\} < \infty, \\ D_2 &= \left\{ a = (a_m) \in \omega : \sum_{i=m}^{\infty} \sup_l \left| \sum_{m=0}^{l} f_m(-1)^{i-m} \binom{i}{i-m} a_i \right| \text{ exists for each } m \in \mathbb{N} \right\}, \\ D_3 &= \left\{ a = (a_m) \in \omega : \lim_{l\to\infty} \sum_{m=0}^{l} \left| \sum_{m=0}^{l} f_m(-1)^{i-m} \binom{i}{i-m} a_i \right| \text{ exist} \right\}. \end{aligned}$$

Then,

$$\begin{aligned} ((P,f,q)_0)^\beta &= D_1 \cap D_2, \\ ((P,f,q)_c)^\beta &= D_1 \cap D_2 \cap D_3, \\ ((P,f,q)_\infty)^\beta &= D_2 \cap D_3. \end{aligned}$$

Proof. We will give the proof for $(P,f,q)_0$. Let us consider the equation

$$\begin{aligned} \sum_{m=0}^{l} a_m\varphi_m &= \sum_{m=0}^{l} \left(\sum_{i=m}^{l} f_m(-1)^{i-m} \binom{i}{i-m} \sigma_i \right) a_m \\ &= \sum_{m=0}^{l} \left(\sum_{i=m}^{l} f_m(-1)^{i-m} \binom{i}{i-m} a_i \right) \sigma_m \\ &= (D\sigma)_l, \end{aligned} \tag{19}$$

where

$$D=(d_{lm})=\begin{cases}\sum_{i=m}^{l}\left(f_m(-1)^{i-m}\binom{i}{i-m}\right)a_i, & 0\leqslant m\leqslant l,\\ 0, & m>l,\end{cases}\quad \text{for}\quad l\in\mathbb{N}. \tag{20}$$

Thus, we infer from Lemma 2 with equality(19) that

$$a\psi=(a_m\psi_m)\in cs,\qquad \text{whenever}\quad \psi=(\psi_m)\in(P,f,q)_0$$

and

$$D\sigma\in c,\qquad \text{whenever}\quad \sigma=(\sigma_m)\in c_0.$$

Using relations (15) and (16), we conclude that $\lim\limits_{l\to\infty} d_{lm}$ exists for each $m\in\mathbb{N}$ and

$$\sup_{l\in\mathbb{N}}\sum_{m=0}^{l}\left\{\left|\sum_{i=m}^{l} f_m(-1)^{i-m}\binom{i}{i-m}a_i\right|\right\}<\infty,$$

which shows $((P,f,q)_0)^\beta=D\cap D_2$.

The proof for the spaces $(P,f,q)_c,(P,f,q)_\infty$ is same, so we omit it. □

THEOREM 9. *The sequence spaces $(P,f,q)_\infty$, $(P,f,q)_c$ and $(P,f,q)_0$ all have D_1 as their γ-dual.*

Proof. Consider the space $(P,f,q)_0$. Let us assume the equality

$$\begin{aligned}\left|\sum_{m=0}^{l}a_m\psi_m\right| &= \left|\sum_{m=0}^{l}a_m\left[\sum_{i=0}^{l}f_m(-1)^{m-i}\binom{m}{m-i}\sigma_i\right]\right|\\ &= \left|\sum_{m=0}^{l}\left[\sum_{i=m}^{l}f_m(-1)^{i-m}\binom{i}{i-m}a_i\right]\sigma_m\right|\\ &\leqslant \sum_{m=0}^{l}\left|\left[\sum_{i=m}^{l}f_m(-1)^{i-m}\binom{i}{i-m}a_i\right]\right||\sigma_m|.\end{aligned}$$

Now taking supremum over $l\in\mathbb{N}$, we get

$$\begin{aligned}\sup_{l\in\mathbb{N}}\left|\sum_{m=0}^{l}a_m\psi_m\right| &\leqslant \sup_{l\in\mathbb{N}}\left[\sum_{m=0}^{l}\left|\sum_{i=m}^{l}f_m(-1)^{i-m}\binom{i}{i-m}a_i\right|\right]|\sigma_m|\\ &\leqslant \|\sigma\|_{c_0}\sup_{l}\sum_{m=0}^{l}\left(\sup_{l}\left|\sum_{i=m}^{l}f_m(-1)^{i-m}\binom{i}{i-m}a_i\right|\right)\\ &\leqslant \infty.\end{aligned}$$

This means that $a=(a_m)\in\{(P,f,q)_0\}^\gamma$, so $D_1\subset\{(P,f,q)_0\}^\gamma$. Conversely, let $a=(a_m)\in\{(P,f,q)_0\}^\gamma$ and $\psi\in(P,f,q)_0$, we have

$$\sum_{m=0}^{l}\left[\sum_{i=m}^{l} f_m(-1)^{i-m}\binom{i}{i-m}a_i\right]\sigma_m\in\ell_\infty,$$

whenever $a\psi=(a_m\psi_m)\in bs$. Consequently, in the class (c_0,ℓ_∞) contains the matrix D provided in equation (20). Thus, the condition

$$\sup_l\left[\sum_{m=0}^{l}\left|\sum_{i=m}^{l} f_m(-1)^{i-m}\binom{i}{i-m}a_i\right|\right]<\infty,$$

is satisfied, which implies that $a=(a_m)\in D_1$. In other terms,

$$\{(P,f,q)_0\}^\gamma\subset D_1.$$

This complete the proof of the theorem. □

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