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ON SOME CRITERIA FOR STARLIKENESS OF RATIONAL FUNCTIONS

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Abstract. In the present paper, we investigate a new class of analytic functions $\mathscr{V}_{\alpha,\mu}(A,B)$ and subclasses of it, which consists of analytic functions f where satisfies the condition

$$\mathscr{V}_{\alpha,\mu}(A,B)=\left\{f\in\mathscr{A}:(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)\prec\frac{1+Az}{1+Bz}\right\}.$$

By making use of differential subordination, we find conditions on the parameters of α,μ,A and B which guarantee univalance or starlikeness of the members of that class. Our results will generalize or improve the earlier results obtained by other researches. Also we provide some new criteria for rational functions to be univalent or starlike.

1. Introduction

Let $\mathscr{H}$ denote the class of all analytic functions f in the unit disk $D=\{z\in\mathbb{C}:|z|<1\}$. For $n\geqslant 1$, a positive integer let

$$\mathscr{A}_n=\left\{f\in\mathscr{H}:\quad f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k\right\},\tag{1}$$

and

$$\mathscr{A}_{n,b}=\left\{f\in\mathscr{H}:\quad f(z)=z+bz^{n+1}+\sum_{k=n+2}^{\infty}a_kz^k\right\}\tag{2}$$

with $\mathscr{A}_1=\mathscr{A}$, where $\mathscr{A}$ is referred to as the normalized analytic functions in the unit disk, and by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of functions in D which are univalent. Also for $-1\leqslant B<A\leqslant 1$ suppose that

$$\mathscr{S}^*(A,B)=\left\{f\in\mathscr{A}:\quad\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},\quad z\in D\right\},\tag{3}$$

such that $\mathscr{S}^*(1,-1)=\mathscr{S}^*$ is called the well known class of all starlike functions with respect to the origin. Now we introduce the classes

$$\mathscr{U}(\lambda,\mu):=\left\{f\in\mathscr{A}:\frac{f(z)}{z}\neq 0\quad\text{and}\quad\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}-1\right|<\lambda,\ z\in D\right\},$$

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and $\mathscr{U}(\lambda) := \mathscr{U}(\lambda,1)$. It is known [2,6], (see also [3,18]) that functions in $\mathscr{U}(\lambda)$ are univalent if $0<\lambda\leqslant 1$ but not necessarily univalent if $\lambda>1$. This class and other relevant subclasses are extensively studied by authors in the [1,3,4,7,9–14,16,18] and references there in. Obradović [7] and, Pannusamy and Singh [16] proved that

$$\mathscr{U}(\lambda,\mu)\subset\mathscr{S}^* \quad \text{if} \quad \mu<1 \quad \text{and} \quad 0\leqslant\lambda\leqslant\frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}.$$

In 2007, R. Fournier and S. Ponnusamy [4] extended it as follows:

$$\mu\in\mathbb{C} \quad \text{with} \quad Re(\mu)<1 \quad \text{then} \quad \mathscr{U}(\lambda,\mu)\subset\mathscr{S}^* \quad \text{iff} \quad 0\leqslant\lambda\leqslant\frac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}}.$$

Recently Obradović and Tuneski [10] (see also [8,9]) studied coefficients of the class $\mathscr{U}(\lambda)$. If reader desire to obtain more information about classes $\mathscr{U}(\lambda,\mu)$ could check out other articles which recently are published [11–14].

In this paper we extended classes $\mathscr{U}(\lambda,\mu)$ as:

$$\begin{aligned}&\mathscr{V}_{\alpha,\mu}(A,B)\\&=\left\{f\in\mathscr{A}:\frac{f(z)}{z}\neq 0,(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)\prec\frac{1+Az}{1+Bz},\ z\in D\right\}\end{aligned} \tag{4}$$

where $-1\leqslant B<A\leqslant 1$, $\alpha>0$, and μ is complex number. Also we define $\mathscr{V}_{\alpha,\mu,n}(A,B) = \mathscr{V}_{\alpha,\mu}(A,B)\bigcap\mathscr{A}_n$, and $\mathscr{V}_{\alpha,\mu,n,b}(A,B)=\mathscr{V}_{\alpha,\mu}(A,B)\bigcap\mathscr{A}_{n,b}$ where $\mathscr{A}_n,\mathscr{A}_{n,b}$ are defined in (1), (2), respectively.

We note that in general the functions belonging to this class are not necessary univalent. For showing that let $-1<B<A\leqslant 1$, $|\lambda|<1$, $\beta<\frac{1}{1+|\lambda|}$, $\mu=\beta|\lambda|$ and α is chosen such that $|\lambda||1-\frac{\alpha}{\mu}|<\frac{A-B}{1+|B|}$. Now let us consider the function

$$f(z)=\frac{z}{(1+\lambda z)^{\frac{1}{\mu}}}$$

which is belongs to the class $\mathscr{V}_{\alpha,\mu}(A,B)$, because we have

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)=1+\lambda\left(1-\frac{\alpha}{\mu}\right)z\prec\frac{1+Az}{1+Bz}.$$

But it easy to see that $f'(z)=0$ for $z=\frac{-\mu}{\lambda(\mu-1)}\in D$, and this shows that f is not univalent in the unit disk. In the articles presented in this content, the domain of classes are considered the disk with center at one and the radius is less than one. The question is whether it is possible to change the class domain. In this article, we will try to answer this question and study it. Therefore we define the classes $\mathscr{V}_{\alpha,\mu,n}(A,B)$ and $\mathscr{V}_{\alpha,\mu,n,b}(A,B)$ and study properties such as starlikeness of elements of those classes. In section 2 we find conditions on the parameters α,μ,A,B,b and n such that guarantee the starlikeness.

The basic tool for proving our results is the following lemma due to Miller and Mocanu [5].

LEMMA 1. *Let the function $w(z)$ is defined by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

be analytic in D. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in D$, then there exists a real number $k \geqslant n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k.$$

2. Main results

We will prove in the following theorem the elements of class $\mathscr{V}_{\alpha,\mu}(A,B)$ are starlike for suitable choices of α, μ, A and B.

THEOREM 1. *Let $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$, $-1 < B < A \leqslant 1$, $\alpha \geqslant 1$, and μ is complex number with $\Re\mu < \alpha n$. If*

$$(A-B)^2 \left\{ \frac{1}{(1-AB)^2} + \frac{|\mu|^2}{(1-|B|)^2 |\alpha n - \mu|^2} \right\} < 1,$$

then $f \in \mathscr{S}^*(\delta)$, where $\delta = \frac{\alpha-1}{\alpha}$.

Proof. Let $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$ and define $p(z) = (\frac{z}{f(z)})^{\mu}$. From $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$ we have

$$p(z) - \frac{\alpha}{\mu} z p'(z) \prec \frac{1+Az}{1+Bz},$$

and this is equivalent to

$$\left| p(z) - \frac{\alpha}{\mu} z p'(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}. \tag{5}$$

In the first step we show that $p(z) \prec 1 + \lambda_1 z$, where $\lambda_1 = \frac{(A-B)|\mu|}{(1-|B|)|\alpha n - \mu|}$. For this let the function w is defined by $p(z) = 1 + \lambda_1 w(z)$. It is clear that the function w is analytic in the unit disk and $w(0) = w'(0) = \dots = w^{(n-1)}(0) = 0$. We claim that $|w(z)| < 1$ for all z in the unit disk, otherwise by using Lemma 1 there exists a point $z_0 \in D$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = k w(z_0)$ with $k \geqslant n$. Now

$$\begin{aligned}
\left| p(z_0) - \frac{\alpha}{\mu} z_0 p'(z_0) - \frac{1-AB}{1-B^2} \right| &= \left| \lambda_1 w(z_0) - \frac{\alpha}{\mu} m \lambda_1 w'(z_0) - \frac{AB-B^2}{1-B^2} \right| \\
&\geqslant |\lambda_1| \frac{|\alpha m - \mu|}{|\mu|} - \frac{|B|(A-B)}{1-B^2} \\
&\geqslant |\lambda_1| \frac{|\alpha n - \mu|}{|\mu|} - \frac{|B|(A-B)}{1-B^2} \\
&= \frac{A-B}{1-|B|} - \frac{|B|(A-B)}{1-B^2} \\
&= \frac{A-B}{1-B^2},
\end{aligned}$$

and this is contradiction with (5). Hence $p(z) \prec 1+\lambda_1 z$, where $\lambda_1 = \frac{(A-B)|\mu|}{(1-|B|)|\alpha n-\mu|}$. Also $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$ implies that there exists Shawars function w such that

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = \frac{1+Aw(z)}{1+Bw(z)},$$

and so we have

$$\frac{1}{1-\delta}\left(\frac{zf'(z)}{f(z)} - \delta\right) = \frac{\frac{1+Aw(z)}{1+Bw(z)}}{\left(\frac{z}{f(z)}\right)^{\mu}}.$$

Now f is starlike of order δ if and only if

$$\left|\arg \frac{\frac{1+Aw(z)}{1+Bw(z)}}{\left(\frac{z}{f(z)}\right)^{\mu}}\right| < \frac{\pi}{2},$$

and it is sufficient to be

$$\left|\arg \frac{1+Aw(z)}{1+Bw(z)}\right| + \left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \frac{\pi}{2}. \tag{6}$$

But $|\arg \frac{1+Aw(z)}{1+Bw(z)}| < \arcsin \frac{A-B}{1-AB}$ and $|\arg(\frac{z}{f(z)})^{\mu}| < \arcsin \lambda_1$, so inequality (6) holds if

$$(A-B)^2 \left\{ \frac{1}{(1-AB)^2} + \frac{|\mu|^2}{(1-|B|)^2|\alpha n-\mu|^2} \right\} < 1.$$

Thus the proof is complete. □

By putting $\mu = n = 1$, $\alpha = 2$ and $B = 0$ in the Theorem 1 we obtain

COROLLARY 1. *Let $f \in \mathscr{A}$ and satisfies in the condition*

$$\left| -\frac{z}{f(z)} + 2\left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \frac{\sqrt{2}}{2},$$

then $f \in S^*(\frac{1}{2})$.

By putting $\mu = \alpha = 1$, $n = 2$ and $B = 0$ in the Theorem 1 we obtain

COROLLARY 2. *Let $f \in \mathscr{A}_2$ and satisfies in the condition*

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \frac{\sqrt{2}}{2},$$

then $f \in S^*$.

By putting $\mu = \alpha = 1$, $n = 2$ and $B = -A$ with $0 < A < 1$ and solving the condition related to A we obtain $A \simeq .295597$. Now using Theorem 1 we obtain

COROLLARY 3. *Let $f \in \mathscr{A}_2$ and satisfies in the condition*

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec \frac{1+.295597z}{1-.295597z},$$

then $f \in S^*$.

THEOREM 2. *Let $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$ where $-1 < B < A \leqslant 1, \alpha > \frac{2}{3}$ and μ is complex number such that $\frac{\alpha(1-\alpha)}{N(2\alpha-1)(1-|B|)} < \Re\mu < \alpha n$. If $0 \leqslant \delta < 1$ and*

$$(A-B) \leqslant \begin{cases} \frac{\alpha(1-|B|)\sqrt{2\alpha(1-\delta)-1}}{\sqrt{\alpha^2+|\mu|^2N^2(2\alpha(1-\delta)-1)(1-|B|)^2}} & \text{if } \ 0 \leqslant \delta < \frac{|\mu|N(2\alpha-1)(1-|B|)-\alpha(1-\alpha)}{2\alpha|\mu|N(1-|B|)+\alpha^2}, \\ \frac{\alpha^2(1-|B|)(1-\delta)}{|\mu|N(\alpha\delta-\alpha+1)(1-|B|)+\alpha} & \text{if } \ \frac{|\mu|N(2\alpha-1)(1-|B|)-\alpha(1-\alpha)}{2\alpha|\mu|N(1-|B|)+\alpha^2}| \leqslant \delta < 1, \end{cases}$$

then $f \in \mathscr{S}^*(\delta)$ where $N = \int_0^1 \frac{t^{n-1-\frac{\Re\mu}{\alpha}}}{1-|B|t^n}dt$.

Proof. Let us define $p(z) = \left(\frac{z}{f(z)}\right)^\mu$. Then from the definition of the class of $\mathscr{V}_{\alpha,\mu,n,b}(A,B)$ for $f \in \mathscr{V}_{\alpha,\mu,n,b}(A,B)$, we have

$$p(z) - \frac{\alpha}{\mu}zp'(z) \prec \frac{1+Az}{1+Bz} = 1+\lambda w_1(z),$$

where $w_1(z) = \frac{w(z)}{1+Bw(z)}$ and $w(z)$ is an analytic function with $|w(z)| < 1$ and $w(0) = w'(0) = \ldots = w^{(n-1)}(0) = 0$ also $\lambda = A-B$. It is easy to check that

$$\left(\frac{z}{f(z)}\right)^\mu = 1-\lambda\frac{\mu}{\alpha}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt,$$

and

$$\frac{zf'(z)}{f(z)} = \frac{\alpha-1}{\alpha} + \frac{1}{\alpha}\frac{1+\lambda w_1(z)}{1-\lambda\frac{\mu}{\alpha}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt},$$

so

$$\frac{1}{1-\delta}\left(\frac{zf'(z)}{f(z)}-\delta\right) = \frac{1+\lambda\mu\frac{\delta-\frac{\alpha-1}{\alpha}}{\alpha(1-\delta)}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt + \frac{\lambda}{\alpha(1-\delta)}w_1(z)}{1-\lambda\frac{\mu}{\alpha}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt}.$$

Now Re $\left(\frac{zf'(z)}{f(z)}\right) > \delta$ is equivalent to the condition

$$\frac{1+\lambda\mu\frac{\delta-\frac{\alpha-1}{\alpha}}{\alpha(1-\delta)}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt + \frac{\lambda}{\alpha(1-\delta)}w_1(z)}{1-\lambda\frac{\mu}{\alpha}\int_0^1 \frac{w_1(tz)}{t^{1+\frac{\mu}{\alpha}}}dt} \neq -iT, \qquad \text{for all } \ T \in \mathbb{R} \ \text{ and } \ z \in D,$$

or

$$\lambda\left[\frac{1}{(1+iT)\alpha^2(1-\delta)}\left[\mu\int_0^1\frac{w_1(z)}{t^{1+\frac{\mu}{\alpha}}}dt[(\alpha\delta-\alpha+1)-iT\alpha(1-\delta)]+\alpha w_1(z)\right]\right]\neq -1.$$

Suppose that B_n denote the class of all Schwarz functions w such that $w(0)=w'(0)=\ldots=w^{(n-1)}(0)=0$, and let

$$M=\sup\left|\frac{1}{(1+iT)\alpha^2(1-\delta)}\left[\mu\int_0^1\frac{w_1(z)}{t^{1+\frac{\mu}{\alpha}}}dt[(\alpha\delta-\alpha+1)-iT\alpha(1-\delta)]+\alpha w_1(z)\right]\right|,$$

where sup is taken on $z\in D$, $w\in B_n$, $T\in\mathbb{R}$. In view of $|w_1(z)|<\frac{|z|^n}{1-|B||z|^n}$, we obtain

$$M<\sup_{T\in\mathbb{R}}\left\{\frac{1}{\alpha^2(1-\delta)\sqrt{1+T^2}}\left[|\mu|N\sqrt{(\alpha\delta-\alpha+1)^2+T^2\alpha^2(1-\delta)^2}+\frac{\alpha}{1-|B|}\right]\right\},$$

where

$$N=\int_0^1\frac{t^{n-1-\frac{\Re\mu}{\alpha}}}{1-|B|t^n}dt.$$

Now it is enough to show that $\lambda M<1$. Let

$$\varphi(x)=\frac{1}{\sqrt{1+x}}\left[|\mu|N\sqrt{(\alpha\delta-\alpha+1)^2+x\alpha^2(1-\delta)^2}+\frac{\alpha}{1-|B|}\right],\qquad x\in[0,+\infty).$$

With simple calculation we obtain

$$\varphi'(x)=\frac{|\mu|N[-1+2\alpha(1-\delta)]-\frac{\alpha}{1-|B|}\sqrt{(\alpha\delta-\alpha+1)^2+x\alpha^2(1-\delta)^2}}{2\sqrt{(\alpha\delta-\alpha+1)^2+x\alpha^2(1-\delta)^2}\sqrt{1+x^2}(1+x)}.$$

Case 1. Let $0\leqslant\delta<\frac{|\mu|N(2\alpha-1)(1-|B|)-\alpha(1-\alpha)}{2\alpha|\mu|N(1-|B|)+\alpha^2}$. Then we see that φ has its only critical point in the positive real line at

$$x_0=\frac{1}{\alpha^2(1-\delta)^2}\left(\frac{|\mu|^2N^2(2\alpha(1-\delta)-1)^2(1-|B|)^2}{\alpha^2}-(\alpha\delta-\alpha+1)^2\right).$$

Furthermore, we can see that $\varphi'(x)>0$ for $0\leqslant x<x_0$ and $\varphi'(x)<0$ for $x>x_0$. Hence $\varphi'(x)$ attains its maximum value at x_0 and

$$\varphi(x)\leqslant\varphi(x_0)=\frac{\alpha(1-\delta)[|\mu|^2N^2(2\alpha(1-\delta)-1)(1-|B|)^2+\alpha^2]}{\sqrt{2\alpha(1-\delta)-1}\sqrt{\alpha^2+|\mu|^2N^2(2\alpha(1-\delta)-1)(1-|B|)^2}(1-|B|)}.$$

Case 2. Let $\delta>\frac{|\mu|N(2\alpha-1)(1-|B|)-\alpha(1-\alpha)}{2\alpha|\mu|N(1-|B|)+\alpha^2}$. Then one can observe that $\varphi'(x)<0$ for $x\geqslant 0$ and so the function φ gets its maximum value at $x=0$ and

$$\varphi(x)\leqslant\varphi(0)=\frac{|\mu|N(\alpha\delta-\alpha-1)(1-|B|)+\alpha}{1-|B|}.$$

Thus the proof is complete. $\square$

By putting $n=2$, $B=\frac{-1}{4}$ and $\mu=\alpha$ with $\mu>\frac{8+3\ln 3}{8+6\ln 3}$ we obtain $N=\frac{\ln 3}{2}$ and so by using the above theorem we have

COROLLARY 4. *Let* $f\in\mathscr{V}_{\mu,\mu,2}(A,\frac{-1}{4})$ *and* $\mu>\frac{8+3\ln 3}{8+6\ln 3}$. *If*

$$\left(A+\frac{1}{4}\right)\leqslant\begin{cases}\frac{6\sqrt{2\mu(1-\delta)-1}}{\sqrt{64+9(\ln 3)^2(2\mu(1-\delta)-1)}} & \text{if } \ 0\leqslant\delta<\frac{\mu(8+6\ln 3)-(3\ln 3+8)}{\mu(8+6\ln 3)},\\ \frac{6\mu(1-\delta)}{3\ln 3|(\mu\delta-\mu+1)+8} & \text{if } \ \frac{\mu(8+6\ln 3)-(3\ln 3+8)}{\mu(8+6\ln 3)}\leqslant\delta<1,\end{cases}$$

then $f\in\mathscr{S}^*(\delta)$.

Also by putting $B=0$, $n=2$ and $\mu=\alpha$, we obtain $N=1$ and so in view of Theorem 2 we have

COROLLARY 5. *Let* $f\in\mathscr{V}_{\mu,\mu,2}(\lambda,0)$ *and* $\mu>\frac{2}{3}$. *If*

$$\lambda\leqslant\begin{cases}\frac{6\sqrt{2\mu(1-\delta)-1}}{\sqrt{64+9(\ln 3)^2(2\mu(1-\delta)-1)}} & \text{if } \ 0\leqslant\delta<\frac{\mu(8+6\ln 3)-(3\ln 3+8)}{\mu(8+6\ln 3)},\\ \frac{6\mu(1-\delta)}{3\ln 3|(\mu\delta-\mu+1)+8} & \text{if } \ \frac{\mu(8+6\ln 3)-(3\ln 3+8)}{\mu(8+6\ln 3)}\leqslant\delta<1,\end{cases}$$

then $f\in\mathscr{S}^*(\delta)$.

Also we remark that by putting $B=0$ in the Theorem 2 we obtain Theorem 2.1 of [1] and by putting $\alpha=1$ and $B=0$ we obtain Theorem 2 of [19].

THEOREM 3. *Let* $\frac{1+(1-|B|)nN}{1+2(1-|B|)nN}\leqslant\alpha\leqslant 1$, *$n$ is natural number, b is complex number such that* $n|b|<1$ *and* $\mu=\alpha n$ *with* $-1<B<A\leqslant 1$. *Also let* $f\in\mathscr{V}_{\alpha,\mu,n,b}(A,B)$ *if*

$$(A-B)\leqslant\begin{cases}\frac{\sqrt{1-t_0^2(2\alpha\delta+1-2\alpha)}-n|b|\alpha}{\frac{1}{1-|B|}t_0+nN} & \text{if } \ 0\leqslant\delta\leqslant K,\\ \frac{\alpha[(1-\delta)-n|b|(-\alpha+1+\alpha\delta)]}{\frac{1}{1-|B|}+nN(-\alpha+1+\alpha\delta)} & \text{if } \ K<\delta<\dfrac{1+(\alpha-1)n|b|}{\alpha n|b|+1},\end{cases}$$

where $K=\frac{nN(2\alpha-1)(1-|B|)+\alpha-1+n|b|\alpha^2}{\alpha[1+n|b|\alpha+2nN(1-|B|)]}$, $N=\int_0^1\frac{dt}{1-|B|t^{n+1}}$,

$$t_0=\frac{\frac{1}{1-|B|}nN(2\alpha\delta-2\alpha+1)+n|b|\alpha\sqrt{M}}{n^2(2\alpha-1-2\alpha\delta)[-N^2(2\alpha-1-2\alpha\delta)+|b|^2\alpha^2]},$$

and

$$M=(2\alpha-1-2\alpha\delta)\left[\frac{1}{(1-|B|)^2}+n^2N^2(2\alpha-1-2\alpha\delta)-n^2|b|^2\alpha^2\right].$$

Then $f\in\mathscr{S}^*(\delta)$.

Proof. From the definition of the class $\mathscr{V}_{\alpha,\mu,n,b}(A,B)$ for $f \in \mathscr{V}_{\alpha,\mu,n,b}(A,B)$, we have

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)=\frac{1+Aw(z)}{1+Bw(z)}=1+\lambda w_1(z), \tag{7}$$

where $\lambda = A-B$ and $w_1(z)=\frac{w(z)}{1+Bw(z)}$ with

$$w \in \mathscr{B}_n := \left\{w \in \mathscr{H},\ w(0)=w'(0)=\ldots=w^{(n)}(0)=0,\ \ |w(z)|<1\right\}.$$

Using the same argument of Theorems 1 and 2 we observe that

$$\left(\frac{z}{f(z)}\right)^{\alpha n}=1-\alpha nbz^n-\lambda n\int_0^1\frac{w_1(tz)}{t^{n+1}}dt,\ \ \lambda=A-B.$$

From (7) we obtain

$$\frac{zf'(z)}{f(z)}=\frac{\alpha-1}{\alpha}+\frac{1}{\alpha}\frac{1+\lambda w_1(z)}{1-\alpha nbz^n-\lambda n\int_0^1\frac{w_1(tz)}{t^{n+1}}dt}.$$

So

$$\frac{1}{1-\delta}\left[\frac{zf'(z)}{f(z)}-\delta\right]$$

$$=\frac{1+\frac{nb}{1-\delta}(-\alpha+1+\alpha\delta)z^n+\frac{\lambda n}{\alpha(1-\delta)}(-\alpha+1+\alpha\delta)\int_0^1\frac{w_1(tz)}{t^{n+1}}dt+\frac{\lambda}{\alpha(1-\delta)}w_1(z)}{1-\alpha nbz^n-\lambda n\int_0^1\frac{w_1(tz)}{t^{n+1}}dt}.$$

Now $\Re\frac{zf'(z)}{f(z)}>\delta$ if $\mathrm{Re}\,\frac{1}{1-\delta}\left(\frac{zf'(z)}{f(z)}-\delta\right)>0$, or for all $T\in\mathbb{R}$,

$$\frac{\alpha(1-\delta)+\alpha nb(-\alpha+1+\alpha\delta)z^n+\lambda n(-\alpha+1+\alpha\delta)\int_0^1\frac{w_1(tz)}{t^{n+1}}dt+\lambda w_1(z)}{\alpha(1-\delta)\left(1-\alpha nbz^n-\lambda n\int_0^1\frac{w_1(tz)}{t^{n+1}}dt\right)}\neq -iT,$$

which is equivalent to

$$\lambda\left[\frac{w_1(z)+n(-\alpha+1+\alpha\delta-iT(1-\delta)\alpha)\int_0^1\frac{w_1(tz)}{t^{n+1}}dt}{(1+iT)(1-\delta)\alpha+nb\alpha(-\alpha+1+\alpha\delta-iT(1-\delta)\alpha)z^n}\right]\neq -1.$$

Now for all $T\in R$ and $z\in D$, we define

$$M=\sup\left|\frac{w_1(z)+n(-\alpha+1+\alpha\delta-iT(1-\delta)\alpha)\int_0^1\frac{w_1(tz)}{t^{n+1}}dt}{(1+iT)(1-\delta)\alpha+nb\alpha(-\alpha+1+\alpha\delta-iT(1-\delta)\alpha)z^n}\right|,$$

than

$$M \leqslant \frac{\frac{1}{1-|B|}+n\sqrt{(-\alpha+1+\alpha\delta)^2+T^2\alpha^2(1-\delta)^2}\int_0^1 \frac{dt}{1-|B|t^{n+1}}}{\alpha(1-\delta)\sqrt{1+T^2}-n|b|\alpha\sqrt{(-\alpha+1+\alpha\delta)^2+T^2\alpha^2(1-\delta)^2}}.$$

Now for completing the proof it is sufficient to show that $\lambda M < 1$. For this we define

$$\varphi(x) = \frac{1}{\alpha}\left(\frac{\frac{1}{1-|B|}+nN\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}}{(1-\delta)\sqrt{1+x}-n|b|\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}}\right), \quad x \in [0,+\infty), \tag{8}$$

Differentiating $\varphi(x)$ with respect to x, we get

$$\varphi'(x) = \frac{nN(1-\delta)(2\alpha(1-\delta)-1)-\frac{1-\delta}{1-|B|}\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}+\frac{n|b|\alpha^2(1-\delta)^2}{1-|B|}\sqrt{1+x}}{2\sqrt{1+x}\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}(\alpha[(1-\delta)\sqrt{1+x}-n|b|\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}])^2}.$$

Now if $b \neq 0$ then denominator of $\varphi(x)$ is positive for all $x \geqslant 0$, provided $\delta < \frac{1+(\alpha-1)n|b|}{\alpha n|b|+1}$, $0 < \alpha \leqslant 1$ and $n|b| \leqslant 1$. For completing the discussion we consider different steps.

Case 1. Let $\frac{2\alpha-1}{2\alpha} < \delta < \frac{1+(\alpha-1)n|b|}{\alpha n|b|+1}$, than it is clear that $\varphi'(x) < 0$, so

$$\varphi(x) \leqslant \varphi(0) = \frac{1}{\alpha}\left(\frac{\frac{1}{1-|B|}+nN(-\alpha+1+\alpha\delta)}{1-\delta-n|b|(-\alpha+1+\alpha\delta)}\right),$$

therefore $f \in \mathscr{S}^*(\delta)$ when $(A-B)\frac{\frac{1}{1-|B|}+nN(-\alpha+1+\alpha\delta)}{\alpha[(1-\delta)-n|b|(-\alpha+1+\alpha\delta)]} < 1$, and

$$\frac{2\alpha-1}{2\alpha} < \delta < \frac{1+(\alpha-1)n|b|}{\alpha n|b|+1}.$$

Case 2. Let $\frac{nN(2\alpha-1)(1-|B|)+\alpha-1+n|b|\alpha^2}{\alpha[1+n|b|\alpha+2nN(1-|B|)]} < \delta < \frac{2\alpha-1}{2\alpha}$ and define

$$\begin{aligned} N(x) = {} & nN(2\alpha(1-\delta)-1)-\frac{1}{1-|B|}\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2} \\ & +\frac{n|b|\alpha^2(1-\delta)}{1-|B|}\sqrt{1+x}, \end{aligned}$$

than

$$N'(x) = \frac{\alpha^2(1-\delta)}{2(1-|B|)}\left[\frac{-(1-\delta)}{\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}}+\frac{n|b|}{\sqrt{1+x}}\right].$$

Now when

$$\frac{(1-\delta)}{\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}} > \frac{n|b|}{\sqrt{1+x}},$$

or

$$n^2|b|^2(-\alpha+1+\alpha\delta)^2-(1-\delta)^2 < x(1-\delta)^2(1-\alpha^2\delta^2), \tag{9}$$

then $N'(x)<0$. But the right side of (9) is positive, and the left side is negative because

$$n^2|b|^2(-\alpha+1+\alpha\delta)^2-(1-\delta)^2 < (-\alpha+1+\alpha\delta)^2-(1-\delta)^2,$$

and the right side of the above relation is negative for $\delta < \frac{\alpha}{\alpha+1}$. Since $\frac{2\alpha-1}{2\alpha} < \frac{\alpha}{\alpha+1}$ for $0<\alpha\leqslant 1$. Thus for $\delta < \frac{2\alpha-1}{2\alpha}$ the right side of (9) is negative and so $N(x)$ is decreasing and

$$N(x) \leqslant N(0) = nN(2\alpha(1-\delta)-1) - \frac{-\alpha+1+\alpha\delta-n|b|\alpha^2(1-\delta)}{1-|B|} < 0.$$

So by considering the case 1 and case 2, we conclude that $f\in\mathscr{S}^*(\delta)$ when

$$(A-B)\frac{\frac{1}{1-|B|}+nN(-\alpha+1+\alpha\delta)}{\alpha[(1-\delta)-n|b|(-\alpha+1+\alpha\delta)]} < 1,$$

for

$$\frac{nN(2\alpha-1)(1-|B|)+\alpha-1+n\,|b|\,\alpha^2}{\alpha[1+n|b|\alpha+2nN(1-|B|)]} < \delta < \frac{1+(\alpha-1)n|b|}{\alpha n|b|+1}.$$

Now let $0<\delta<\frac{nN(2\alpha-1)(1-|B|)+\alpha-1+n|b|\alpha^2}{\alpha[1+n|b|\alpha+2nN(1-|B|)]}$ and define

$$t = \frac{1}{\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}}.$$

Substituting t in (8) we obtain

$$\varphi(x)=\psi(t) := \frac{\frac{t}{1-|B|}+nN}{\sqrt{1-(-\alpha+1+\alpha\delta)^2t^2+\alpha^2(1-\delta)^2t^2}-n\alpha|b|},$$

when $0<t<\frac{1}{-\alpha+1+\alpha\delta}$, so $Max_{x\in[0,\infty)}\varphi(x) = Max_{t\in(0,\frac{1}{-\alpha+1+\alpha\delta})}\psi(t)$. Differentiating of $\psi(t)$ we obtain

$$\psi'(t) = \frac{R(t)}{\sqrt{1-(-\alpha+1+\alpha\delta)^2+t^2\alpha^2(1-\delta)^2}\left(\sqrt{1-(-\alpha+1+\alpha\delta)^2+t^2\alpha^2(1-\delta)^2}-n|b|\alpha\right)^2},$$

where

$$R(t) = \frac{1}{1-|B|} - nNt(2\alpha-1-2\alpha\delta) - n|b|\alpha\sqrt{1+t^2(2\alpha-1-2\alpha\delta)}.$$

It is easy to see that $R(0)=\frac{1}{1-|B|}-n|b|\alpha>0$. Since

$$R'(t) = -(2\alpha-1-2\alpha\delta)\left[nN\sqrt{1+t^2(2\alpha-1-2\alpha\delta)}+n|b|\alpha\right]$$

and $2\alpha-1-2\alpha\delta>0$, so $R'(t)<0$ and $R(t)$ decreases. Also

$$\begin{aligned}
&R\left(\frac{1}{-\alpha+1+\alpha\delta}\right)\\
&=\frac{1}{1-|B|}-nN\frac{2\alpha-1-2\alpha\delta}{-\alpha+1+\alpha\delta}-n|b|\alpha^2\frac{1-\delta}{-\alpha+1+\alpha\delta}\\
&=\alpha(1+n|b|\alpha+2nN(1-|B|))\left[\delta-\frac{nN(2\alpha-1)(1-|B|)+\alpha-1+n|b|\alpha^2}{\alpha(1+n|b|\alpha+2nN(1-|B|))}\right]\\
&\quad -nN|B|(-2\alpha\delta+2\alpha-1)-\alpha\delta|B|+|B|(\alpha-1)\\
&<0.
\end{aligned}$$

Hence we conclude that $R(t)$ have a root in $[0,\frac{1}{-\alpha+1+\alpha\delta})$. But $R(t_0)=0$, where

$$t_0=\frac{\frac{1}{1-|B|}nN(2\alpha\delta-2\alpha+1)+n|b|\alpha\sqrt{M}}{n^2(2\alpha-1-2\alpha\delta)[-N^2(2\alpha-1-2\alpha\delta)+|b|^2\alpha^2]},$$

with

$$M=(2\alpha-1-2\alpha\delta)\left[\frac{1}{(1-|B|)^2}+n^2N^2(2\alpha-1-2\alpha\delta)-n^2|b|^2\alpha^2\right].$$

With substituting t_0 in $\psi(t)$, we have

$$\psi(t_0)=\frac{\frac{1}{1-|B|}t_0+nN}{\sqrt{1-t_0^2(2\alpha\delta+1-2\alpha)}-n|b|\alpha},$$

and the proof is complete. $\square$

In the proof of Theorem 3 we do not consider the case $b=0$. Using the symbols of that theorem we assert

COROLLARY 6. *Let* $\frac{1+(1-|B|)nN}{1+2(1-|B|)nN}\leqslant\alpha\leqslant 1$, *$n$ is natural number,* $\mu=\alpha n$ *and* $-1<B<A\leqslant 1$. *If* $f\in\mathscr{V}_{\alpha,\mu,n,b}(A,B)$ *where* $b=0$, *than* $f\in\mathscr{S}^*(\delta)$ *when*

$$(A-B)\leqslant\begin{cases}\frac{(1-|B|)\sqrt{2\alpha(1-\delta)-1}}{\sqrt{1+n^2N^2(2\alpha(1-\delta)-1)(1-|B|)^2}} & \text{if } 0\leqslant\delta<\frac{(1-|B|)nN(2\alpha-1)+\alpha-1}{\alpha(1+2(1-|B|)nN)},\\ \frac{\alpha(1-\delta)(1-|B|)}{1+(1-|B|)nN(-\alpha+1+\alpha\delta)} & \text{if } \frac{(1-|B|)nN(2\alpha-1)+\alpha-1}{\alpha(1+2(1-|B|)nN)}\leqslant\delta<1,\end{cases}$$

where $N=\int_0^1\frac{dt}{1-|B|t^{n+1}}$.

Proof. The same as proof of Theorem 3, let $f\in\mathscr{V}_{\alpha,\mu,n,b}(A,B)$, and $b=0$ in (8). Then differentiating (8) we obtain

$$\varphi'(x)=\frac{nN(1-\delta)(2\alpha(1-\delta)-1)-\frac{1-\delta}{1-|B|}\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}}{2\sqrt{1+x}\sqrt{(-\alpha+1+\alpha\delta)^2+x\alpha^2(1-\delta)^2}\alpha^2(1-\delta)^2(1+x)}.$$

If $nN(2\alpha(1-\delta)-1)-\frac{-\alpha+1+\alpha\delta}{1-|B|}<0$, than $\varphi'(x)<0$ for all positive x and so $\varphi(x)$ is decreasing and $\varphi(x)\leqslant\varphi(0)$. Therefore

$$\varphi(x)\leqslant\varphi(0)=\frac{1}{\alpha(1-\alpha)}\left(\frac{1}{1-|B|}+nN(-\alpha+1+\alpha\delta)\right),\qquad x\in[0,\infty).$$

Now if $nN(2\alpha(1-\delta)-1)-\frac{-\alpha+1+\alpha\delta}{1-|B|}>0$, than $\varphi'(x)$ for $x\geqslant x_0$ is negative and $\varphi'(x)$ for $x\leqslant x_0$ is positive so $\varphi(x)$ gets its maximum value in x_0, where

$$x_0=\frac{n^2N^2(2\alpha(1-\delta)-1)^2(1-|B|)^2-(-\alpha+1+\alpha\delta)^2}{\alpha^2(1-\delta)^2}.$$

Hence

$$\varphi(x)\leqslant\varphi(x_0)=\frac{\frac{1}{1-|B|}+n^2N^2(2\alpha(1-\delta)-1)(1-|B|)}{\sqrt{\alpha^2(1-\delta)^2+n^2N^2(2\alpha(1-\delta)-1)^2(1-|B|)^2-(-\alpha+1+\alpha\delta)^2}},$$

and the proof is complete. $\square$

By setting $n=1$, $\alpha=1$, $\mu=1$, $B=0$ and $A=\lambda$ in Theorem 3 we obtain the following theorem of [17] (see theorem 1.2).

COROLLARY 7. *If $f\in\mathscr{U}(\lambda)$ and $a=\frac{|f''(0)|}{2}<1$, then $f\in\mathscr{S}^*(\delta)$ whenever $0<\lambda\leqslant\lambda(\delta)$, where*

$$\lambda(\delta)=\begin{cases}\frac{\sqrt{(1-2\delta)(2-a^2-2\delta)}-a(1-2\delta)}{2(1-\delta)} & \text{if}\quad 0\leqslant\delta<\dfrac{1+a}{3+a},\\[2ex] \frac{1-\delta(1+a)}{1+\delta} & \text{if}\quad \frac{1+a}{3+a}\leqslant\delta<\dfrac{1}{1+a}.\end{cases}$$

Also by letting $\mu=\alpha=\frac{2}{3}$ and $b=B=0$ and $n=1$ in the corollary 6 we obtain

COROLLARY 8. *Let $f\in\mathscr{A}$, if*

$$\left|\frac{1}{3}\left(\frac{z}{f(z)}\right)^{\frac{2}{3}}+\frac{2}{3}\left(\frac{z}{f(z)}\right)^{\frac{4}{3}}f'(z)-1\right|<\frac{1}{2},$$

then $f\in S^*$.

Furthermore by letting $\alpha=\mu=n=1$, $b=\delta=0$ and $B=\frac{-1}{4}$, we obtain $N=\ln 3$, $A\simeq./328837$. Hence using corollary 6 we have

COROLLARY 9. *Let $f\in\mathscr{A}$. If*

$$\left|\left(\frac{z}{f(z)}\right)^{2}f'(z)-1/15435\right|<./61742,$$

then $f\in S^*$.

COROLLARY 10. *Let* $f \in \mathscr{V}_{\frac{1}{2},1,2,b}(A,0)$ *for* $|b|<1$, *than* $f \in \mathscr{S}^*(\delta)$ *when*

$$|A| < \frac{1-|b|-\delta(1+|b|)}{4+2\delta} \qquad \textit{and} \qquad \delta < \frac{1-|b|}{1+|b|}.$$

THEOREM 4. *Let* $\alpha \geqslant 1$, n *is natural number and* $\mu = \alpha n$, *also let*

$$0 \leqslant \lambda < \lambda^* = \frac{-n^2\alpha b + \sqrt{1+n^2-n^2\alpha^2 b^2}}{1+n^2}.$$

If b *is real number such that* $\alpha b + \lambda < \frac{1}{n}$ *and* $f \in \mathscr{V}_{\alpha,\mu,n,b}(\lambda,0)$, *then* $f \in \mathscr{S}^*(\frac{\alpha-1}{\alpha})$.

Proof. Let $f \in \mathscr{V}_{\alpha,\mu,n,b}(\lambda,0)$, then in view of definition of that class we have

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1+\lambda w(z), \tag{10}$$

where $w \in \mathscr{B}_n$. Furthermore from (10) we obtain

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \alpha n b z^n - \lambda n \int_0^1 \frac{w(tz)}{t^{n+1}} dt,$$

so

$$\left|\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < \alpha n|b| + \lambda n = n(\alpha|b|+\lambda).$$

Now the above relation implies

$$\left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| < \arcsin(n(\alpha|b|+\lambda)) \quad \text{and} \quad |\arg(1+\lambda w(z))| < \arcsin\lambda. \tag{11}$$

Also from (10) we have

$$\frac{zf'(z)}{f(z)} - \frac{\alpha-1}{\alpha} = \frac{1+\lambda w(z)}{\alpha\left(\frac{z}{f(z)}\right)^{\mu}}.$$

Since in view of (11) we can write

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)} - \frac{\alpha-1}{\alpha}\right)\right| &< |\arg(1+\lambda w(z))| + \left|\arg\left(\frac{z}{f(z)}\right)^{\mu}\right| \\ &< \arcsin\lambda + \arcsin(n(\alpha|b|+\lambda)).\end{aligned} \tag{12}$$

Now it is easy to check that $\arcsin\lambda + \arcsin(n(\alpha|b|+\lambda)) < \frac{\pi}{2}$, if

$$\lambda \leqslant \lambda^* = \frac{-n^2\alpha b + \sqrt{1+n^2-n^2\alpha^2 b^2}}{1+n^2},$$

so the proof is complete. $\square$

By setting $\mu = \alpha = n = 1$ we get the result obtained in [15]. Also by setting $\alpha = \mu = 2$ and $n = 1$ in the Theorem 4 we obtain

COROLLARY 11. *Let $|b| < \frac{1}{4}$ and $f \in \mathscr{A}_{1,b}$ satisfies the condition*

$$\left| -\left(\frac{z}{f(z)}\right)^2 + 2\left(\frac{z}{f(z)}\right)^3 f'(z) - 1 \right| < \frac{-2b + \sqrt{2-4b^2}}{2},$$

then $f \in \mathscr{S}^*(\frac{1}{2})$.

Furthermore by setting $\mu = n = 2$ and $\alpha = 1$ in the Theorem 4 we have

COROLLARY 12. *Let $|b| < \frac{\sqrt{5}}{2}$ and $f \in \mathscr{A}_{2,b}$ satisfies the condition*

$$\left| \left(\frac{z}{f(z)}\right)^3 f'(z) - 1 \right| < \frac{-4b + \sqrt{5-4b^2}}{5},$$

then $f \in \mathscr{S}^*$.

Finally we prove

THEOREM 5. *Let $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$, $-1 < B < A \leqslant 1$, $0 < \alpha \leqslant 1$, and μ is complex number with $\Re\mu < \alpha n$. If*

$$\frac{A-B}{1-|B|} \leqslant \frac{\alpha|\alpha n - \mu|}{|\alpha n - \mu| + |\mu|(1+\alpha)},$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{(A-B)(|\alpha n - \mu| + |\mu|)}{\alpha[|\alpha n - \mu|(1-|B|) - (A-B)|\mu|]}.$$

Proof. Let $f \in \mathscr{V}_{\alpha,\mu,n}(A,B)$ and define

$$p(z) = \left(\frac{z}{f(z)}\right)^{\mu} \quad \text{and} \quad Q(z) = \frac{zf'(z)}{f(z)} - 1,$$

then from the definition of the class $\mathscr{V}_{\alpha,\mu,n}(A,B)$ and using the same argument of the Theorem 1 we have

$$p(z) \prec 1 + \frac{(A-B)|\mu|}{(1-|B|)|\alpha n - \mu|} z, \tag{13}$$

and

$$|p(z)(\alpha Q(z) + 1) - 1| < \frac{A-B}{1-|B|}. \tag{14}$$

Now in view of (13) and (14) we have

$$\begin{aligned} |Q(z)| &= \left| \frac{[p(z)(\alpha Q(z)+1) - 1] - (p(z) - 1)}{\alpha p(z)} \right| \\ &\leqslant \frac{|p(z)(\alpha Q(z)+1) - 1| + |p(z) - 1|}{\alpha |p(z)|} \\ &\leqslant \frac{\frac{A-B}{1-|B|} + \frac{(A-B)|\mu|}{(1-|B|)|\alpha n - \mu|}}{\alpha\left(1 - \frac{(A-B)|\mu|}{(1-|B|)|\alpha n - \mu|}\right)} \leqslant 1. \end{aligned}$$

So

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{(A-B)(|\alpha n-\mu|+|\mu|)}{\alpha[|\alpha n-\mu|(1-|B|)-(A-B)|\mu|]},$$

and the proof is complete. □

By setting $\mu=\alpha=1$, $n=2$ and $B=-A$ where $0<A<1$ in the Theorem 5 we obtain

COROLLARY 13. *Let $f\in\mathscr{A}_2$ and $0<A$ be real number such that $\frac{2A}{1-A}\leqslant\frac{1}{3}$. If*

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-\frac{1+A^2}{1-A^2}\right|<\frac{2A}{1-A^2},$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{4A}{1-3A}.$$

Also by setting $\mu=\alpha=1$, $n=2$ and $B=0$ in the Theorem 5 we obtain

COROLLARY 14. *Let $f\in\mathscr{A}_2$ and $0<A\leqslant\frac{1}{3}$. If*

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<A,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{2A}{1-A}.$$

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