Journal of
Classical
Analysis
Volume 27, Number 1 (2026), 107–118
doi:10.7153/jca-2026-27-08

A FOUR–DIMENSIONAL TAUBERIAN THEOREM FOR BLOCK–STRETCHED SEQUENCES

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Abstract. This paper presents a Tauberian theorem for four-dimensional RH-regular summability methods applied to double sequences and their block stretchings. The work extends Dawson's lemma and Maddox's theorem from single sequence theory to the double sequence setting, establishing necessary conditions for matrices that sum every block stretching. The main result characterizes block-finite matrices and provides a complete Tauberian-type theorem for block-stretched double sequences.

1. Introduction

In this paper, we present a Tauberian theorem for four-dimensional RH-regular summability methods applied to double sequences and their block stretchings. The study of convergence properties of double sequences and their transformations has been an active area of research since Pringsheim's seminal work [7] in 1900. In single sequence theory, Buck [1] showed that if a regular matrix sums every subsequence of a sequence, then that sequence must be convergent. Maddox [6] improved this by proving that if a non-Schur matrix sums every subsequence of a sequence, then that sequence is convergent.

We extend two key theorems from single sequence theory to the double sequence setting: Dawson's 1976 theorem [2] on stretching transformations, which showed that for divergent sequences, certain matrix properties must hold when summing all possible stretchings, and Maddox's theorem [6], which characterized matrices that sum every subsequence of a sequence.

Block stretching of double sequences was introduced and studied in [3, 4], establishing fundamental properties and initial convergence results. Building on this foundation, our paper extends classical Tauberian theory to handle block-stretched double sequences.

The structure of this paper is as follows: We begin with essential definitions and preliminary results, including Pringsheim's notion of convergence for double sequences and the concept of RH-regular four-dimensional matrices as characterized by Robison and Hamilton [5, 8]. Using the established framework of block stretching, we then

Mathematics subject classification (2020): 40B05, 40C05.
Keywords and phrases: Double sequences, four-dimensional matrices, Tauberian theorems, block stretching, RH-regular matrices, Pringsheim convergence.
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develop our theoretical tools through two-level block stretching processes and corresponding index mapping functions. The core of the paper presents and proves our main Tauberian theorem characterizing the behavior of RH-regular matrices that sum block-stretched double sequences.

2. Preliminaries and definitions

In this section, we begin by recalling some fundamental concepts about double sequences and their convergence properties.

DEFINITION 1. ([7]) A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P\text{-}\lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k,l > N$. We shall describe such an x more briefly as "P-convergent".

DEFINITION 2. A double sequence x is divergent in the Pringsheim sense (P-divergent) provided that x does not converge in the Pringsheim sense.

REMARK 1. For a double sequence $x = \left(x_{k,l}\right)$, P-divergence means it does not P-converge to any finite limit L. More precisely, this means

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \quad \text{such that} \quad \forall N \in \mathbb{N}, \exists k,l > N \quad \text{with} \quad \left|x_{k,l} - L\right| \geqslant \varepsilon.$$

A key consequence is that if x is P-divergent, then in particular it cannot converge to zero. Thus, there exists some $\varepsilon > 0$ such that

$$\forall N \in \mathbb{N}, \exists k,l > N \quad \text{with} \quad \left|x_{k,l}\right| \geqslant \varepsilon.$$

This condition ensures the existence of infinitely many terms satisfying $\left|x_{k,l}\right| \geqslant \varepsilon$. Moreover, these terms can be arranged into a subsequence $\{x_{i_p,j_q}\}_{p,q=1,1}^{\infty,\infty}$ where both indices i_p and j_q form strictly increasing sequences. Such a subsequence can be constructed inductively:

(i) First, choose (i_1, j_1) such that $\left|x_{i_1,j_1}\right| \geqslant \varepsilon$.

(ii) Given (i_k, j_k), choose (i_{k+1}, j_{k+1}) with $i_{k+1} > i_k$, $j_{k+1} > j_k$ and $\left|x_{i_{k+1},j_{k+1}}\right| \geqslant \varepsilon$.

The existence of such terms at each step is guaranteed by the P-divergence property.

Having defined the basic sequence properties, we now introduce matrix methods.

DEFINITION 3. Let A denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn-th term of Ax is as follows

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

DEFINITION 4. ([8]) The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

THEOREM 1. ([5, 8]) *The four-dimensional matrix A is RH-regular if and only if*

RH_1 : $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2 : $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1;$

RH_3 : $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4 : $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5 : $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*

RH_6 : *there exist finite positive integers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| \leqslant A.$

3. Block stretching and index mappings

In this section, we develop the essential tools for our main results. We first recall the concept of stretching by blocks for double sequences and then introduce the corresponding index mapping functions that allow us to track how elements transform under block stretching operations. This machinery will be crucial for establishing our Tauberian theorem in the subsequent sections.

3.1. Stretching by blocks

We begin with the fundamental concept of stretching a double sequence by blocks, which was introduced and studied in [3, 4].

DEFINITION 5. (Block stretching) Let $x = (x_{i,j})$ be a double sequence and $\{a_i\}_{i=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ be sequences of positive integers. Define the cumulative indices:

$$\begin{aligned} R_i &= a_1 + a_2 + \cdots + a_i && (R_0 = 0) \\ S_j &= b_1 + b_2 + \cdots + b_j && (S_0 = 0). \end{aligned}$$

The stretched double sequence $y = (y_{n,k})$ induced by $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ is defined as:

$$y_{n,k} = x_{i,j} \quad \text{if} \quad R_{i-1} < n \leqslant R_i \quad \text{and} \quad S_{j-1} < k \leqslant S_j.$$

In other words, each entry $x_{i,j}$ is expanded into a block of size $a_i \times b_j$ in the stretched sequence, with all entries in this block having the value $x_{i,j}$.

3.2. Two-level block stretching process

Building on the basic stretching operation, we now define a two-level stretching process. Let $x=(x_{i,j})$ be a double sequence. We define two sequential levels of block stretching:

First-level stretching: The first transformation produces an intermediate double sequence $z=(z_{k,l})$ using two sequences of positive integers $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$. Define cumulative indices

$$R_i=\sum_{k=1}^{i} a_k \quad (R_0=0) \quad \text{and} \quad S_j=\sum_{k=1}^{j} b_k \quad (S_0=0).$$

The stretched sequence $z=(z_{k,l})$ is defined by

$$z_{k,l}=x_{i,j} \quad \text{if} \quad R_{i-1}<k\leqslant R_i \quad \text{and} \quad S_{j-1}<l\leqslant S_j.$$

Second-level stretching: The second transformation maps $z=(z_{k,l})$ to the final double sequence $y=(y_{p,q})$ using sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$. Define cumulative indices

$$U_i=\sum_{k=1}^{i} u_k \quad (U_0=0) \quad \text{and} \quad V_j=\sum_{k=1}^{j} v_k \quad (V_0=0).$$

The resulting sequence $y=(y_{p,q})$ is defined by

$$y_{p,q}=z_{k,l} \text{ if } U_{k-1}<p\leqslant U_k \quad \text{and} \quad V_{l-1}<q\leqslant V_l.$$

3.3. Index mapping functions

To track the transformation of indices from $y=(y_{p,q})$ back to $x=(x_{i,j})$, we define two mapping functions:

DEFINITION 6. (Row mapping) The function $\phi:\mathbb{N}\to\mathbb{N}$ maps a row index p in y to its corresponding row index in x:

$$\phi(p)=i' \quad \text{where} \quad \exists i\in\mathbb{N}: U_{i-1}<p\leqslant U_i \quad \text{and} \quad R_{i'-1}<i\leqslant R_{i'}.$$

DEFINITION 7. (Column mapping) The function $\psi:\mathbb{N}\to\mathbb{N}$ maps a column index q in y to its corresponding column index in x:

$$\psi(q)=j' \quad \text{where} \quad \exists j\in\mathbb{N}: V_{j-1}<q\leqslant V_j \quad \text{and} \quad S_{j'-1}<j\leqslant S_{j'}.$$

3.4. Block correspondence sets

For analyzing the structure of stretched blocks, we define sets that collect indices mapping to specific entries in x:

DEFINITION 8. (Row index set) For each $i' \in \mathbb{N}$, define

$$K(i') = \bigcup_{\{i: R_{i'-1} < i \leqslant R_{i'}\}} \{p \in \mathbb{N} : U_{i-1} < p \leqslant U_i\}.$$

This set contains all row indices in y corresponding to row i' in x.

DEFINITION 9. (Column index set) For each $j' \in \mathbb{N}$, define

$$L(j') = \bigcup_{\{j: S_{j'-1} < j \leqslant S_{j'}\}} \{q \in \mathbb{N} : V_{j-1} < q \leqslant V_j\}.$$

This set contains all column indices in y corresponding to column j' in x.

4. Extensions of Dawson's lemma to four-dimensional matrices and stretching by blocks

Our main results extend Dawson's classical lemma on matrix summability of stretched sequences to the setting of four-dimensional matrices and block stretching of double sequences. After establishing some preliminary definitions about block-finite matrices, we prove a key lemma about block stretching transformations that leads to our main theorem.

DEFINITION 10. (Block-finite matrix) A four-dimensional matrix $B = \left(b_{m,n,k,l}\right)$ is said to be block-finite if, for each pair of indices (m,n), there exist finite positive integers $K_{m,n}$ and $L_{m,n}$ such that

$$b_{m,n,k,l} = 0 \quad \text{for all} \quad k > K_{m,n} \quad \text{and} \quad l > L_{m,n}.$$

In other words, for each (m,n), there are only finitely many non-zero entries $b_{m,n,k,l}$.

LEMMA 2. (Extended Dawson) *Let $x = (x_{i,j})$ be a double sequence and $A = \left(a_{m,n,k,l}\right)$ be an RH-regular four-dimensional matrix. Then there exists a block stretching z of x and a four-dimensional block-finite matrix B such that if y is any block stretching of z, then Ay and By either both P-converge or both P-diverge.*

Proof. For each $i, j \in \mathbb{N}$, define $M_{i,j}$ as

$$M_{i,j} = \sum_{k=1}^{i} \sum_{l=1}^{j} |x_{k,l}| + 1.$$

This definition ensures $M_{i,j} > |x_{i,j}|$ for all i,j. Construct two sequences of positive integers $\{n_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$ inductively. Let $n_0 = m_0 = 1$, and define

$$R_i = \sum_{k=0}^{i} n_k \quad \text{with} \quad R_0 = 0 \quad \text{and} \quad S_j = \sum_{k=0}^{j} m_k \quad \text{with} \quad S_0 = 0.$$

For $i = j = 1$, by condition RH_5, there exists $N_1 \in \mathbb{N}$ such that for all $r_1^{(1)}, s_1^{(1)}, r_2^{(1)}, s_2^{(1)} \in \mathbb{N}$ with $r_1^{(1)}, s_1^{(1)}, r_2^{(1)}, s_2^{(1)} \geqslant N_1$ such that

$$\sum_{k=r_1^{(1)}}^{s_1^{(1)}} \sum_{l=r_2^{(1)}}^{s_2^{(1)}} |a_{1,1,k,l}| < \frac{1}{2^2 M_{2,2}}.$$

Define $n_1 = m_1 = N_1$. Thus, $R_1 = S_1 = N_1$. Assume that for some $p \geqslant 1$, we have defined $n_1, \ldots, n_p$ and $m_1, \ldots, m_p$, and consequently $R_1, \ldots, R_p$ and $S_1, \ldots, S_p$. For the $(p+1)^{\text{th}}$ step, by condition RH_5, there exists $N_{p+1} \in \mathbb{N}$ such that for all $r_1^{(p+1)}, s_1^{(p+1)}, r_2^{(p+1)}, s_2^{(p+1)} \in \mathbb{N}$ satisfying

$$\begin{aligned} R_p &< r_1^{(p+1)} \leqslant s_1^{(p+1)} \leqslant R_p + N_{p+1}, \\ S_p &< r_2^{(p+1)} \leqslant s_2^{(p+1)} \leqslant S_p + N_{p+1}, \end{aligned}$$

and for each $m,n = 1,2,\ldots,p+1$, we have

$$\sum_{k=r_1^{(p+1)}}^{s_1^{(p+1)}} \sum_{l=r_2^{(p+1)}}^{s_2^{(p+1)}} |a_{m,n,k,l}| < \frac{1}{2^{2(p+1)} M_{p+2,p+2}}.$$

Define $n_{p+1} = m_{p+1} = N_{p+1}$. Thus, $R_{p+1} = R_p + N_{p+1}$ and $S_{p+1} = S_p + N_{p+1}$. This construction ensures that for all $i,j \geqslant 1$, and for all $r_1^{(i)}, s_1^{(i)}, r_2^{(j)}, s_2^{(j)} \in \mathbb{N}$ satisfying $R_{i-1} < r_1^{(i)} \leqslant s_1^{(i)} \leqslant R_i$ and $S_{j-1} < r_2^{(j)} \leqslant s_2^{(j)} \leqslant S_j$, and for each $m,n \leqslant \max(i,j)$,

$$\sum_{k=r_1^{(i)}}^{s_1^{(i)}} \sum_{l=r_2^{(j)}}^{s_2^{(j)}} |a_{m,n,k,l}| < \frac{1}{2^{i+j} M_{i+1,j+1}}.$$

For each $p \geqslant 1$, by condition RH_5, choose N_p such that for all $m,n \leqslant p$:

$$\sum_{k=R_{p-1}+1}^{R_{p-1}+N_p} \sum_{l=S_{p-1}+1}^{S_{p-1}+N_p} |a_{m,n,k,l}| < \frac{1}{2^{2p} M_{p+1,p+1}}.$$

Set $n_p = m_p = N_p$, and update

$$R_p = R_{p-1} + n_p, \quad S_p = S_{p-1} + m_p.$$

Construct $z = (z_{k,l})$ by block stretching of x induced by $\{n_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$:

$$z_{k,l} = x_{i,j} \quad \text{if} \quad R_{i-1} < k \leqslant R_i \quad \text{and} \quad S_{j-1} < l \leqslant S_j.$$

We define the block-finite matrix $B = \left(b_{m,n,k,l}\right)$ to match A strictly within the indices $k < R_m$ and $l < S_n$, and zero elsewhere,

$$b_{m,n,k,l} = \begin{cases} a_{m,n,k,l}, & \text{if} \quad k < R_m \quad \text{and} \quad l < S_n; \\ 0, & \text{otherwise.} \end{cases}$$

To establish the relationship between A and the block-finite matrix B, we examine the difference between their corresponding elements. Consider

$$\begin{aligned}
&\left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l} - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{m,n,k,l} \right| \text{ (by definition of } b_{m,n,k,l}) \\
&= \left| \left(\sum_{k=1}^{R_m - 1} \sum_{l=S_n}^{\infty} a_{m,n,k,l} \right) + \left(\sum_{k=R_m}^{\infty} \sum_{l=1}^{S_n - 1} a_{m,n,k,l} \right) + \left(\sum_{k=R_m}^{\infty} \sum_{l=S_n}^{\infty} a_{m,n,k,l} \right) \right| \\
&\leqslant \left(\sum_{i=1}^{m-1} \sum_{j=n}^{\infty} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right) \\
&\quad + \left(\sum_{i=m}^{\infty} \sum_{j=1}^{n-1} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right) \\
&\quad + \left(\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right).
\end{aligned}$$

Recall from the construction of the sequences $\{R_i\}$ and $\{S_j\}$ that for all $i \geqslant m$ and $j \geqslant n$:

$$\sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} |a_{m,n,k,l}| < \frac{1}{2^{i+j} M_{i+1,j+1}}.$$

Therefore,

$$\begin{aligned}
&\left(\sum_{i=1}^{m-1} \sum_{j=n}^{\infty} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right) + \left(\sum_{i=m}^{\infty} \sum_{j=1}^{n-1} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right) \\
&\quad + \left(\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} \left| \sum_{k=R_{i-1}+1}^{R_i} \sum_{l=S_{j-1}+1}^{S_j} a_{m,n,k,l} \right| \right) \\
&< \left(\sum_{i=1}^{m-1} \sum_{j=n}^{\infty} 2^{-(i+j)} \right) + \left(\sum_{i=m}^{\infty} \sum_{j=1}^{n-1} 2^{-(i+j)} \right) + \left(\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} 2^{-(i+j)} \right) \\
&= \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m+n-2}} \right) + \left(\frac{1}{2^{m-1}} - \frac{1}{2^{m+n-2}} \right) + \frac{1}{2^{m+n-2}} = \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}} - \frac{1}{2^{m+n-2}}.
\end{aligned}$$

This bound approaches 0 as $m,n\to\infty$, establishing the relationship between A and B.

Let y be any block stretching of z induced by the two sequences of positive integers $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$. Define

$$U_i=\sum_{t=1}^{i}u_t \quad \text{with} \quad U_0=0 \quad \text{and} \quad V_j=\sum_{t=1}^{j}v_t \quad \text{with} \quad V_0=0.$$

Entries of y satisfy

$$y_{p,q}=z_{i,j}=x_{\phi(p),\psi(q)},$$

where $\phi(p)$ and $\psi(q)$ map indices of y to x through intermediate blocks in z. Now

$$\begin{aligned}|Ay-By|&=\left|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}a_{m,n,k,l}y_{k,l}-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}b_{m,n,k,l}y_{k,l}\right|\\&\leqslant\underbrace{\sum_{k\geqslant R_m}\sum_{l<S_n}|a_{m,n,k,l}y_{k,l}|}_{\mathrm{I}}+\underbrace{\sum_{k\geqslant R_m}\sum_{l\geqslant S_n}|a_{m,n,k,l}y_{k,l}|}_{\mathrm{II}}+\underbrace{\sum_{k<R_m}\sum_{l\geqslant S_n}|a_{m,n,k,l}y_{k,l}|}_{\mathrm{III}}.\end{aligned}$$

Term I: First, we establish the index mapping: when $k\geqslant R_m$, this implies $i'\geqslant\phi(R_m)$, and when $l<S_n$, this implies $j'<\psi(S_n)$. Using the summability condition from RH_5, which ensures $\sum_{k,l=1.1}^{\infty,\infty}\left|a_{m,n,k,l}\right|$ is P-convergent and thus allows us to bound partial sums over blocks $K(i')\times L(j')$, together with the uniform boundedness from RH_6, which provides that $\sum_{k,l>K_2}\left|a_{m,n,k,l}\right|\leqslant K_1$ for some finite $K_1,K_2>0$, ensuring our bounds remain valid as indices increase, we can conclude that for indices satisfying $i'\geqslant\phi(R_m)$ and $j'<\psi(S_n)$, our inductive construction guarantees

$$\sum_{k\in K(i')}\sum_{l\in L(j')}|a_{m,n,k,l}|<\frac{1}{2^{i'+j'}M_{i'+1,j'+1}}.$$

Moreover, since $|y_{k,l}|=|x_{i',j'}|<M_{i',j'}$, we can bound Term I:

$$\mathrm{I}\leqslant\sum_{i'\geqslant\phi(R_m)}\sum_{j'=1}^{\psi(S_n)-1}\frac{M_{i',j'}}{2^{i'+j'}M_{i'+1,j'+1}}.$$

Observing that $\dfrac{M_{i',j'}}{M_{i'+1,j'+1}}\leqslant 1$, we can simplify

$$\mathrm{I}\leqslant\sum_{i'\geqslant\phi(R_m)}\sum_{j'=1}^{\infty}\frac{1}{2^{i'+j'}}=\sum_{i'\geqslant\phi(R_m)}\frac{1}{2^{i'}}=\frac{1}{2^{\phi(R_m)-1}}.$$

Term II: The index mapping shows that $k\geqslant R_m$ implies $i'\geqslant\phi(R_m)$, and $l\geqslant S_n$ implies $j'\geqslant\psi(S_n)$. Using these bounds and applying geometric series summation

$$\mathrm{II}\leqslant\sum_{i'\geqslant\phi(R_m)}\sum_{j'\geqslant\psi(S_n)}\frac{1}{2^{i'+j'}}=\frac{1}{2^{\phi(R_m)+\psi(S_n)-2}}.$$

Term III: Here, $k < R_m$ implies $i' < \phi(R_m)$, and $l \geqslant S_n$ implies $j' \geqslant \psi(S_n)$. Applying geometric series summation again

$$\mathrm{III} \leqslant \sum_{i'=1}^{\phi(R_m)-1} \sum_{j' \geqslant \psi(S_n)} \frac{1}{2^{i'+j'}} = \frac{1}{2^{\psi(S_n)-1}} \left(1 - \frac{1}{2^{\phi(R_m)-1}}\right).$$

Combining all three terms

$$|Ay - By| \leqslant \frac{1}{2^{\phi(R_m)-1}} + \frac{1}{2^{\psi(S_n)-1}}.$$

As $m,n \to \infty$, we have $\phi(R_m) \to \infty$ and $\psi(S_n) \to \infty$, which implies $|Ay - By| \to 0$.

Since we have established $|Ay - By| \to 0$, this ensures that Ay and By must share the same P-convergence behavior. Moreover, the block-finite property of matrix B guarantees that By convergence depends only on finite sums, which A preserves due to its RH-regularity. Therefore, we can conclude that Ay P-converges if and only if By does. $\square$

5. Analogous to Maddox's theorem

Having established our extension of Dawson's lemma, we now present the main theorem of this paper, which provides a double sequence analogue of Maddox's classical theorem on summing subsequences. This characterization identifies necessary conditions for four-dimensional matrices that sum every block stretching of a double sequence.

THEOREM 3. (Maddox-type theorem for block stretching) *Let $x = (x_{i,j})$ be a P-divergent double sequence and $A = \left(a_{m,n,k,l}\right)$ be a four-dimensional matrix which sums every block stretching of x. Then there exist positive integers N and M such that:*

i) $P\text{-}\lim_{m,n} a_{m,n,k,l} = c_{k,l}$ *for each $k > N$ and $l > M$,*

ii) $P\text{-}\lim_{m,n} \sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0,$

iii) $\sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} |c_{k,l}|$ *is P-convergent.*

Proof. Assume condition (i) fails. Then for any choice of finite positive integers N, M, there exists $k > N$, $l > M$ such that the double sequence $(a_{m,n,k,l})_{m,n}$ does not P-converge to $c_{k,l}$. Then there exists $\delta > 0$ and increasing sequences $\{m_p\}_{p=1}^{\infty}$ and $\{n_q\}_{q=1}^{\infty}$ of positive integers such that

$$|a_{m_p,n_q,k,l} - c_{k,l}| \geqslant \delta \quad \text{for all} \quad p,q.$$

Since x is P-divergent, there exists $\varepsilon > 0$ and sequences $\{i_r\}_{r=1}^{\infty}$ and $\{j_s\}_{s=1}^{\infty}$ of positive integers where $|x_{i_r,j_s}| \geqslant \varepsilon$ for all r,s. We construct a fixed block stretching y of x using sequences of positive integers $\{u_i\}$ and $\{v_j\}$ as follows:

(1) Choose u_{i_r} and v_{j_s} so x_{i_r,j_s} maps to position (k,l) in y.

(2) For all other indices i and j, set $u_i = v_j = 1$.

(3) The stretched sequence y at position (k,l) equals x_{i_r,j_s} and equals 0 at all other positions.

Then for all p,q, we have

$$\left|(Ay)_{m_p,n_q}\right| \geqslant \left|a_{m_p,n_q,k,l}\right| \cdot \left|y_{k,l}\right| \geqslant \delta\varepsilon.$$

Therefore $(Ay)_{m,n}$ cannot P-converge, contradicting that A sums every block stretching of x.

Assume condition (ii) fails. Then there exists $\varepsilon > 0$ and sequences $\{m_p\}_{p=1}^{\infty}$, $\{n_q\}_{q=1}^{\infty}$ of positive integers where

$$\sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} \left|a_{m_p,n_q,k,l} - c_{k,l}\right| \geqslant \varepsilon \quad \text{for all} \quad p,q.$$

Since x is P-divergent, there exists $\delta > 0$ and sequences $\{i_r\}_{r=1}^{\infty}$, $\{j_s\}_{s=1}^{\infty}$ of positive integers where $\left|x_{i_r,j_s}\right| \geqslant \delta$ for all r,s. We construct a block stretching y of x as follows:

(1) Choose u_{i_r} and v_{j_s} so x_{i_r,j_s} maps to positions (k,l) where $k > N$ and $l > M$.

(2) For all other indices i and j, set $u_i = v_j = 1$.

(3) The stretched sequence y equals x_{i_r,j_s} at these chosen positions and equals 0 elsewhere.

Then for all p,q, we have

$$\left|(Ay)_{m_p,n_q}\right| \geqslant \sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} \left|a_{m_p,n_q,k,l} - c_{k,l}\right| \cdot \left|y_{k,l}\right| \geqslant \delta\varepsilon.$$

This shows $(Ay)_{m,n}$ does not P-converge, contradicting our assumption that A sums every block stretching of x.

Suppose, contrary to our assertion in (iii), that $\sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} \left|c_{k,l}\right|$ is not P-convergent. Since x is P-divergent, there exists $\varepsilon > 0$ and sequences $\{i_r\}_{r=1}^{\infty}$ and $\{j_s\}_{s=1}^{\infty}$ of positive integers such that $\left|x_{i_r,j_s}\right| \geqslant \varepsilon$ for all $r,s \in \mathbb{N}$. We construct a block stretching y of x that produces a contradiction to the hypothesis that A sums every block stretching of x. Let $\{u_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ be sequences of positive integers defined as follows:

For each pair (i_r, j_s), choose u_{i_r} and v_{j_s} such that x_{i_r,j_s} maps to position $(k_{r,s}, l_{r,s})$ in y where $k_{r,s} > N$ and $l_{r,s} > M$. For all other indices i and j, set $u_i = v_j = 1$. This construction ensures:

(1) $y_{k_{r,s},l_{r,s}} = x_{i_r,j_s}$ with $\left|y_{k_{r,s},l_{r,s}}\right| \geqslant \varepsilon$ for all $r,s \in \mathbb{N}$.

(2) $y_{k,l} = 0$ for all (k,l) not corresponding to any $(k_{r,s}, l_{r,s})$.

By condition (ii) of the theorem

$$P\text{-}\lim_{m,n} \left| \sum_{k=N+1}^{\infty} \sum_{l=M+1}^{\infty} (a_{m,n,k,l} - c_{k,l}) y_{k,l} \right| = 0.$$

Therefore, $(Ay)_{m,n}$ converges in Pringsheim's sense if and only if $\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} c_{k,l}y_{k,l}$ converges. However, by our construction

$$\left|\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} c_{k,l}y_{k,l}\right| = \left|\sum_{r,s=1}^{\infty} c_{k_{r,s},l_{r,s}} x_{i_r,j_s}\right| \geqslant \varepsilon \sum_{r,s=1}^{\infty} |c_{k_{r,s},l_{r,s}}|.$$

Since $\{(k_{r,s}, l_{r,s})\}$ forms a subsequence of indices with $k_{r,s} > N$ and $l_{r,s} > M$, and we assumed $\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} |c_{k,l}|$ is not P-convergent, the series $\sum_{r,s=1}^{\infty} |c_{k_{r,s},l_{r,s}}|$ must also diverge. Consequently, $\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} c_{k,l}y_{k,l}$ cannot converge, implying $(Ay)_{m,n}$ diverges. This contradicts our hypothesis that A sums every block stretching of x. Therefore, $\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} |c_{k,l}|$ must be P-convergent. □

Our result in Theorem 3 can be reformulated as a classical Tauberian theorem:

COROLLARY 4. *Let $\mathscr{D}$ be the set of all four-dimensional matrices $A = \left(a_{m,n,k,l}\right)$ for which there exist positive integers N and M such that:*

i) $P\text{-}\lim_{m,n} a_{m,n,k,l} = c_{k,l}$ *exists for each $k > N$ and $l > M$,*

ii) $P\text{-}\lim_{m,n} \sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} |a_{m,n,k,l} - c_{k,l}| = 0,$

iii) $\sum_{k=N+1}^{\infty}\sum_{l=M+1}^{\infty} |c_{k,l}|$ *is P-convergent.*

If $A \notin \mathscr{D}$ sums every block stretching of a double sequence $x = (x_{i,j})$, then x is P-convergent.

Proof. We proceed by contradiction. Suppose x is P-divergent. Then by Theorem 3, any matrix that sums every block stretching of x must satisfy conditions (i)–(iii), and therefore must belong to $\mathscr{D}$. But we are given that A sums every block stretching of x and $A \notin \mathscr{D}$. This is a contradiction. Therefore, our assumption that x is P-divergent must be false. Since every double sequence is either P-convergent or P-divergent, we conclude that x must be P-convergent. □

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(Received March 1, 2025)

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