

ON A CLASS OF QUOTIENT FUNCTIONS

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Abstract. In this manuscript, we consider a class of quotient functions having fixed second coefficient described on open unit disk. We obtain the growth and distortion theorems for such class. Moreover, we examine the radius of lemniscate starlikeness, lune starlikeness, parabolic starlikeness, sine starlikeness, exponential starlikeness and several other radius estimates for such class.

1. Preliminary literature

Bieberbach theorem gives the bound on second coefficient in the Maclaurin series expansion of the univalent functions which has a significant part in univalent function theory. For that reason, functions with fixed second coefficients have attracted many authors over the decades. Initially, Gronwall [8] investigated a subclass of one-to-one analytic functions with fixed second coefficient. The author determined the growth and distortion theorems for such classes. In [15], Lee *et al.* discussed some useful applications on differential subordination for such functions. Recently, Kumar *et al.* [14] computed the sharp bound on initial coefficients of Ma-Minda type univalent functions with fixed second coefficient. Several authors have worked in the direction of growth and distortion estimates, subordination and radius problems for functions with fixed second coefficient [12, 22]. The class of the analytic functions f which are normalized by the condition $f(0) = f'(0) - 1 = 0$ and defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{A}$ and the class of all functions $f \in \mathcal{A}$ of the form

$$f(z) = z + nbz^2 + \cdots \quad (z \in \mathbb{D})$$

is denoted by $\mathcal{A}_{nb}$ where b is the fixed number satisfying $0 \leq b \leq 1$, $n \in \mathbb{N}$. Let $\mathcal{A}_b := \mathcal{A}_{1b}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which contains all univalent functions. Let $\mathcal{P}$ be the class of analytic functions having positive real part. Let $\mathcal{S}^*$ and $\mathcal{K}$ represent the subclasses of $\mathcal{S}$ containing starlike and convex functions, respectively. Analytically $f \in \mathcal{S}^*$ if $zf'(z)/f(z) \in \mathcal{P}$ and $f \in \mathcal{K}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$ for all $z \in \mathbb{D}$. The function $f \in \mathcal{A}$ is acknowledged as close to convex if there exists a function $g \in \mathcal{K}$ such that $f'(z)/g'(z)$ has positive real part for all $z \in \mathbb{D}$. These functions were studied initially by Kaplan [11]. Let P be a property and $\mathcal{M}$ be a set of functions. Then the real number $R_P(\mathcal{M}) = \sup\{r > 0 : f \in \mathcal{M} \text{ has the property } P \text{ in the disk } \mathbb{D}_r\}$

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is radius of the property P for the set $\mathcal{M}$. For the class $\mathcal{S}$, the sharp radius of convexity $R_{\mathcal{K}}(\mathcal{S})$ is $2 - \sqrt{3}$ due to Nevanlinna and radius of starlikeness $\mathcal{R}_{\mathcal{S}^*}(\mathcal{S}) = \tanh \frac{\pi}{4} \approx 0.65579$ due to Grunsky. In [3], Ali *et al.* discussed radius problem for the functions f fulfilling one of the conditions (i) $\operatorname{Re}(f(z)/g(z)) > 0$ (ii) $|(f(z) - g(z))/g(z)| < 1$ under some restrictions on g . These functions are closely associated with close-to-convex functions. Recently, Sebastian *et al.* [25] computed certain best possible radius estimates for the classes of functions $f \in \mathcal{A}$ fulfilling one of the following conditions: (i) $\operatorname{Re} f(z)/g(z) > 0$ where $(1 + (1/z))g(z)$ has positive real part (ii) $|(f(z) - g(z))/g(z)| > 0$ where $(1 + (1/z))g(z)$ has positive real part where $g \in \mathcal{A}$, or (iii) $(1 + (1/z))f(z)$ has positive real part. Motivated by the aforementioned works, we consider a class of quotient functions having fixed second coefficients which is defined as:

The functions f and g whose Taylor series expansions are given by

$$f(z) = z + a_2 z^2 + \cdots \quad \text{and} \quad g(z) = z + g_2 z^2 + \cdots$$

such that $(1+z)g(z)/z$ and $f(z)/g(z)$ are in $\mathcal{P}$ for all $z \in \mathbb{D}$. Then

$$\frac{1+z}{z}g(z) = 1 + (1+g_2)z + \cdots \in \mathcal{P}.$$

Thus, we get $|1+g_2| \leq 2$, that is $-3 \leq g_2 \leq 1$. Also, $f(z)/g(z) = 1 + (a_2 - g_2)z + \cdots \in \mathcal{P}$ which gives $|a_2 - g_2| \leq 2$, that is $-5 \leq a_2 \leq 3$. Now, we consider the functions f and g with fixed second coefficient and having following series expansion

$$f(z) = z + 5bz^2 + \cdots \quad \text{and} \quad g(z) = z + 3cz^2 + \cdots$$

where $|b| \leq 1$ and $|c| \leq 1$.

DEFINITION 1. For $|b| \leq 1$ and $|c| \leq 1$, let $\mathcal{F}_{b,c}$ be defined as

$$\mathcal{F}_{b,c} := \left\{ f \in \mathcal{A}_{5b} : \frac{f(z)}{g(z)} \in \mathcal{P}, \frac{1+z}{z}g(z) \in \mathcal{P}, g \in \mathcal{A}_{3c}, z \in \mathbb{D} \right\}.$$

Consider the functions f_0 and g_0 which are defined as

$$f_0(z) = \frac{z(1-z)^2(1+z)}{(1-(5b-3c)z+z^2)(1-(3c+1)z+z^2)} \quad \text{and} \quad (1)$$

$$g_0(z) = \frac{z(1-z)}{(1-(3c+1)z+z^2)}. \quad (2)$$

For the function f_0 with the choice of g_0 given by (2), we have

$$\frac{f_0(z)}{g_0(z)} = \frac{1-w_1(z)}{1+w_1(z)} \quad \text{and} \quad \frac{1+z}{z}g_0(z) = \frac{1-w_2(z)}{1+w_2(z)}$$

where

$$w_1(z) = \frac{z(z - (5b-3c)/2)}{(1 - (5b-3c)z/2)} \quad \text{with} \quad |5b-3c| \leq 2$$

and

$$w_2(z) = \frac{z(z - (3c + 1)/2)}{(1 - (3c + 1)z/2)} \quad \text{with} \quad |3c + 1| \leq 2$$

are analytic functions satisfying the conditions of Schwarz lemma in $\mathbb{D}$ and thus $f_0(z)/g_0(z) \in \mathcal{P}$ and $((1+z)g_0(z))/z \in \mathcal{P}$ in $\mathbb{D}$. Hence the function $f_0 \in \mathcal{F}_{b,c}$. Since the function f_0 given by (1) belong to the class $\mathcal{F}_{b,c}$, the class $\mathcal{F}_{b,c}$ is non empty.

In 1992, Ma and Minda [16] studied the covering, growth and distortion theorem for the unified class $\mathcal{S}^*(\varphi)$ of starlike functions which contains the functions f satisfying $zf'(z)/f(z) \prec \varphi(z)$, $z \in \mathbb{D}$ where φ is the analytic function satisfying the inequality $\operatorname{Re}(\varphi(z)) > 0$ and $\prec$ shows subordination. The function φ plays an important role in the study of theory of second order differential subordination [5]. The class $\mathcal{S}^*(\varphi)$ contains various well known subclasses of starlike functions. For instance, $S^*((1+Az)/(1+Bz)) := S^*[A, B]$; $(-1 \leq B < A \leq 1)$. The class $S^*[A, B]$ is known as the class of Janowski starlike functions [10]. For $0 \leq \alpha < 1$, the class $S^*[A = 1 - 2\alpha, B = -1] := S^*(\alpha)$ accommodate starlike functions of order α [23]. The class $S_p := S^*(1 + (2/\pi^2)(\log((1 - \sqrt{z})/(1 + \sqrt{z})))^2)$ contains parabolic starlike functions which are associated with region $\{w \in \mathbb{C} : |w - 1| < \operatorname{Re}(w)\}$ [24]. The class $S^*(\sqrt{1+z}) := S_L^*$ contains lemniscate starlike functions [28]. The class $S^*(e^z) := S_e^*$ is associated with the region $\{w \in \mathbb{C} : |\log w| < 1\}$ [20]. The functions $f \in S_e^*$ are known as exponential starlike functions. The class $S^*(1 + (4/3)z + (2/3)z^2) := S_c^*$ contains the functions f which are associated with cardioid domain [27]. The class $S^*(z + \sqrt{1+z^2}) := S_{\mathcal{C}}^*$ contains lune starlike functions which are related to region $\{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}$ [7]. The class $S^*(1 + \sin z) := S_{\sin}^*$ is related to the Sine function [6]. The geometric properties of the class $S^*(1 + (z(k+z))/(k(k-z))) := S_R^*$, where $k = \sqrt{2} + 1$ were studied in [13]. The class $S^*(1 + z - (z^3/3)) := S_{N_e}^*$ is associated with two cusped kidney shaped region $\Omega_{N_e} = \{u + ib : ((u-1)^2 + v^2 - 4/9)^3 - 4v^2/3 < 0\}$ [30].

In 1970, Tepper [29] obtained sharp bounds on $|f(z)|$, $|f'(z)|$ and $|zf'(z)/f(z)|$ for the class of functions starlike of order α where $0 \leq \alpha \leq 1$. Recently, the authors [5] investigated the growth and distortion theorem for a unified class of normalized analytic functions. In [1], Ali *et al.* determined certain radius of starlikeness for the functions $f \in \mathcal{A}_b$. For more literature on radius problems, see [4, 9, 17, 19].

In this manuscript, we establish growth and disortion theorems for the class $\mathcal{F}_{b,c}$. Further, we determine radius estimates of the functions belonging to the class $\mathcal{F}_{b,c}$ to be in the subclasses $S^*(\alpha)$, S_L^* , S_p , S_e^* , S_c^* , $S_{\sin}^*$, $S_{\mathcal{C}}^*$, S_R^* and $S_{N_e}^*$ of starlike functions. We also show some connections of obtained radius estimates with the existing ones.

The class of analytic functions $p(z) = 1 + b_1z + b_2z^2 + \dots$ fulfilling $\operatorname{Re}(p(z)) > \alpha$ where $0 \leq \alpha < 1$, is denoted by $\mathcal{P}(\alpha)$. It is noted that $\mathcal{P}(0) = \mathcal{P}$. For $p \in \mathcal{P}(\alpha)$ [21, p. 170], we have $|b_1| \leq 2(1 - \alpha)$. Denote by $P_b(\alpha)$ the subclass of $\mathcal{P}(\alpha)$ which contains following type functions

$$p(z) = 1 + 2b(1 - \alpha)z + \dots, \quad |b| \leq 1$$

and let $\mathcal{P}_b := \mathcal{P}_b(0)$. In order to establish our main results, the following results are required:

LEMMA 1. [18, Lemma 2, p. 212] Let $|b| \leq 1$. If $p \in \mathcal{P}_b(0)$, then for $|z| = r < 1$,

$$|p(z)| \leq \frac{1 + b|z| + |z|^2}{1 - |z|^2}.$$

LEMMA 2. [18, Corollary 1, p. 213] If $p \in \mathcal{P}_b(\alpha)$, then for $|z| = r < 1$,

$$\operatorname{Re}(p(z)) \geq \frac{1 + b\alpha|z| - (1 - 2\alpha)|z|^2}{1 + b|z| + |z|^2}.$$

LEMMA 3. [18, Theorem 2, p. 213] Let $|b| \leq 1$ and $0 \leq \alpha < 1$. If $p \in \mathcal{P}_b(\alpha)$, then for $|z| = r < 1$,

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2} \frac{|b|r^2 + 2r + |b|}{(1 - 2\alpha)r^2 + 2(1 - \alpha)|b|r + 1}.$$

2. Growth and distortion Theorems

THEOREM 1. Let $d = |5b - 3c| \leq 2$ and $c' = |3c + 1| \leq 2$. Let $c' + d = t$ and $c'd = s$ and let $|z| = r < 1$. Then, for the function $f \in \mathcal{F}_{b,c}$, we have

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}.$$

The result is sharp.

Proof. Let $f \in \mathcal{F}_{b,c}$ and $g \in \mathcal{A}_{3c}$ such that

$$\frac{f(z)}{g(z)} \in \mathcal{P} \quad \text{and} \quad \frac{1+z}{z}g(z) \in \mathcal{P}, \quad (z \in \mathbb{D}). \quad (3)$$

The functions $p, h: \mathbb{D} \rightarrow \mathbb{C}$ are defined as

$$p(z) = \frac{1+z}{z}g(z) = 1 + (1+3c)z + \cdots \quad (4)$$

and

$$h(z) = \frac{f(z)}{g(z)} = 1 + (5b-3c)z + \cdots \quad (5)$$

In view of (3), (4) and (5), it is observed that the functions $p \in \mathcal{P}_{(3c+1)/2}$ and $h \in \mathcal{P}_{(5b-3c)/2}$. Thus, we have

$$f(z) = \frac{zp(z)h(z)}{1+z} \quad (6)$$

so that

$$\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)} + \frac{1}{1+z}. \quad (7)$$

The bilinear mapping $\frac{1}{1+z}$ maps $|z| \leq r$ onto the disk

$$\left| \frac{1}{1+z} - \frac{1}{1-r^2} \right| \leq \frac{r}{1-r^2}. \quad (8)$$

On taking $\alpha = 0$ in Lemma 3, we have

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{r}{1-r^2} \frac{c'r^2 + 4r + c'}{r^2 + c'r + 1} \quad \text{and} \quad \left| \frac{zh'(z)}{h(z)} \right| \leq \frac{r}{1-r^2} \frac{dr^2 + 4r + d}{r^2 + dr + 1}. \quad (9)$$

Using (7), (9) and (8), we get

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - \frac{1}{1-r^2} \right| &\leq \frac{r}{1-r^2} \left(\frac{c'r^2 + 4r + c'}{r^2 + c'r + 1} + \frac{dr^2 + 4r + d}{r^2 + dr + 1} + 1 \right) \\ &= \frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}. \end{aligned} \quad (10)$$

In view of (10) we get the desired inequality

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{1}{1-r^2} - \frac{r}{1-r^2} \left(\frac{c'r^2 + 4r + c'}{r^2 + c'r + 1} + \frac{dr^2 + 4r + d}{r^2 + dr + 1} + 1 \right) \quad (11)$$

$$= \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}. \quad (12)$$

Next, we show the sharpness of this inequality for the function f_0 given by (1). Let $m = -(5b-3c) \geq 0$, $n = -(3c+1) \geq 0$, $m+n=p$ and $mn=q$. Then

$$\frac{zf'_0(z)}{f_0(z)} = \frac{-(1+p)z^5 - (7+p+2q)z^4 - (2+5p+q)z^3 - (6+p+q)z^2 - z + 1}{(1-z^2)(z^2 + mz + 1)(z^2 + nz + 1)}. \quad (13)$$

At $z = r$, we have

$$\begin{aligned} \operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right) &= \frac{-(1+p)r^5 - (7+p+2q)r^4 - (2+5p+q)r^3 - (6+p+q)r^2 - r + 1}{(1-r^2)(r^2 + mr + 1)(r^2 + nr + 1)}. \end{aligned}$$

This proves the sharpness. $\square$

THEOREM 2. Let $d = |5b-3c| \leq 2$ and $c' = |3c+1| \leq 2$. Let $c' + d = t$ and $c'd = s$ and let $|z| = r < 1$. Then, for $f \in \mathcal{F}_{b,c}$, we have

$$\left| \frac{zf'(z)}{f(z)} \right| \leq \frac{(1+t)r^5 + (9+t+2s)r^4 + (2+7t+s)r^3 + (10+t+3s)r^2 + (1+2t)r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}$$

and

$$\left| \frac{zf'(z)}{f(z)} \right| \geq \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)}.$$

The result is sharp.

Proof. Let $f \in \mathcal{F}_{b,c}$ and $g \in \mathcal{A}_{3c}$. Consider

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} \right| &= \left| \frac{zf'(z)}{f(z)} - \frac{1}{1-r^2} + \frac{1}{1-r^2} \right| \\ &\leq \left| \frac{zf'(z)}{f(z)} - \frac{1}{1-r^2} \right| + \frac{1}{1-r^2} \\ &\leq \frac{r}{1-r^2} \left(\frac{c'r^2+4r+c'}{r^2+c'r+1} + \frac{dr^2+4r+d}{r^2+dr+1} + 1 \right) + \frac{1}{1-r^2} \\ &= \frac{(1+t)r^5 + (9+t+2s)r^4 + (2+7t+s)r^3 + (10+t+3s)r^2 + (1+2t)r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)}. \end{aligned} \quad (14)$$

Hence, we get the desired bound. Define the functions $f_0, g_0 : \mathbb{D} \rightarrow \mathbb{C}$ as

$$\begin{aligned} f_0(z) &= \frac{z(1+(5b-3c)z+z^2)(1+(3c+1)z+z^2)}{(1-z^2)^2(1+z)}, \\ g_0(z) &= \frac{z(1+(3c+1)z+z^2)}{(1-z^2)(1+z)}. \end{aligned} \quad (15)$$

For the function f_0 with the choice of g_0 given by (15), we have

$$\frac{f_0(z)}{g_0(z)} = \frac{1+w_1(z)}{1-w_1(z)} \quad \text{and} \quad \frac{1+z}{z}g_0(z) = \frac{1+w_2(z)}{1-w_2(z)}$$

where

$$w_1(z) = \frac{z(z+(5b-3c)/2)}{(1+(5b-3c)z/2)} \quad \text{with} \quad |5b-3c| \leq 2$$

and

$$w_2(z) = \frac{z(z+(3c+1)/2)}{(1+(3c+1)z/2)} \quad \text{with} \quad |3c+1| \leq 2$$

are analytic functions satisfying the conditions of Schwarz lemma in open unit disk $\mathbb{D}$ and thus $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ and $\operatorname{Re}((1+z)g_0(z)/z) > 0$ in $\mathbb{D}$. Hence the function $f_0 \in \mathcal{F}_{b,c}$. For $m = -(5b-3c) \geq 0$, $n = -(3c+1) \geq 0$, $p = m+n$ and $q = mn$, we have

$$\frac{zf'_0(z)}{f_0(z)} = \frac{-(1+p)z^5 + (9+p+2q)z^4 - (2+7p+q)z^3 + (10+p+3q)z^2 - (1+2p)z + 1}{(1-z^2)(1-mz+z^2)(1-nz+z^2)}.$$

At $z = -r$, we have

$$\frac{zf'_0(z)}{f_0(z)} = \frac{(1+p)r^5 + (9+p+2q)r^4 + (2+7p+q)r^3 + (10+p+3q)r^2 + (1+2p)r + 1}{(1-r^2)(1+mr+r^2)(1+nr+r^2)}.$$

Thus the result is sharp. Also, we have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} \right| &\geq \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \\ &\geq \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)}. \end{aligned}$$

The result is sharp for the function f_0 given by (1). $\square$

THEOREM 3. Let $d = |5b - 3c| \leq 2$ and $c' = |3c + 1| \leq 2$. Let $c' + d = t$ and $c'd = s$ and let $|z| = r < 1$. For $f \in \mathcal{F}_{b,c}$, we have

$$|f(z)| \leq \frac{r(1+c'r+r^2)(1+dr+r^2)}{(1-r^2)^2(1-r)}$$

and

$$|f(z)| \geq r \left(\frac{e^r}{1+r} \exp(F(r)) \right)$$

where

$$F(r) = - \int_0^r \frac{u}{1-u^2} \left(\frac{c'u^2+4u+c'}{u^2+c'u+1} + \frac{du^2+4u+d}{u^2+du+1} \right) du.$$

Proof. Using (6) and Lemma 1, we get

$$|f(z)| = \frac{|z||p(z)||h(z)|}{|1+z|} \leq \frac{r(1+c'r+r^2)(1+dr+r^2)}{(1-r^2)^2(1-r)}. \quad (16)$$

For the function f_0 given by (15) and $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$ and $z = -r$, we have

$$|f_0(z)| = \frac{r(1+mr+r^2)(1+nr+r^2)}{(1-r^2)^2(1-r)}.$$

If $z = re^{i\theta}$, then

$$\frac{\partial}{\partial r} \log \left| \frac{f(z)}{z} \right| = \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) - 1.$$

Using (11), we get

$$\begin{aligned} \frac{\partial}{\partial r} \log \left| \frac{f(z)}{z} \right| &\geq \frac{1}{1-r^2} - \frac{r}{1-r^2} \left(\frac{c'r^2+4r+c'}{r^2+c'r+1} + \frac{dr^2+4r+d}{r^2+dr+1} + 1 \right) - 1 \\ &= \frac{-r}{1+r} - \frac{r}{1-r^2} \left(\frac{c'r^2+4r+c'}{r^2+c'r+1} + \frac{dr^2+4r+d}{r^2+dr+1} \right) \\ &= 1 - \frac{1}{1+r} - \frac{r}{1-r^2} \left(\frac{c'r^2+4r+c'}{r^2+c'r+1} + \frac{dr^2+4r+d}{r^2+dr+1} \right). \end{aligned}$$

Integrating from $u = 0$ to $u = |z| = r < 1$, we get

$$\log \left| \frac{f(z)}{z} \right| \geq r - \log(1+r) - \int_0^r \frac{u}{1-u^2} \left(\frac{c'u^2 + 4u + c'}{u^2 + c'u + 1} + \frac{du^2 + 4u + d}{u^2 + du + 1} \right) du.$$

On taking exponents, we get

$$|f(z)| \geq r \left(\frac{e^r}{1+r} \exp(F(r)) \right) \quad (17)$$

where

$$F(r) = - \int_0^r \frac{u}{1-u^2} \left(\frac{c'u^2 + 4u + c'}{u^2 + c'u + 1} + \frac{du^2 + 4u + d}{u^2 + du + 1} \right) du. \quad \square$$

THEOREM 4. Let $d = |5b - 3c| \leq 2$ and $c' = |3c + 1| \leq 2$. Let $c' + d = t$ and $c'd = s$ and let $|z| = r < 1$. Then for the function $f \in \mathcal{F}_{b,c}$, we have

$$|f'(z)| \leq \frac{(1+t)r^5 + (9+t+2s)r^4 + (2+7t+s)r^3 + (10+t+3s)r^2 + (1+2t)r + 1}{(1-r)^4(1+r)^3}$$

and

$$|f'(z)| \geq r \left(\frac{e^r}{1+r} \exp(F(r))G(r) \right)$$

where

$$F(r) = - \int_0^r \frac{u}{1-u^2} \left(\frac{c'u^2 + 4u + c'}{u^2 + c'u + 1} + \frac{du^2 + 4u + d}{u^2 + du + 1} \right) du$$

and

$$G(r) = \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}.$$

Proof. Consider the following arrangement of the function $f \in \mathcal{F}_{b,c}$

$$|f'(z)| = \left| \frac{zf'(z)}{f(z)} \right| \left| \frac{f(z)}{z} \right| \quad (18)$$

Using inequalities defined by (14) and (16), we get the desired upper bound on $|f'(z)|$. For the function f_0 given by (15) and $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$ and $z = -r$, we have

$$|f'(z)| = \frac{(1+p)r^5 + (9+p+2q)r^4 + (2+7p+q)r^3 + (10+p+3q)r^2 + (1+2p)r + 1}{(1-r)^4(1+r)^3}$$

which proves the sharpness.

Using (12), (17) and (18), we get

$$\begin{aligned} |f'(z)| &\geq \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \left| \frac{f(z)}{z} \right| \\ &\geq \frac{e^r}{1+r} \exp(F(r))G(r) \end{aligned}$$

where

$$F(r) = - \int_0^r \frac{u}{1-u^2} \left(\frac{c'u^2 + 4u + c'}{u^2 + c'u + 1} + \frac{du^2 + 4u + d}{u^2 + du + 1} \right) du$$

and

$$G(r) = \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)}. \quad \square$$

3. Radius estimates

Next, we obtain the best possible Ma-Minda type radius estimates for the class $\mathcal{F}_{b,c}$.

THEOREM 5. *Let $d = |5b - 3c| \leq 2$ and $c' = |3c + 1| \leq 2$. Let $c' + d = t$ and $c'd = s$. For the class $\mathcal{F}_{b,c}$, the following radius results hold:*

1. *Let $y_1(r) = -\alpha r^6 + (1 + (1 - \alpha)t)r^5 + (7 - \alpha + t + (2 - \alpha)s)r^4 + (2 + 5t + s)r^3 + (6 + \alpha + t + (1 + \alpha)s)r^2 + (1 + \alpha t)r + \alpha - 1$. Then the sharp $R_{S^*(\alpha)}(\mathcal{F}_{b,c})$ is the smallest root $r_1 \in (0, 1)$ of the equation $y_1(r) = 0$.*
2. *Let $y_2(r) = \sqrt{2}r^6 + (1 + (1 + \sqrt{2})t)r^5 + (9 + \sqrt{2} + t + \sqrt{2}(\sqrt{2} + 1)s)r^4 + (2 + 7t + s)r^3 + (10 - \sqrt{2} + t + (3 - \sqrt{2})s)r^2 + (1 + \sqrt{2}(\sqrt{2} - 1)t)r + 1 - \sqrt{2}$. Then the $R_{S_L^*}(\mathcal{F}_{b,c})$ is the smallest root $r_2 \in (0, 1)$ of the equation $y_2(r) = 0$. The radius is sharp for $3c + 1 \leq 0$ and $5b - 3c \leq 0$.*
3. *Let $y_3(r) = -r^6 + (2 + t)r^5 + (13 + 2t + 3s)r^4 + (4 + 10t + 2s)r^3 + (13 + 2t + 3s)r^2 + (2 + t)r - 1$. Then the $R_{S_p}(\mathcal{F}_{b,c})$ is the smallest root $r_3 \in (0, 1/\sqrt{3})$ of the equation $y_3(r) = 0$. For $3c + 1 \leq 0$ and $5b - 3c \leq 0$, the radius is sharp.*
4. *Let $y_4(r) = -r^6 + (e + (e - 1)t)r^5 + (7e - 1 + et + (2e - 1)s)r^4 + (2e + 5et + es)r^3 + (6e + 1 + et + (e + 1)s)r^2 + (e + t)r + 1 - e$. Then the $R_{S_e^*}(\mathcal{F}_{b,c})$ is the smallest root $r_4 \in (0, \sqrt{(1 - 2e + e^2)/(1 + e^2)})$ of the equation $y_4(r) = 0$. The radius is sharp for $3c + 1 \leq 0$ and $5b - 3c \leq 0$.*
5. *Let $y_5(r) = -r^6 + (3 + 2t)r^5 + (20 + 3t + 5s)r^4 + (6 + 15t + 3s)r^3 + (19 + 3t + 4s)r^2 + (3 + t)r - 2$. The $R_{S_c^*}(\mathcal{F}_{b,c})$ is the smallest root $r_5 \in (0, \sqrt{2/5})$ of the equation $y_5(r) = 0$. For $3c + 1 \leq 0$ and $5b - 3c \leq 0$, the radius is sharp.*

6. Let $y_6(r) = (1 + \sin 1)r^6 + (1 + (2 + \sin 1)t)r^5 + (10 + \sin 1 + t + (3 + \sin 1)s)r^4 + (2 + 7t + s)r^3 + (9 - \sin 1 + t + (2 - \sin 1)s)r^2 + (1 + (1 - \sin 1)t)r - \sin 1$. Then the $R_{S_{\sin}}^*(\mathcal{F}_{b,c})$ is the smallest root $r_6 \in (0, 1)$ of the equation $y_6(r) = 0$. The radius is sharp for $3c + 1 \leq 0$ and $5b - 3c \leq 0$.
7. Let $y_7(r) = -(\sqrt{2} - 1)r^6 + (1 + \sqrt{2}(\sqrt{2} - 1)t)r^5 + (8 - \sqrt{2} + t + (3 - \sqrt{2})s)r^4 + (2 + 5t + s)r^3 + (5 + \sqrt{2} + t + \sqrt{2}s)r^2 + (1 + (\sqrt{2} - 1)t)r - 2 + \sqrt{2}$. Then the $R_{S_{\sqrt{2}}}^*(\mathcal{F}_{b,c})$ is the smallest root $r_7 \in (0, 1)$ of the equation $y_7(r) = 0$. The radius is sharp for $3c + 1 \leq 0$ and $5b - 3c \leq 0$.
8. Let $y_8(r) = -(2\sqrt{2} - 2)r^6 + (1 + (3 - 2\sqrt{2})t)r^5 + (9 - 2\sqrt{2} + t + (4 + 2\sqrt{2})s)r^4 + (2 + 5t + s)r^3 + (4 + 2\sqrt{2} + t + (2\sqrt{2} - 1)s)r^2 + (1 + (2\sqrt{2} - 2)t)r + 2\sqrt{2} - 3$. The $R_{S_R}^*(\mathcal{F}_{b,c})$ is the smallest root $r_8 \in (0, \sqrt{(2 - \sqrt{2})/2})$ of the equation $y_8(r) = 0$. The radius is sharp for $3c + 1 \leq 0$ and $5b - 3c \leq 0$.
9. Let $y_9(r) = 5r^6 + (3 + 8t)r^5 + (32 + 3t + 11s)r^4 + (6 + 21t + 3s)r^3 + (25 + 3t + 4s)r^2 + (3 + t)r - 2$. Then the $R_{S_{Ne}}^*(\mathcal{F}_{b,c})$ is the smallest root $r_9 \in (0, 1)$ of the equation $y_9(r) = 0$. For $3c + 1 \leq 0$ and $5b - 3c \leq 0$, the radius is sharp.

Proof. Let $f \in \mathcal{F}_{b,c}$ and $g \in \mathcal{A}_{3c}$ such that

$$\frac{f(z)}{g(z)} \in \mathcal{P} \quad \text{and} \quad \frac{1+z}{z}g(z) \in \mathcal{P}, \quad (z \in \mathbb{D}).$$

1. In view of (12), it is noted that

$$\begin{aligned} & \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \\ & \geq \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \\ & \geq \alpha \end{aligned} \tag{19}$$

whenever $y_1(r) \leq 0$. For $0 \leq \alpha < 1$, $y_1(0) = \alpha - 1 < 0$ and $y_1(1) = 16 + 8t + 4s > 0$. In view of the intermediate value theorem the equation $y_1(r) = 0$ has a root $r_1 \in (0, 1)$. The result is sharp for the function f_0 defined in (1). Let $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$, $m + n = p$ and $mn = q$. Using (19) and (13), at $z = r_1$, we have

$$\frac{zf_0'(z)}{f_0(z)} = \alpha,$$

which shows the sharpness of the result.

2. From (10), it follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r^6 + (1+2t)r^5 + (10+t+3s)r^4 + (2+7t+s)r^3 + (9+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)}. \quad (20)$$

Since the centre of the above disc is 1, by using [2, Lemma 2.2, p. 6559], the function $f \in \mathcal{F}_{b,c}$ belongs to the class S_L^* if

$$\frac{r^6 + (1+2t)r^5 + (10+t+3s)r^4 + (2+7t+s)r^3 + (9+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \sqrt{2}-1 \quad (21)$$

equivalently $y_2(r) \leq 0$. Since $y_2(0) = 1 - \sqrt{2} < 0$ and $y_2(1) = 24 + 12t + 6s > 0$. Thus by intermediate value theorem the equation $y_2(r) = 0$ has a root $r_2 \in (0, 1)$.

The estimate is sharp for the function f_0 defined in (15). For $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$, $p = m + n$ and $q = mn$, we have

$$\frac{zf_0'(z)}{f_0(z)} = \frac{-(1+p)z^5 + (9+p+2q)z^4 - (2+7p+q)z^3 + (10+p+3q)z^2 - (1+2p)z + 1}{(1-z^2)(1-mz+z^2)(1-nz+z^2)}. \quad (22)$$

For the function f_0 (21) together with (22) at $z = -r_2$, yields

$$\begin{aligned} & \left| \left(\frac{zf_0'(z)}{f_0(z)} \right)^2 - 1 \right| \\ &= \left| \left(\frac{(1+p)r_2^5 + (9+p+2q)r_2^4 + (2+7p+q)r_2^3 + (10+p+3q)r_2^2 + (1+2p)r_2 + 1}{(1-r_2^2)(1+mr_2+r_2^2)(1+nr_2+r_2^2)} \right)^2 - 1 \right| \\ &= \left| \left(\frac{r_2^6 + (1+2p)r_2^5 + (10+p+3q)r_2^4 + (2+7p+q)r_2^3 + (9+p+2q)r_2^2 + (1+p)r_2 + 1}{(1-r_2^2)(1+mr_2+r_2^2)(1+nr_2+r_2^2)} \right)^2 - 1 \right| \\ &= 1. \end{aligned}$$

Thus, the result is sharp.

3. It is easy to see that $y_3(0) = -1 < 0$ and $y_3(1/\sqrt{3}) = 2(d(4(3+5\sqrt{3})) + 3(6+\sqrt{3})c' + 4(4(4+\sqrt{3}) + (3+5\sqrt{3})c'))/27 > 0$. Thus by intermediate value theorem the equation $y_3(r) = 0$ has the smallest root $r_3 \in (0, 1/\sqrt{3})$. Also, a simple calculation shows that $1/2 \leq 1/(1-r^2) \leq 3/2$ for $0 \leq r \leq 1/3$. In view of the fact that the center of the disc (10) lies in $[1/2, 3/2]$, by lemma [26, p. 321], $f \in S_p$ if

$$\frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \frac{1}{1-r^2} - \frac{1}{2}.$$

That is

$$\frac{(1+t)r^5 + (7+t+2s)r^4 + (2+5t+s)r^3 + (6+t+s)r^2 + r - 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \leq -\frac{1}{2}.$$

A simple calculation gives

$$\begin{aligned} & \frac{-r^6 + r^5 + (6+t+s)r^4 + (2+5t+s)r^3 + (7+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \\ & \leq \frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \quad (23) \end{aligned}$$

equivalently $y_3(r) \leq 0$.

The result is sharp for the function f_0 defined in (1). Let $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$, $p = m + n$ and $q = mn$. A simple calculation involving (13) and (23) for $z = r_3$ gives

$$\begin{aligned} & \operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right) \\ &= \frac{-(1+p)r_3^5 - (7+p+2q)r_3^4 - (2+5p+q)r_3^3 - (6+p+q)r_3^2 - r_3 + 1}{(1-r_3^2)(r_3^2 + mr_3 + 1)(r_3^2 + nr_3 + 1)} \\ &= \frac{-r_3^6 + r_3^5 + (6+p+q)r_3^4 + (2+5p+q)r_3^3 + (7+p+2q)r_3^2 + (1+p)r_3}{(1-r_3^2)(r_3^2 + mr_3 + 1)(r_3^2 + nr_3 + 1)} \\ &= \left| \frac{r_3^6 - r_3^5 - (6+p+q)r_3^4 - (2+5p+q)r_3^3 - (7+p+2q)r_3^2 - (1+p)r_3}{(1-r_3^2)(r_3^2 + mr_3 + 1)(r_3^2 + nr_3 + 1)} \right| \\ &= \left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| \end{aligned}$$

which proves the sharpness.

4. It is easy to verify that $1/e \leq 1/(1-r^2) \leq (e+e^{-1})/2$ for $r \leq \sqrt{(1-2e+e^2)/(1+e^2)}$. By [20, Lemma 2.2, p. 368], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class S_e^* if

$$\frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \leq \frac{1}{1-r^2} - \frac{1}{e}.$$

It is easy to see that

$$\frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \leq \frac{1}{e} \quad (24)$$

equivalently $y_4(r) \leq 0$. Then $y_4(0) = 1 - e < 0$ and $y_4\left(\sqrt{(1-2e+e^2)/(1+e^2)}\right) = 9.51072 + 4.85068t + 2.42573s > 0$. It follows by the intermediate value theorem that the equation $y_4(r) = 0$ has a root $r_4 \in \left(0, \sqrt{(1-2e+e^2)/(1+e^2)}\right)$.

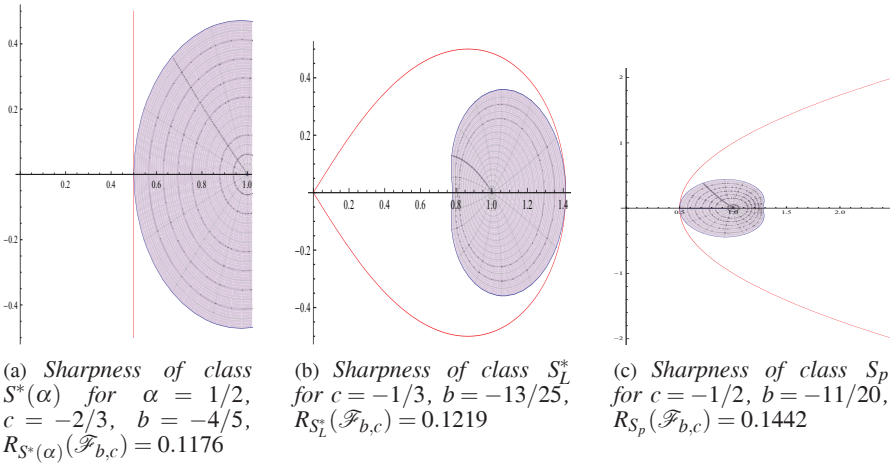


Figure 1: Sharpness of starlike functions associated with starlike function of order α , lemniscate and parabola for particular choices of α, b, c .

The estimate is sharp for the function f_0 defined in (1) for $m = -(5b - 3c) \geq 0$ and $n = -(3c + 1) \geq 0$. Let $p = m + n$ and $q = mn$ then at $z = r_4$, using (13) and (24), we have

$$\begin{aligned} & \left| \log \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left| \log \left(\frac{-(1+p)r_4^5 - (7+p+2q)r_4^4 - (2+5p+q)r_4^3 - (6+p+q)r_4^2 - r_4 + 1}{(1-r_4^2)(r_4^2 + mr_4 + 1)(r_4^2 + nr_4 + 1)} \right) \right| \\ &= 1. \end{aligned}$$

5. An easy calculation shows that $1/3 \leq 1/(1-r^2) \leq 5/3$ for $r \leq 2/5$. By making use of [27, Lemma 2.5, p.926], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class S_c^* if

$$\frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \leq \frac{1}{1-r^2} - \frac{1}{3}.$$

That is

$$\frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2 + c'r + 1)(r^2 + dr + 1)} \leq \frac{1}{3} \quad (25)$$

which gives $y_5(r) \leq 0$. It is noted that $y_5(0) = -2 < 0$ and $y_5(\sqrt{2/5}) = 3(364 + 49\sqrt{10} + (70 + 61\sqrt{10})t + 10(10 + \sqrt{10})s)/125 > 0$. It follows by the intermediate value theorem that the equation $y_5(r) = 0$ has a root $r_5 \in (0, \sqrt{2/5})$. For $m = -(5b - 3c) \geq 0$ and $n = -(3c + 1) \geq 0$ the estimate is sharp for the function

f_0 defined in (1). Using (13) and (25) for $p = m + n$, $q = mn$ and $z = r_5$, we have

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left| \frac{-(1+p)r_5^5 - (7+p+2q)r_5^4 - (2+5p+q)r_5^3 - (6+p+q)r_5^2 - r_5 + 1}{(1-r_5^2)(r_5^2+mr_5+1)(r_5^2+nr_5+1)} \right| \\ &= \frac{1}{3} \in \partial\Omega_c \end{aligned}$$

where Ω_c is the region bounded by the cardioid $\{x+iy: 81x^4 - 324x^3 + 162x^2y^2 + 270x^2 - 324xy^2 - 84x + 81y^4 - 54y^2 + 9 = 0\}$. This proves the sharpness.

6. Using [6, Lemma 3.3, p.219], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class $S_{\sin}^*$ if

$$\begin{aligned} & \frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \\ & \leq \sin 1 - \frac{r^2}{1-r^2}. \end{aligned}$$

A simple calculation gives

$$\begin{aligned} & \frac{(1+t)r^5 + (9+t+2s)r^4 + (2+7t+s)r^3 + (10+t+3s)r^2 + (1+2t)r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \\ & \leq \sin 1 + 1 \end{aligned} \tag{26}$$

which implies $y_6(r) \leq 0$. Proceeding as above a root r_6 of $y_6(r) = 0$ lies in $(0, 1)$. The result is sharp for the function f_0 defined in (15) for $m = -(5b - 3c) \geq 0$ and $n = -(3c + 1) \geq 0$. It follows by (22) and (26) for $z = -r_6$,

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left| \frac{(1+p)r_6^5 + (9+p+2q)r_6^4 + (2+7p+q)r_6^3 + (10+p+3q)r_6^2 + (1+2p)r_6 + 1}{(1-r_6^2)(1+mr_6+r_6^2)(1+nr_6+r_6^2)} \right| \\ &= 1 + \sin 1 \end{aligned}$$

which proves sharpness.

7. By [7, Lemma 2.1, p. 3], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class $S_{\mathbb{C}}^*$ if

$$\begin{aligned} & \frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \\ & \leq 1 - \sqrt{2} + \frac{1}{1-r^2}. \end{aligned}$$

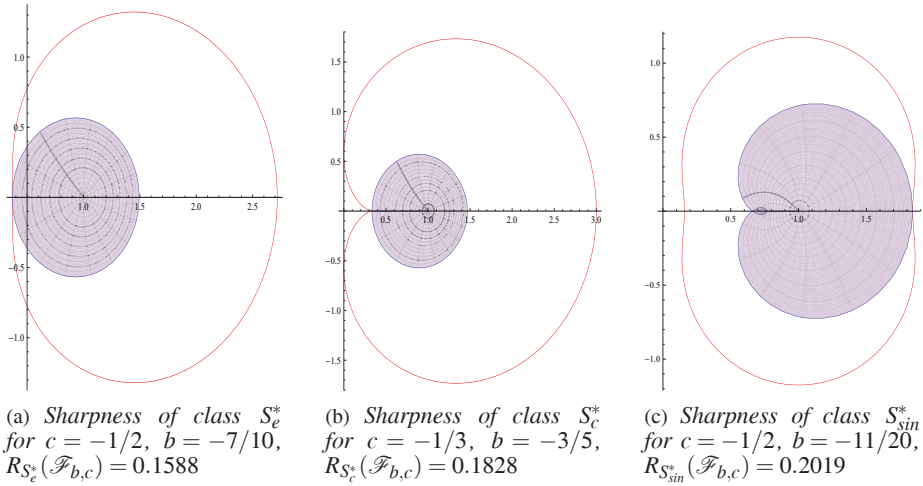


Figure 2: Sharpness of starlike functions associated with exponential, cardioid and sin function for particular choices of b, c .

That is

$$\frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \sqrt{2} - 1 \quad (27)$$

which simplifies to $y_7(r) \leq 0$. In view of the intermediate value theorem, a root of the equation $y_7(r) = 0$, r_7 lies in $(0, 1)$. The bound is best possible for the function f_0 defined by (1). For $m = -(5b - 3c) \geq 0$, $n = -(3c + 1) \geq 0$, $p = m + n$ and $q = mn$ using (13) and (27) at $z = r_7$, we have

$$\begin{aligned} & \left| \left(\frac{zf_0'(z)}{f_0(z)} \right)^2 - 1 \right| \\ &= \left| \left(\frac{-(1+p)r_7^5 - (7+p+2q)r_7^4 - (2+5p+q)r_7^3 - (6+p+q)r_7^2 - r_7 + 1}{(1-r_7^2)(r_7^2+mr_7+1)(r_7^2+nr_7+1)} \right)^2 - 1 \right| \\ &= \left| \left(\frac{1}{1-r_7^2} - \frac{(1+p)r_7^5 + (8+p+2q)r_7^4 + (2+6p+q)r_7^3 + (8+p+2q)r_7^2 + (1+p)r_7 + 1}{(1-r_7^2)(r_7^2+mr_7+1)(r_7^2+nr_7+1)} \right)^2 - 1 \right| \\ &= 2 \left| \frac{zf_0'(z)}{f_0(z)} \right|. \end{aligned}$$

8. An easy calculation shows that $2(\sqrt{2}-1) \leq 1/(1-r^2) \leq \sqrt{2}$ for $r \leq \sqrt{(2-\sqrt{2})/2}$. By [13, Lemma 2.2 p. 202], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class S_R^* if

$$\frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \frac{1}{1-r^2} + 2 - 2\sqrt{2}.$$

It is easy to see that

$$\frac{-(1+t)r^5 - (7+t+2s)r^4 - (2+5t+s)r^3 - (6+t+s)r^2 - r + 1}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq 2\sqrt{2} - 2 \quad (28)$$

which implies $y_8(r) \leq 0$. Then $y_8(0) = -3 + 2\sqrt{2} < 0$ and $y_8(\sqrt{(2-\sqrt{2})/2}) = 0.25(12.9668 + 6.5102t + 3.1782s) > 0$. In view of the intermediate value theorem, a root of the equation $y_8(r) = 0$, r_8 lies in $(0, \sqrt{(2-\sqrt{2})/2})$. The obtained estimate is best possible for the function f_0 defined by (1). For $m = -(5b-3c) \geq 0$, $n = -(3c+1) \geq 0$. Let $p = m+n$ and $q = mn$. Using (13) and (28) at $z = r_8$, we have

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left| \frac{-(1+p)r_8^5 - (7+p+2q)r_8^4 - (2+5p+q)r_8^3 - (6+p+q)r_8^2 - r_8 + 1}{(1-r_8^2)(r_8^2+mr_8+1)(r_8^2+nr_8+1)} \right| \\ &= 2(\sqrt{2}-1) = \psi(-1) \end{aligned}$$

where $\psi(z) = 1 + \frac{z}{k} \left(\frac{k+z}{k-z} \right) = 1 + \frac{1}{k}z + \frac{2}{k^2}z^2 + \dots$, $k = \sqrt{2} + 1$.

9. Using [30, Lemma 2.2, p. 86], the function f in the class $\mathcal{F}_{b,c}$ belongs to the class $S_{N_c}^*$ if

$$\frac{(1+t)r^5 + (8+t+2s)r^4 + (2+6t+s)r^3 + (8+t+2s)r^2 + (1+t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \frac{5}{3} - \frac{1}{1-r^2}.$$

A simple calculation gives

$$\frac{(1+t)r^5 + (9+t+2s)r^4 + (2+7t+s)r^3 + (10+t+3s)r^2 + (1+2t)r}{(1-r^2)(r^2+c'r+1)(r^2+dr+1)} \leq \frac{5}{3} \quad (29)$$

equivalently, $y_9(r) \leq 0$. It follows by the intermediate value theorem that a root r_9 of the equation $y_9(r) = 0$ lies in $(0, 1)$. The estimate is sharp for the function

f_0 defined in (15) $m = -(5b - 3c) \geq 0$ and $n = -(3c + 1) \geq 0$. Let $p = m + n$ and $q = mn$. Using (22) and (29) for $z = -r_9$, we have

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left| \frac{(1+p)r_9^5 + (9+p+2q)r_9^4 + (2+7p+q)r_9^3 + (10+p+3q)r_9^2 + (1+2p)r_9 + 1}{(1-r_9^2)(1+mr_9+r_9^2)(1+nr_9+r_9^2)} \right| \\ &= \frac{5}{3} \text{ which lies on the boundary of } \Omega_{N_e}. \end{aligned}$$

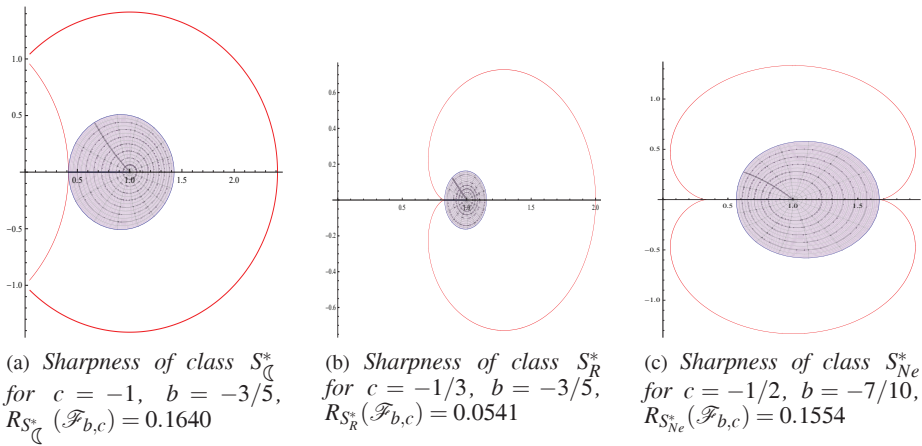


Figure 3: Sharpness of starlike functions associated with the function $z + \sqrt{1+z^2}$, the rational function and the nephroid for particular choices of b, c .

This proves the sharpness. $\square$

REMARK 1. On taking $b = -1$ and $c = -1$, parts (1)–(8) of Theorem 5 reduces to [25, Theorem 2.1].

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