

STRONG CONVERGENCE OF ISHIKAWA ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce a modified Ishikawa iterative process for approximating a fixed point of nonexpansive mappings in Hilbert spaces. we establish the strong convergence theorem of the general iteration scheme under some mild conditions. Our results extend and improve the results announced by many others.

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E , and $T : C \rightarrow C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T ; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self mapping $f : C \rightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in C.$$

We use Π_C to denote the collection of all contractions on C . That is, $\Pi_C = \{f : C \rightarrow C \text{ a contraction}\}$. A operator A is strong positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.1}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [1, 13-14] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H :

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.2}$$

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where C is the fixed point set of a nonexpansive mapping S and b is a given point in H . In [14,15], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n b, \quad n \geq 0,$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [5] introduced a new iterative scheme by the viscosity approximation method [7]:

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.3}$$

where C is the fixed point set of a nonexpansive mapping S , h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$.)

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [6] and is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^\infty$ is in the interval $[0, 1]$.

The second iteration process is referred to as Ishikawa's iteration process [3] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases} \tag{1.5}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. But both (1.4) and (1.5) have only weak convergence, in general (see [2] for an example). For example, Reich [8], shows that if E is a uniformly convex and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by processes (1.4) converges weakly to a point in $F(T)$. (An extension of this result to processes (1.5) can be found in [11].) Therefore, many authors attempt to modify (1.4) and (1.5) to have strong convergence.

In this paper, motivated by Kim and Xu [4], Su and Qin [10], Marino and Xu [5] and Xu [12], we introduce a composite iteration scheme as follows

$$\begin{cases} x_0 = x \in C \text{ arbitrarily chosen,} \\ z_n = \gamma_n x_n + (1 - \gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1 - \beta_n)Tz_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{cases} \tag{1.6}$$

where $f \in \Pi_C$ is a contraction and A is a linear bounded operator. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, that $\{x_n\}$ defined by (1.6) converges to a fixed point of T , which solve some variation inequality and is also the optimality condition for the minimization problem (1.3).

Now, we consider some special cases of the iterative scheme. If $\gamma_n = 1$ in (1.6), then (1.6) reduces to

$$\begin{cases} x_0 = x \in C \text{ arbitrarily chosen,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n. \end{cases} \tag{1.7}$$

When $\{\gamma_n\} = \gamma = 1$ and $A = I$ in (1.6), we have that (1.6) collapses to

$$\begin{cases} x_0 = x \in C \text{ arbitrarily chosen,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \end{cases} \tag{1.8}$$

which was considered by Yao et al. [16]. When $A = I$, $\{\gamma_n\} = \gamma = 1$ and $f(y) = u \in C$ for all $y \in C$ in (1.6), we have that (1.6) reduces to

$$\begin{cases} x_0 = x \in C \text{ arbitrarily chosen,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.9}$$

which was considered by Kim and Xu [4].

Our purpose in this paper is to introduce the composite iteration scheme for approximating a fixed point of nonexpansive mappings, which solve some variational inequality. We establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.6). The results improve and extend results of Kim and Xu [4], Marino and Xu [5], Su and Qin [10], Yao et al. [16] and many others.

LEMMA 1.1. *In a Hilbert space H , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle, \quad x, y \in H.$$

LEMMA 1.2. (Suzuki [9]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

LEMMA 1.3. (Xu [13, 14]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty;$

(ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n \rightarrow \infty} \alpha_n = 0.$

LEMMA 1.4. (Marino and Xu [5]). Assume that A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}.$

LEMMA 1.5. (Marino and Xu [5]). Let H be a Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma} / \alpha$. Let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \rightarrow 0$ to a fixed point \bar{x} of T , which solves the variational inequality

$$\langle (A - \gamma f)\bar{x}, \bar{x} - z \rangle \leq 0, z \in F(T).$$

2. Main results

THEOREM 2.1. Let C be a closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma} / \alpha$. Given a map $f \in \Pi_C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n \rightarrow \infty} \alpha_n = 0;$

(C2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1;$

(C3) $\gamma_n - \beta_n > a$, for some $a \in [0, 1);$

(C4) $\lim_{n \rightarrow \infty} |\gamma_{n+1} - \gamma_n| = 0.$

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} x_0 = x \in C \text{ arbitrarily chosen,} \\ z_n = \gamma_n x_n + (1 - \gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1 - \beta_n)Tz_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F(T)$, where $q = P_{F(T)}(\gamma f + (I - A))q$ and which also solves some variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \leq 0, p \in F(T).$$

Proof. Since $\alpha_n \rightarrow 0$ by the condition (C1), we may assume, with no loss of generality, that $\alpha_n < (1 - \delta_n)\|A\|^{-1}$ for all n . From Lemma 1.4, we know that if

$0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. First we observe that $\{x_n\}_{n=0}^\infty$ is bounded. Indeed, taking a fixed point p of T and noticing that

$$\|z_n - p\| \leq \gamma_n \|x_n - p\| + (1 - \gamma_n) \|Tx_n - p\| \leq \|x_n - p\|,$$

and

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|Tz_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ap\| + (1 - \alpha_n \bar{\gamma}) \|y_n - p\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &= [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|. \end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|Ap - \gamma f(p)\|}{\bar{\gamma} - \gamma\alpha}\},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\|x_{n+1} - x_n\| \rightarrow 0. \tag{2.1}$$

Observing that

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n) Tx_n, \\ z_{n-1} = \gamma_{n-1} x_{n-1} + (1 - \gamma_{n-1}) Tx_{n-1}. \end{cases}$$

We obtain

$$z_n - z_{n-1} = (1 - \gamma_n)(Tx_n - Tx_{n-1}) + \gamma_n(x_n - x_{n-1}) + (\gamma_{n-1} - \gamma_n)(Tx_{n-1} - x_{n-1}).$$

It follows that

$$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| M_1, \tag{2.2}$$

where M_1 is a constant such that $M_1 = \sup_{n \geq 1} \|x_{n-1} - Tx_{n-1}\|$ for all n . Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. Now, we compute $l_{n+1} - l_n$. That is,

$$x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n, \quad n \geq 0.$$

Observing that

$$\begin{aligned}
 l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} \\
 &\quad - \frac{\alpha_n\gamma f(x_n) + (I - \alpha_nA)y_n - \beta_nx_n}{1 - \beta_n} \\
 &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1 - \beta_n} \\
 &\quad + \frac{y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{y_n - \beta_nx_n}{1 - \beta_n} \\
 &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1 - \beta_n} \\
 &\quad + Tz_{n+1} - Tz_n,
 \end{aligned}$$

we have

$$\begin{aligned}
 \|l_{n+1} - l_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|Ay_n - \gamma f(x_n)\| \\
 &\quad + \|z_{n+1} - z_n\|.
 \end{aligned} \tag{2.3}$$

Substitute (2.2) into (2.3) yields that

$$\begin{aligned}
 \|l_{n+1} - l_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|Ay_n - \gamma f(x_n)\| \\
 &\quad + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|M_1.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| \\
 &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|Ay_n - \gamma f(x_n)\| + |\gamma_{n-1} - \gamma_n|M_1.
 \end{aligned}$$

Observe condition (C1), (C4) and take the limits as $n \rightarrow \infty$ gets

$$\limsup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

We can obtain $\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0$ easily by Lemma 1.2. Since

$$x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n),$$

we have that (2.1) holds. Since $x_{n+1} - y_n = \alpha_n(\gamma f(x_n) - Ay_n)$, we have that

$$\lim_{n \rightarrow \infty} x_{n+1} - y_n = 0. \tag{2.4}$$

Observing that

$$\|y_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{2.5}$$

On the other hand, we have

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|x_n - y_n\| + \|y_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \|y_n - Tz_n\| + \|Tz_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \beta_n \|x_n - Tx_n\| + \|Tz_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \beta_n \|x_n - Tx_n\| + \|z_n - x_n\| \\ &\leq \|x_n - y_n\| + \beta_n \|x_n - Tx_n\| + (1 - \gamma_n) \|Tx_n - x_n\|, \end{aligned}$$

which implies

$$(\gamma_n - \beta_n) \|Tx_n - x_n\| \leq \|x_n - y_n\|.$$

From condition (C3) and (2.5), we obtain

$$\|Tx_n - x_n\| \rightarrow 0. \tag{2.6}$$

Observe that $P_{F(T)}(\gamma f + (I - A))$ is a contraction. Indeed, for $\forall x, y \in H$, we have

$$\begin{aligned} \|P_{F(S)}(\gamma f + (I - A))(x) - P_{F(S)}(\gamma f + (I - A))(y)\| &\leq \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\ &\leq \gamma \|f(x) - f(y)\| + \|I - A\| \|x - y\| \\ &\leq \gamma \alpha \|x - y\| + (1 - \bar{\gamma}) \|x - y\| \\ &< \|x - y\|. \end{aligned}$$

Banach’s Contraction Mapping Principle guarantees that $P_{F(S)}(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_{F(S)}(\gamma f + (I - A))(q)$. Next, we claim that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0, \tag{2.7}$$

where $q = \lim_{t \rightarrow 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$. From x_t solves the fixed point equation

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t.$$

Thus we have

$$\|x_t - x_n\| = \|(I - tA)(Tx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|.$$

It follows from Lemma 1.1 that

$$\begin{aligned} \|x_t - x_n\|^2 &= \|(I - tA)(Tx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\ &\leq (1 - \bar{\gamma}t)^2 \|Tx_t - x_n\|^2 + 2t \langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\ &\leq (1 - 2\bar{\gamma}t + (\bar{\gamma}t)^2) \|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t \langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t \langle Ax_t - Ax_n, x_t - x_n \rangle, \end{aligned} \tag{2.8}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Tx_n\|) \|x_n - Tx_n\| \rightarrow 0, \text{ as } n \rightarrow 0. \tag{2.9}$$

It follows from (1.1) that

$$\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{\bar{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t). \quad (2.10)$$

Let $n \rightarrow \infty$ in (2.10) and note (2.9) yields

$$\limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{t}{2} M, \quad (2.11)$$

where $M > 0$ is a constant such that $M \geq \bar{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0, 1)$ and $n \geq 1$. Taking $t \rightarrow 0$ from (2.11), we have

$$\limsup_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0.$$

Since H is a Hilbert space the order of $\limsup_{t \rightarrow 0}$ and $\limsup_{n \rightarrow \infty}$ is exchangeable, and hence (2.7) holds. Now from Lemma 1.1, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|((1 - \delta_n)I - \alpha_n A)(y_n - q) + \delta_n(x_n - q) + \alpha_n(\gamma f(x_n) - Aq)\|^2 \\ &\leq \|((1 - \delta_n)I - \alpha_n A)(y_n - q) + \delta_n(x_n - q)\|^2 + 2\alpha_n \langle \gamma f(x_n) - Aq, x_{n+1} - q \rangle \\ &\leq [(1 - \delta_n - \alpha_n \bar{\gamma}) \|y_n - q\| + \delta_n \|x_n - q\|]^2 + 2\alpha_n \langle \gamma f(x_n) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma \alpha \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + \alpha_n \gamma \alpha (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \frac{(1 - \alpha_n \bar{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &= \frac{(1 - 2\alpha_n \bar{\gamma} + \alpha_n \alpha \gamma)}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 + \frac{\alpha_n \bar{\gamma}^2}{1 - \alpha_n \gamma \alpha} \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\ &\leq [1 - \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha}] \|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha} [\frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle + \frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)} M_2], \end{aligned}$$

where M_2 is a appropriate constant such that $M_2 = \sup_{n \rightarrow \infty} \|x_n - q\|$ for all n . Put $l_n = \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \alpha \gamma}$ and $t_n = \frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(q) - Aq, x_{n+1} - q \rangle + \frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)} M_2$. That is,

$$\|x_{n+1} - q\|^2 \leq (1 - l_n) \|x_n - q\|^2 + l_n t_n. \quad (2.12)$$

It follows from condition (C1) and (2.11) that

$$\lim_{n \rightarrow \infty} l_n = 0, \quad \sum_{n=1}^{\infty} l_n = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} t_n \leq 0.$$

Apply Lemma 1.3 to (2.12) to conclude $x_n \rightarrow q$. This completes the proof. \square

As corollaries of Theorem 2.1, we have the following corollaries immediately.

COROLLARY 2.2. *Let C be a closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given a map $f \in \Pi_C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ in $(0,1)$, the following conditions are satisfied*

$$(C1) \quad \sum_{n=0}^\infty \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Let $\{x_n\}_{n=1}^\infty$ be the composite process defined by (1.7). Then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F(T)$, where $q = P_{F(T)}(\gamma f + (I - A))q$ and which also solve the variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \leq 0, \quad p \in F(T).$$

Proof. Taking $\{\gamma_n\} = 1$ in Theorem 2.1, we can conclude the desired conclusion easily. \square

COROLLARY 2.3. *Let C be a closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping. Given a map $f \in \Pi_C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ in $(0,1)$, the following conditions are satisfied*

$$(C1) \quad \sum_{n=0}^\infty \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Let $\{x_n\}_{n=1}^\infty$ be the composite process defined by (1.8). Then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F(T)$, where $q = P_{F(T)}f(q)$ and which also solve the variational inequality

$$\langle f(q) - q, q - p \rangle \leq 0, \quad p \in F(T).$$

Proof. Taking $\{\gamma_n\} = \gamma = 1$ and $A = I$ in Theorem 2.1, we can conclude the desired conclusion easily. \square

REFERENCES

- [1] F. DEUTSCH, I. YAMADA, *Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings*, Numer. Funct. Anal. Optim. 19 (1998) 33–56.
- [2] A. GENEL, J. LINDENSTRAUSS, *An example concerning fixed points*, Israel J. Math. 22 (1975) 81–86.
- [3] S. ISHIKAWA, *Fixed points by a new iteration method*, Proc. Am. Math. Soc. 44 (1974) 147–150.
- [4] T. H. KIM, H. K. XU, *Strong convergence of modified Mann iterations*, Nonlinear Anal. 61 (2005) 51–60.
- [5] G. MARINO AND H. K. XU, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. 318 (2006) 43–52.
- [6] W. R. MANN, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953) 506–510.
- [7] A. MOUDAFI, *Viscosity approximation methods for fixed points problems*, J. Math. Anal. Appl. 241 (2000) 46–55.
- [8] S. REICH, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 67 (1979) 274–276.

- [9] T. SUZUKI, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. 305 (2005) 227–239.
- [10] Y. F. SU, X. L. QIN, *Strong convergence theorems for nonexpansive mapping*, J. Syst Sci Complexity 20 (2007) 85–94
- [11] K. K. TAN, H. K. XU, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (2) (1993) 301–308.
- [12] H. K. XU, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. 314 (2006) 631–643.
- [13] H. K. XU, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. 66 (2002) 240–256.
- [14] H. K. XU, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. 116 (2003) 659–678.
- [15] I. YAMADA, *The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings*, in: D. Butnariu, Y. Censor, S. Reich (Eds.), *Inherently Parallel Algorithm for Feasibility and Optimization*, Elsevier, 2001, pp. 473–504.
- [16] Y. YAO, R. D. CHEN, Y. C. Y., *Strong convergence and certain control conditions for modified Mann iteration*, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.01.009.

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