

WEIGHTED L_p -NORM INEQUALITIES IN CONVOLUTIONS AND THEIR APPLICATIONS

NGUYEN DU VI NHAN AND DINH THANH DUC

(communicated by S. Saitoh)

Abstract. In this paper, we give some new type of convolution inequalities in weighted $L_p(\mathbb{R}^2, dx dy)$ spaces and their important applications to partial differential equations and integral transforms. Especially, we will see their applications to non-homogeneous linear differential equations.

1. Introduction

It is well known, for the Fourier convolution

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x - \xi)g(\xi)d\xi,$$

the Young's inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \quad f \in L_p(\mathbb{R}), \quad g \in L_q(\mathbb{R}), \quad (1.1)$$

$$r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0),$$

is fundamental (see [9]). In a series of papers, S. Saitoh (see [5], [6], [7], [8]) derived new type norm inequalities in convolutions in weighted L_p ($p > 1$) spaces.

PROPOSITION 1. ([5]) *For two non-vanishing functions $\rho_j \in L_1(\mathbb{R})$ ($j = 1, 2$), the L_p ($p > 1$) weighted convolution inequality*

$$\left\| ((F_1 \rho_1) * (F_2 \rho_2)) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \leq \|F_1\|_{L_p(\mathbb{R}, |\rho_1|)} \|F_2\|_{L_p(\mathbb{R}, |\rho_2|)} \quad (1.2)$$

holds for $F_j \in L_p(\mathbb{R}, |\rho_j|)$ ($j = 1, 2$). Equality holds here if and only if

$$F_j(x) = C_j e^{\alpha x} \quad (1.3)$$

where α is a constant such that $e^{\alpha x} \in L_p(\mathbb{R}, |\rho_j|)$ ($j = 1, 2$) (otherwise C_1 or $C_2 = 0$). Here

$$\|F\|_{L_p(\mathbb{R}, |\rho|)} = \left\{ \int_{-\infty}^{+\infty} |F(x)|^p |\rho(x)| dx \right\}^{\frac{1}{p}}.$$

In this paper, we will consider some convolution inequalities in weighted $L_p(\mathbb{R}^2, dx dy)$ spaces and their important applications.

Mathematics subject classification (2000): 44A35, 35A22, 26D20.

Key words and phrases: Convolution, inequality, weighted L_p norm, Green function, integral transform, partial differential equations.

2. New type of convolution inequality

The backgrounds in our fundamental weighted $L_p(\mathbb{R}^2, dx dy)$ convolution inequality will be given by Hölder's inequality, Fubini's theorem and by changing the variables in integral.

THEOREM 1. *For two non-vanishing functions $\rho_j(x, y)$ ($j = 1, 2$) belonging to $L_1(\mathbb{R}^2, dx dy)$ and for $p > 1$ we have L_p weighted convolution inequality*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(\xi, \tau) \rho_1(\xi, \tau) F_2(x-\xi, y-\tau) \rho_2(x-\xi, y-\tau) d\xi d\tau \right|^p}{\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho_1(\xi, \tau)| |\rho_2(x-\xi, y-\tau)| d\xi d\tau \right)^{p-1}} dx dy$$

$$\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi d\tau \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^p |\rho_2(\xi, \tau)| d\xi d\tau \quad (2.4)$$

for $F_j(\xi, \tau) \in L_p(\mathbb{R}^2, |\rho_j(\xi, \tau)| d\xi d\tau)$ ($j = 1, 2$). Equality holds for F_j if and only if F_j are represented in the form

$$F_j(\xi, \tau) = C_j e^{\alpha\xi + \beta\tau}; \quad C_j : \text{constant}, \quad (2.5)$$

where α, β are two constants such that $F_j(\xi, \tau) \in L_p(\mathbb{R}^2, |\rho_j(\xi, \tau)| d\xi d\tau)$ ($j = 1, 2$).

Proof. Applying Hölder inequality and Fubini's theorem and by changing the variables in integrals we obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(\xi, \tau) \rho_1(\xi, \tau) F_2(x-\xi, y-\tau) \rho_2(x-\xi, y-\tau) d\xi d\tau \right|^p}{\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho_1(\xi, \tau)| |\rho_2(x-\xi, y-\tau)| d\xi d\tau \right)^{p-1}} dx dy$$

$$\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| |F_2(x-\xi, y-\tau)|^p |\rho_2(x-\xi, y-\tau)| d\tau d\xi dx dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi d\tau \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^p |\rho_2(\xi, \tau)| d\xi d\tau.$$

Equality holds if and only if for a function $k(x, y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}$ such that

$$F_1(\xi, \tau) F_2(x-\xi, y-\tau) = k(x, y) \quad \forall (\xi, \tau) \in \mathbb{R} \times \mathbb{R}$$

that is

$$F_1(x_1, x_2) F_2(y_1, y_2) = k(x_1 + y_1, x_2 + y_2) \quad \forall (x_j, y_j) \in \mathbb{R}^2 (j = 1, 2). \quad (2.6)$$

From function equation (2.6), we have the equality problem in (2.5). \square

REMARK 1. In the inequality (2.4), we can represent the following convolution form:

$$\left\| ((F_1 \rho_1) * (F_2 \rho_2)) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \leq \|F_1\|_{L_p(\mathbb{R}^2, |\rho_1|)} \|F_2\|_{L_p(\mathbb{R}^2, |\rho_2|)}. \quad (2.7)$$

REMARK 2. The essentials of Theorem 1 are clear and are based on Hölder inequality and Fubini's theorem and the transform of integrals. So, we will be able to obtain many variations of Theorem. For example, in many cases the convolution will be given in the form

$$\rho_2(\xi, \tau) \equiv 1, \text{ and } F_2(x - \xi, y - \tau) = G(x - \xi, y - \tau)$$

for some Green's functions $G(x - \xi, y - \tau)$. Then, we have the inequality

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi, \tau) \rho(\xi, \tau) G(x - \xi, y - \tau) d\xi d\tau \right|^p dx dy \\ & \leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau \\ & \quad \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |G(\xi, \tau)|^p d\xi d\tau, \end{aligned} \quad (2.8)$$

for an $L_1(\mathbb{R}^2, d\xi d\tau)$ function ρ , and for functions F and G with finite integrals in the right hand side in (2.8).

Moreover we have the following fundamental inequality which is applied to non-homogeneous linear differential equations

THEOREM 2. For two non-vanishing functions $\rho_j(x, y)$ ($j = 1, 2$) belonging to $L_1(\mathbb{R}^2, dxdy)$, and for $p > 1, q > 1, p^{-1} + q^{-1} = 1$, we have the inequality

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{\left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(\xi, \tau) \rho_1(\xi, \tau) F_2(x - \xi, \tau) \rho_2(x - \xi, \tau) d\xi d\tau \right|^p}{\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho_1(\xi, \tau)| |\rho_2(x - \xi, \tau)| d\xi d\tau \right)^{p-1}} dx \\ & \leq \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}} \\ & \quad \cdot \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^q |\rho_2(\xi, \tau)| d\xi \right]^q d\tau \right)^{\frac{1}{q}} \\ & = \left\| \|F_1\|_{L_p(\mathbb{R}, |\rho_1| d\xi)}^p \right\|_{L_p(\mathbb{R}, d\tau)} \left\| \|F_2\|_{L_p(\mathbb{R}, |\rho_2| d\xi)}^p \right\|_{L_q(\mathbb{R}, d\tau)}. \end{aligned} \quad (2.9)$$

where $F_j(j = 1, 2)$ are such that the right hand side of (2.9) is finite. Equality holds for F_j if and only if F_j are represented in the form

$$F_j(\xi, \tau) = C_j(\tau) e^{\alpha \xi}; \quad C_1(\tau) C_2(\tau) = c : \text{constant}, \quad (2.10)$$

and

$$\frac{\left[|C_1(\tau)| \int_{-\infty}^{+\infty} e^{p\alpha \xi} |\rho_1(\xi, \tau)| d\xi \right]^p}{\int_{-\infty}^{+\infty} \left[|C_1(\tau)| \int_{-\infty}^{+\infty} e^{p\alpha \xi} |\rho_1(\xi, \tau)| d\xi \right]^p d\tau}$$

$$= \frac{\left[|C_2(\tau)| \int_{-\infty}^{+\infty} e^{\rho\alpha\xi} |\rho_2(\xi, \tau)| d\xi \right]^q}{\int_{-\infty}^{+\infty} \left[|C_2(\tau)| \int_{-\infty}^{+\infty} e^{\rho\alpha\xi} |\rho_1(\xi, \tau)| d\xi \right]^q d\tau}, \quad (2.11)$$

where α is a constant such that the right hand side of (2.9) is finite.

Proof. Put

$$I = \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(\xi, \tau) \rho_1(\xi, \tau) F_2(x - \xi, \tau) \rho_2(x - \xi, \tau) d\xi d\tau \right|^p.$$

By Hölder inequality, we obtain directly

$$I \leq \left(\int_{-\infty}^{+\infty} (f(\xi))^{\frac{1}{p}} (g(\xi))^{\frac{1}{q}} d\xi \right)^p \leq \left(\int_{-\infty}^{+\infty} g(\xi) d\xi \right)^{p-1} \int_{-\infty}^{+\infty} f(\xi) d\xi, \quad (2.12)$$

where

$$\begin{aligned} f(\xi) &= \int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| |F_2(x - \xi, \tau)|^p |\rho_2(x - \xi, \tau)| d\tau \\ g(\xi) &= \int_{-\infty}^{+\infty} |\rho_1(\xi, \tau)| |\rho_2(x - \xi, \tau)| d\tau. \end{aligned}$$

Therefore, by changing the variables in integrals and Fubini's theorem we get

$$\begin{aligned} K &= \int_{-\infty}^{+\infty} \frac{\left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_1(\xi, \tau) \rho_1(\xi, \tau) F_2(x - \xi, \tau) \rho_2(x - \xi, \tau) d\xi d\tau \right|^p}{\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho_1(\xi, \tau)| |\rho_2(x - \xi, \tau)| d\xi d\tau \right)^{p-1}} dx \\ &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| |F_2(x - \xi, \tau)|^p |\rho_2(x - \xi, \tau)| d\tau d\xi dx \\ &= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi \int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^p |\rho_2(\xi, \tau)| d\xi \right] d\tau. \end{aligned}$$

Then, by Hölder inequality, we have

$$K \leq \left\| \|F_1\|_{L_p(\mathbb{R}, |\rho_1| d\xi)}^p \right\|_{L_p(\mathbb{R}, d\tau)} \left\| \|F_2\|_{L_p(\mathbb{R}, |\rho_2| d\xi)}^p \right\|_{L_q(\mathbb{R}, d\tau)}. \quad (2.13)$$

Equality holds if and only if for a function $k(x)$ in x

$$F_1(\xi, \tau) F_2(x - \xi, \tau) = k(x) \text{ a.e. on } \mathbb{R},$$

and

$$\frac{\left[\int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi \right]^p}{\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F_1(\xi, \tau)|^p |\rho_1(\xi, \tau)| d\xi \right]^p d\tau}$$

$$= \frac{\left[\int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^p |\rho_2(\xi, \tau)| d\xi \right]^q}{\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F_2(\xi, \tau)|^p |\rho_2(\xi, \tau)| d\xi \right]^q d\tau}.$$

Therefore, we have the equalities problem in (2.9) and (2.10). The proof is complete. \square

In the inequality (2.9), in many cases the convolution will be given in the form

$$\rho_2(\xi, \tau) \equiv 1, \text{ and } F_2(x - \xi, \tau) = G(x - \xi, \tau)$$

for some Green's functions $G(x - \xi, \tau)$. Then, we have the inequality

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi, \tau) \rho(\xi, \tau) G(x - \xi, \tau) d\xi d\tau \right|^p dx \\ & \leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}} \\ & \quad \cdot \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |G(\xi, \tau)|^p d\xi \right]^q d\tau \right)^{\frac{1}{q}}, \end{aligned} \quad (2.14)$$

for an $L_1(\mathbb{R}^2, d\xi d\tau)$ function ρ , and for function F and G with finite integrals in the right hand side in (2.14).

In general, in (2.14) we have a generalization

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left| \int_a^b d\tau \int_{-\infty}^{+\infty} F(\xi, \tau) \rho(\xi, \tau) G(x - \xi, \tau) d\xi \right|^p dx \\ & \leq \left(\int_a^b d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_a^b \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}} \\ & \quad \cdot \left(\int_a^b \left[\int_{-\infty}^{+\infty} |G(\xi, \tau)|^p d\xi \right]^q d\tau \right)^{\frac{1}{q}}. \end{aligned} \quad (2.15)$$

In the sequel, we shall show several typical applications. We get not only L_p integral estimates for the solutions of non-homogeneous linear differential equations but also solutions homogeneous linear differential equations in the space \mathbb{R}^2 or \mathbb{R}^3 and several integral transforms ([1], [2], [3], [4], [9]).

3. Applications

3.1. Wave Equation

For the function

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0, \end{cases}$$

we consider the integral transform

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau) \rho(\xi, \tau) d\xi \\ &= \frac{1}{2c} \int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} \theta\left(\frac{t}{2} - \left|\tau - \frac{t}{2}\right|\right) \theta(c(t-\tau) - |x-\xi|) F(\xi, \tau) \rho(\xi, \tau) d\xi, \end{aligned} \quad (3.16)$$

which gives the formal solution $u(x, t)$ of the wave equation

$$u_{tt} = c^2 u_{xx} + F(x, t) \rho(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (c : \text{constant}, > 0), \quad (3.17)$$

satisfying the conditions

$$u(x, 0) = u_t(x, 0) = 0, \quad \text{on } \mathbb{R}. \quad (3.18)$$

Then, we have the inequality

$$\int_{-\infty}^{+\infty} |u(x, t)|^p dx \leq \frac{t^{1+\frac{1}{q}}}{(2c)^{p-1} (1+q)^{\frac{1}{q}}} J, \quad (3.19)$$

where

$$J = \left(\int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_0^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}}$$

is finite.

3.2. Heat Equation

3.2.1. Example 1

We consider the integral transform

$$u(x, t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{F(\xi, \tau) \rho(\xi, \tau)}{2c\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(\xi-x)^2}{4c^2(t-\tau)}\right\} d\xi, \quad (3.20)$$

which gives the solution $u(x, t)$ of the heat equation

$$u_t = c^2 u_{xx} + F(x, t) \rho(x, t), \quad x \in \mathbb{R}, \quad t > 0 \quad (3.21)$$

satisfying the condition

$$u(x, 0) = 0, \quad \text{on } \mathbb{R}. \quad (3.22)$$

For $1 < p < 2$, $q > 1$, $p^{-1} + q^{-1} = 1$, we have the inequality

$$\int_{-\infty}^{+\infty} |u(x, t)|^p dx \leq \frac{t^{\frac{2-p}{2q}}}{\sqrt{p} \left(1 - \frac{p}{2}\right)^{\frac{1}{q}} (2c\sqrt{\pi})^{p-1}} J, \quad (3.23)$$

where

$$J = \left(\int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_0^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}}$$

is finite.

3.2.2. Example 2

In the integral transform

$$u(x, y, t) = \frac{1}{(2c\sqrt{\pi t})^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left\{-\frac{(\xi-x)^2 + (\tau-y)^2}{4c^2t}\right\} F(\xi, \tau)\rho(\xi, \tau)d\xi d\tau, \quad (3.24)$$

which gives the solution $u(x, y, t)$ of the heat equation

$$u_t = c^2(u_{xx} + u_{yy}) \quad (3.25)$$

satisfying the condition

$$u(x, y, 0) = F(x, y)\rho(x, y). \quad (3.26)$$

Then, we not only have the inequality

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dx \leq \frac{1}{\sqrt{p}(pq)^{\frac{1}{2q}}(2c\sqrt{\pi t})^{2p+\frac{1}{p}-2}} K \quad (3.27)$$

but also obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, t)|^p dx dy \leq \frac{1}{p(4c^2\pi t)^{p-1}} I, \quad (3.28)$$

where

$$K = \left(\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}},$$

and

$$I = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau,$$

for $\rho \in L_1(\mathbb{R}^2, d\xi d\tau)$, $F \in L_p(\mathbb{R}^2, |\rho(\xi, \tau)| d\xi d\tau)$.

3.3. Laplace Equation (Poisson Integrals)

3.3.1. Example 1

We consider the Dirichlet problem for the Laplace Equation in a half-space of \mathbb{R}^3 , i.e. the determination of the bounded solution of

$$\Delta_3 u(x, y, t) = 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^2, \quad (3.29)$$

with the boundary condition

$$u(x, y, 0) = F(x, y)\rho(x, y). \quad (3.30)$$

We have the solution of the Dirichlet problem (3.29), (3.30) in the form

$$u(x, y, t) = \frac{t}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{F(\xi, \tau)\rho(\xi, \tau)}{[t^2 + (\xi-x)^2 + (\tau-y)^2]^{\frac{3}{2}}} d\xi d\tau. \quad (3.31)$$

Then, we not only have the inequality

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dx \leq \frac{1}{(2\pi)^p t^{2p+\frac{1}{p}-2}} B\left(\frac{1}{2}, \frac{3p-1}{2}\right) B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{q(3p-1)-1}{2}\right) K, \quad (3.32)$$

but also obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, t)|^p dx dy \leq \frac{1}{(2\pi)^p t^{2p-2}} B\left(\frac{1}{2}, \frac{3p-1}{2}\right) B\left(\frac{1}{2}, \frac{3p-2}{2}\right) I, \quad (3.33)$$

where

$$K = \left(\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}},$$

and

$$I = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau,$$

for $\rho \in L_1(\mathbb{R}^2, d\xi d\tau)$, $F \in L_p(\mathbb{R}^2, |\rho(\xi, \tau)| d\xi d\tau)$.

3.3.2. Example 2

In the conjugate Poisson integral transform

$$u(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi, \tau) \rho(\xi, \tau) \frac{x - \xi}{[t^2 + (\xi - x)^2 + (\tau - y)^2]^{\frac{3}{2}}} d\xi d\tau. \quad (3.34)$$

Then we have

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dx \leq \frac{1}{(2\pi)^p t^{2p+\frac{1}{p}-2}} B\left(\frac{p+1}{2}, \frac{2p-1}{2}\right) B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{q(2p-1)-1}{2}\right) K. \quad (3.35)$$

Moreover we also get

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dy \leq \frac{1}{(2\pi)^p t^{2p+\frac{1}{p}-2}} B\left(\frac{1}{2}, \frac{3p-1}{2}\right) B^{\frac{1}{q}}\left(\frac{qp+1}{2}, \frac{2qp-q-1}{2}\right) K', \quad (3.36)$$

and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, t)|^p dx dy \leq \frac{1}{(2\pi)^p t^{2p-2}} B\left(\frac{1}{2}, p-1\right) B\left(\frac{p+1}{2}, \frac{2p-1}{2}\right) I. \quad (3.37)$$

Here

$$K = \left(\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}},$$

$$K' = \left(\int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\tau \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\tau \right]^p d\xi \right)^{\frac{1}{p}},$$

and

$$I = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau,$$

for $\rho \in L_1(\mathbb{R}^2, d\xi d\tau)$, $F \in L_p(\mathbb{R}^2, |\rho(\xi, \tau)| d\xi d\tau)$.

3.4. Biharmonic equation

3.4.1. Example 1

The solution of the biharmonic equation

$$\Delta_3^2 u(x, y, t) = 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^2, \quad \Delta_3^2 = \Delta_3(\Delta_3) \tag{3.38}$$

with the boundary conditions

$$u(x, y, 0) = F(x, y)\rho(x, y), \quad u_t(x, y, 0) = 0, \tag{3.39}$$

is given by

$$u(x, y, t) = \frac{3t^3}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{F(\xi, \tau)\rho(\xi, \tau)}{[t^2 + (\xi - x)^2 + (\tau - y)^2]^{\frac{5}{2}}} d\xi d\tau. \tag{3.40}$$

Then, we have the following inequalities:

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dx \leq \frac{3^p}{(2\pi)^p t^{2p + \frac{1}{p} - 2}} B\left(\frac{1}{2}, \frac{5p-1}{2}\right) B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{q(5p-1)-1}{2}\right) K, \tag{3.41}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, t)|^p dx dy \leq \frac{3^p}{(2\pi)^p t^{2p-2}} B\left(\frac{1}{2}, \frac{5p-1}{2}\right) B\left(\frac{1}{2}, \frac{5p-2}{2}\right) I, \tag{3.42}$$

where

$$K = \left(\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}},$$

and

$$I = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau,$$

for $\rho \in L_1(\mathbb{R}^2, d\xi d\tau)$, $F \in L_p(\mathbb{R}^2, |\rho(\xi, \tau)| d\xi d\tau)$.

3.4.2. Example 2

The solution of the biharmonic equation

$$\Delta_3^2 u(x, y, t) = 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^2, \quad \Delta_3^2 = \Delta_3(\Delta_3) \quad (3.43)$$

with the boundary conditions

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = F(x, y)\rho(x, y), \quad (3.44)$$

is given by

$$u(x, y, t) = \frac{t^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{F(\xi, \tau)\rho(\xi, \tau)}{[t^2 + (\xi - x)^2 + (\tau - y)^2]^{\frac{3}{2}}} d\xi d\tau. \quad (3.45)$$

Then, we not only have the inequality

$$\int_{-\infty}^{+\infty} |u(x, y, t)|^p dx \leq \frac{1}{(2\pi)^p t^{p+\frac{1}{p}-2}} B\left(\frac{1}{2}, \frac{3p-1}{2}\right) B^{\frac{1}{q}}\left(\frac{1}{2}, \frac{q(3p-1)-1}{2}\right) K, \quad (3.46)$$

but also obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x, y, t)|^p dx dy \leq \frac{1}{(2\pi)^p t^{p-2}} B\left(\frac{1}{2}, \frac{3p-1}{2}\right) B\left(\frac{1}{2}, \frac{3p-2}{2}\right) I, \quad (3.47)$$

where

$$K = \left(\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi \right)^{p-1} \left(\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi \right]^p d\tau \right)^{\frac{1}{p}},$$

and

$$I = \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\rho(\xi, \tau)| d\xi d\tau \right)^{p-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \tau)|^p |\rho(\xi, \tau)| d\xi d\tau,$$

for $\rho \in L_1(\mathbb{R}^2, d\xi d\tau)$, $F \in L_p(\mathbb{R}^2, |\rho(\xi, \tau)| d\xi d\tau)$.

Acknowledgments. The authors wish to express their deep thanks to Professor S. Saitoh for his valuable advice and suggestions.

REFERENCES

- [1] A. V. BITSADZE, *Equations of mathematical physics*, Mir Publishers Moscow, 1980.
- [2] A. V. BITSADZE, D. F. KALINICHENKO, *A collection of problems on the equations of mathematical physics*, Mir Publishers Moscow, 1980.
- [3] YU. A. BRYCHKOV, H. J. GLAESKE, A. P. PRUDNIKOV, VU KIM TUAN, *Multidimensional Integral Transformations*, Gordon and Breach Science Publishers, 1992.
- [4] CHEN JI-XIU, JIANG GUO-YING, PAN YANG-LIAN, QIN TIE-HU, TONG YU-SUN, WU QUAN-SHUI AND XU SHENG-ZHI, *Problems and Solutions in Mathematics*, World Scientific, 1998.
- [5] S. SAITOH, *Weighted L_p -norm inequalities in convolution*, Survey on Classical Inequalities, 225–234 (2000), Kluwer Academic Publishers, The Netherlands.
- [6] S. SAITOH, *A fundamental inequality in the convolution of L_2 functions on the half line*, Proc. Amer. Math. Soc., 91: 285–286, 1984.
- [7] S. SAITOH, *On the convolution of L_2 functions*, Kodai – Math. J., 9: 50–57, 1986.
- [8] S. SAITOH, *Inequalities in the most simple Sobolev space and convolutions of L_2 functions with weights*, Proc. Amer. Math. Soc., 118: 515–520, 1993.
- [9] ELIAS M. STEIN, GUIDO WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton, New Jersey, Princeton University Press, 1971.

(Received May 18, 2007)

Nguyen Du Vi Nhan
Department of Mathematics
Quy Nhon University
Binh Dinh
Vietnam
e-mail: ndvynhan@gmail.com

Dinh Thanh Duc
Department of Mathematics
Quy Nhon University
Binh Dinh
Vietnam
e-mail: ducdinh2002@yahoo.com