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GENERALIZATION OF A BIHARI TYPE INTEGRAL INEQUALITY FOR ABSTRACT LEBESGUE INTEGRAL

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Abstract. In this paper we study some integral inequalities in measure spaces which are natural generalizations of special Bihari type integral inequalities. Explicit upper bounds for the solutions are given. The classical arguments can not be extended to this more general situation, we develop new methods. The results are applied to establish the existence of a solution to the corresponding integral equations.

1. Introduction and the main results

Let $(X,\mathscr{A},\mu)$ be a measure space. $\mathscr{A}$ represents a σ-algebra in the set X. The product of $(X,\mathscr{A},\mu)$ with itself is understood as in [4], and it is denoted by $(X^2,\mathscr{A}^2,\mu^2)$. We say that the function $S : X \to \mathscr{A}$ satisfies the condititon (C) if S has the following three properties (see [3])

(C1) $x \notin S(x)$, $x \in X$,

(C2) if $x_2 \in S(x_1)$, then $S(x_2) \subset S(x_1)$,

(C3) $\{(x_1,x_2) \in X^2 \mid x_2 \in S(x_1)\}$ is μ^2-measurable.

Examples of such functions can be found in [5]. For the moment we consider only two special situations.

EXAMPLE 1.1. Let $X := [0,\infty[$, and let $\mathscr{A}$ be the Lebesgue-measurable subsets of X. Suppose p, $q : X \to X$ are increasing functions such that $0 \leqslant p(x) \leqslant x$ and $x \leqslant q(x)$, $x \in X$. If the functions S_1 and S_2 are defined on X by

$$S_1(x) := [0,p(x)[\qquad \text{and} \qquad S_2(x) :=]q(x),\infty[, \tag{1.1}$$

then they satisfy the condition (C). See [5] for the proof, which is not difficult. Further, it is shown in [5] that the previous examples also make sense in $\mathbb{R}^n$.

Let $(X,\mathscr{A},\mu)$ be a measure space, and let $S : X \to \mathscr{A}$ satisfy the condition (C). In this paper we study integral inequalities of the form

$$y(x) \leqslant f(x) + g(x) \int_{S(x)} y^{\alpha} d\mu, \quad x \in X, \tag{1.2}$$

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and the corresponding integral equations

$$y(x) = f(x) + g(x) \int\limits_{S(x)} y^{\alpha} d\mu, \quad x \in X, \tag{1.3}$$

where $\alpha > 1$, $y : D_y(\subset X) \to \mathbb{R}$ and f, $g : X \to \mathbb{R}$. As the examples just considered show the inequalities (1.2) are related to Bihari type integral inequalities (see [1] and [8]), which have many significant applications, for example, to differential equations, integral equations and difference equations. The equations (1.3) correspond to Volterra type integral equations, which play an important role in practical applications (see [2]).

Before turning to the principal problem of this paper we need some definitions. If v is a function and A is a subset of the domain of v, we denote by $v|A$ the restriction of v to A.

DEFINITION 1.1. Let $(X, \mathscr{A}, \mu)$ be a measure space, let $S : X \to \mathscr{A}$ satisfy the condition (C), and let $\alpha \geqslant 1$.

(a) Let A be a nonempty set from $\mathscr{A}$.

$$\mathscr{L}^{\alpha}(A) := \Big\{ v : A \to \mathbb{R} \mid v \text{ is } \mu\text{-almost measurable on } A, \\ \text{and } |v|^{\alpha} \text{ is } \mu\text{-integrable over } A \Big\}$$

(b) Let A be a nonempty subset of X such that $S(x) \subset A$ for every $x \in A$.

$$\mathscr{L}^{\alpha}_{loc}(A) := \Big\{ v : A \to \mathbb{R} \mid v|S(x) \in \mathscr{L}^{\alpha}(S(x)) \text{ for every} \\ x \in A \text{ with } S(x) \neq \emptyset \Big\}$$

(c) In case $v : D_v(\subset X) \to \mathbb{R}$ the notation $v \in \mathscr{L}^{\alpha}(A)$, or $v \in \mathscr{L}^{\alpha}_{loc}(A)$ mean that $v|A \in \mathscr{L}^{\alpha}(A)$, or $v|A \in \mathscr{L}^{\alpha}_{loc}(A)$.

We are now in a position to define the concept of the solutions of (1.2) and (1.3).

DEFINITION 1.2. We say that a function $y : D_y \to \mathbb{R}$ is a solution of (1.2) or (1.3) if

(i) D_y is a nonempty subset of X such that $S(x) \subset D_y$ for every $x \in D_y$,
(ii) y is nonnegative and $y \in \mathscr{L}^{\alpha}_{loc}(D_y)$,
(iii) y satisfies (1.2) or (1.3) for every $x \in D_y$.

The main goal of this paper is to investigate whether and how one can give a function $b : D_b(\subset X) \to \mathbb{R}$ such that for every solution y of (1.2)

$$y(x) \leqslant b(x), \quad x \in D_y \cap D_b. \tag{1.4}$$

Further, we would like to choose the domain of b as large a subset of X as possible.

We stress that there may be no such function with domain X. To illustrate this, consider the integral inequality

$$y(x) \leqslant 1 + \int\limits_{0}^{x} y^2, \quad x \in [0, \infty[, \tag{1.5}$$

where classical Lebesgue-integral is used. Taking $p(x) := x$, $x \in [0,\infty[$ in (1.1), we can see that (1.5) is a special case of (1.2). It is easily verified that the functions

$$y_n : [0,\infty[\to \mathbb{R}, \quad y_n(x) := \begin{cases} \frac{1}{1-x}, & \text{if} \quad 0 \leqslant x \leqslant 1 - \frac{1}{n} \\ n, & \text{if} \quad 1 - \frac{1}{n} < x \end{cases}, \quad n \in \mathbb{N}^+$$

are solutions of (1.5), thus the domain of every function b satisfying (1.4) must be a subset of $[0,1[$.

In case of $0 < \alpha \leqslant 1$, the analogous problem is quite satisfactorily solved (see [3] and [6]), and in the results $D_b = X$ can be chosen. We can see that in passing from the case $0 < \alpha \leqslant 1$ to the case $\alpha > 1$ the important property of the former that b can be defined on X is lost.

The main results can be more simply formulated if we introduce some shorthand terminology. Suppose $A \in \mathscr{A}$ is a nonempty set such that f and g are nonnegative on A and f, $g \in \mathscr{L}^\alpha(A)$. Then

$$a(A) := \left(\int_A f^\alpha d\mu \right)^{\frac{1}{\alpha}}, \quad b(A) := \left(\int_A g^\alpha d\mu \right)^{\frac{1}{\alpha}}.$$

If $y : D_y \to \mathbb{R}$ is a solution of (1.2), and if $A \in \mathscr{A}$ is a nonempty subset of D_y such that $y \in \mathscr{L}^\alpha(A)$, then

$$z(A) := \int_A y^\alpha d\mu.$$

Lemma 2.1 will show that the equation

$$z = (a(A) + b(A)z)^\alpha, \quad z \geqslant -\frac{a(A)}{b(A)} \quad \text{if } b(A) > 0, \text{ and } z \in [0,\infty[\text{ otherwise}$$

has at most two solutions which are denoted by $z_1(A)$ and $z_2(A)$. They are arranged in order: $z_1(A) < z_2(A)$. If $A = S(x)$ for some $x \in X$, then

$$a(x) := a(S(x)), \quad b(x) := b(S(x)), \quad z(x) := z(S(x))$$
$$z_1(x) := z_1(S(x)), \quad z_2(x) := z_2(S(x)).$$

After these preparations we are in a position to formulate our first result.

THEOREM 1.1. *Let $y : D_y \to \mathbb{R}$ be a solution of (1.2), let f and g be nonnegative on D_y, let f, $g \in \mathscr{L}^\alpha_{loc}(D_y)$, and let*

$$D := \left\{ x \in D_y \mid a^{\alpha-1}(x)b(x) < \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1} \right\}.$$

(a) If $a(x)b(x) = 0$ for every $x \in D_y$, then $D = D_y$.

(b) If $a(x_0)b(x_0) > 0$ for some $x_0 \in D_y$, then D contains a measurable set with positive μ-measure.

(c) For every $x \in D$

$$y(x) \leqslant f(x) + g(x) z_1(x) \leqslant f(x) + g(x)\left(a(x) + b(x)a^{\alpha}(x)\left(1 - \frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}. \tag{1.6}$$

(d) The function defined on D *by the right hand side of (1.6) belongs to* $\mathscr{L}^{\alpha}_{loc}(D)$.

The proof of the previous theorem is by no means a trivial task. In the generality presented here, the standard proofs of similar results, which transform the considered integral inequalities to differential inequalities, can not be applied. The significance of our approach lies in the following facts: the classical problem has been formulated and solved in measure spaces which is an essential generalization; the result can be used in all cases which was investigated previously (inequalities of type (1.2) in $\mathbb{R}^n$, or discrete analogues of (1.2)).

We give another form of the previous result.

THEOREM 1.2. *Consider the integral inequality (1.2). Let* f *and* g *be nonnegative on* X, *let* f, $g \in \mathscr{L}^{\alpha}_{loc}(X)$, *and let*

$$D := \left\{ x \in X \mid a^{\alpha-1}(x)b(x) < \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1} \right\}.$$

(a) The function

$$x \to f(x) + g(x)\left(a(x) + b(x)a^{\alpha}(x)\left(1 - \frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}, \quad x \in D$$

belongs to $\mathscr{L}^{\alpha}_{loc}(D)$.

For every solution $y : D_y \to \mathbb{R}$ *of (1.2)*

(b) If $a(x)b(x) = 0$ *for every* $x \in D_y$, *then* $D_y \subset D$.

(c) If $a(x_0)b(x_0) > 0$ *for some* $x_0 \in D_y$, *then* $D \cap D_y$ *contains a measurable set with positive* μ*-measure.*

(d)

$$y(x) \leqslant f(x) + g(x) z_1(x) \leqslant f(x) + g(x)\left(a(x) + b(x)a^{\alpha}(x)\left(1 - \frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}, \quad x \in D \cap D_y.$$

Regarding the integral equation (1.3) we assert

THEOREM 1.3. *Let* A *be a nonempty subset of* X *such that* $S(x) \subset A$ *for every* $x \in A$, *let* f *and* g *be nonnegative on* A, *let* f, $g \in \mathscr{L}^{\alpha}_{loc}(A)$, *and let*

$$D := \left\{ x \in A \mid a^{\alpha-1}(x)b(x) < \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1} \right\}.$$

Then the integral equation (1.3) has a solution on D.

This assertion contains the result of Theorem 23 in [7], but it gives a new treatment of the problem employing Theorem 1.1 and successive approximations. In addition, we show that there exists a solution of (1.3) on D, while Theorem 23 in [7] implies only the existence of a solution.

2. Preliminary results

We collect here some results that will be used in the proofs of the main theorems. The first lemma is fundamental to our treatment.

LEMMA 2.1. *Let $a,\ b \geqslant 0$, and let $\alpha > 1$.*

(a) Consider the algebraical equation

$$z = (a+bz)^{\alpha}, \tag{2.1}$$

where $z \geqslant -\frac{a}{b}$ if $b>0$, and $z \in [0,\infty[$ otherwise.

(a1) (2.1) has at most two solutions which are denoted by $z_1 = z_1(a,b) \leqslant z_2 = z_2(a,b)$.

(a2) There are exactly two solutions of (2.1) if and only if

$$b > 0 \quad \text{and} \quad a^{\alpha-1}b < \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}. \tag{2.2}$$

In case $a>0$, $0<z_1<z_2$. When $a=0$, then $0=z_1<z_2=b^{-\frac{\alpha}{\alpha-1}}$.

(a3) There is exactly one solution of (2.1) if and only if

$$b = 0 \quad \text{or} \quad a^{\alpha-1}b = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}. \tag{2.3}$$

In the first case, $0 \leqslant z_1 = a^{\alpha}$, and in the other case, $0 < z_1$.

(a4) (2.1) has no solution if and only if

$$a^{\alpha-1}b > \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}. \tag{2.4}$$

(b) Consider the algebraical inequality

$$z \leqslant (a+bz)^{\alpha}, \tag{2.5}$$

where $z \geqslant -\frac{a}{b}$ if $b>0$, and $z \in [0,\infty[$ otherwise.

(b1) Suppose that the condition (2.2) is fulfilled. Then (2.5) holds if and only if $z \in [-\frac{a}{b}, z_1] \cup [z_2,\infty[$.

(b2) In case $b=0$, (2.5) holds if and only if $z \in [0,z_1]$.

(b3) Suppose that the conditions (2.4), or the second part in (2.3) are fulfilled. Then (2.5) holds if and only if $z \in [-\frac{a}{b},\infty[$.

(c) If there is a solution of (2.1), then

$$(a+ba^{\alpha})^{\alpha} \leqslant z_1 \leqslant \left(a+ba^{\alpha}\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}.$$

If there are two solutions of (2.1), then

$$\left(a+ba^{\alpha}\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}<z_2.$$

(d) Let $0\leqslant c\leqslant a$*, and* $0\leqslant d\leqslant b$*. Consider the algebraical equation*

$$z=(c+dz)^{\alpha}, \tag{2.6}$$

where $z\geqslant -\frac{c}{d}$ *if* $d>0$*, and* $z\in[0,\infty[$ *otherwise.*

(d1) If there is a solution of (2.1), then (2.6) also has a solution and $z_1(c,d)\leqslant z_1(a,b)$.

(d2) If there are two solutions of (2.1) and (2.6), respectively, then $z_2(a,b)\leqslant z_2(c,d)$.

Proof. (a) If $a=0$ or $b=0$, then the roots of (2.1) are easily determined. Suppose a, $b>0$, and let

$$f:[-\frac{a}{b},\infty[\to\mathbb{R},\quad f(z):=(a+bz)^{\alpha}-z.$$

The function f is continuous, it has derivatives of all orders on $]-\frac{a}{b},\infty[$, and

$$f'(z)=\alpha b\,(a+bz)^{\alpha-1}-1,\quad f''(z)=\alpha(\alpha-1)b^2\,(a+bz)^{\alpha-2},\quad z>-\frac{a}{b}.$$

The equation $f'(z)=0$ has exactly one solution, namely

$$z_0=\frac{1}{b}\left(\left(\frac{1}{\alpha b}\right)^{\frac{1}{\alpha-1}}-a\right).$$

Since $f''(z)>0$ for all $z\in]-\frac{a}{b},\infty[$, it follows that f is strictly decreasing on $[-\frac{a}{b},z_0]$, strictly increasing on $[z_0,\infty[$, and z_0 is a strict local minimum of f. This implies that if a, $b>0$, then there are at most two solutions of (2.1) and

i. there are exactly two solutions of (2.1) if and only if

$$f(z_0)=\left(\frac{1}{\alpha b}\right)^{\frac{\alpha}{\alpha-1}}-\frac{1}{b}\left(\left(\frac{1}{\alpha b}\right)^{\frac{1}{\alpha-1}}-a\right)<0,$$

that is

$$a^{\alpha-1}b<\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1},$$

ii. there is exactly one solution of (2.1) if and only if $f(z_0)=0$, that is

$$a^{\alpha-1}b=\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1},$$

iii. (2.1) has no solution if and only if $f(z_0)>0$, that is

$$a^{\alpha-1}b>\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}.$$

In cases i. or ii., $f(z_0) = (a+bz_0)^\alpha - z_0 \leqslant 0$. Hence $z_0 > 0$, and therefore, by $f(0) = a^\alpha > 0$, $z_1 > 0$.

(b) This is an immediate consequence of (a).

(c) If $a = 0$ or $b = 0$, then, by (a1) and (a2)

$$(a+ba^\alpha)^\alpha = z_1 = \left(a + ba^\alpha\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^\alpha < z_2$$

whenever z_2 exists.

Suppose a, $b > 0$, and suppose that there are two solutions of (2.1). By the proof of (a), it is enough to show that

$$f\left((a+ba^\alpha)^\alpha\right) > 0 \quad \text{and} \quad f\left(\left(a + ba^\alpha\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^\alpha\right) < 0. \tag{2.7}$$

The first inequality in (2.7) is equivalent to the inequality

$$\left(a + b\left(a+ba^\alpha\right)^\alpha\right)^\alpha > \left(a+ba^\alpha\right)^\alpha$$

which is obvious. The other inequality in (2.7) is equivalent to the inequality

$$ba^{\alpha-1} < \frac{1}{\alpha-1}\left(1-\frac{1}{\alpha}\right)^\alpha$$

which follows from (2.2).

Suppose a, $b > 0$, and suppose that there is only one solution of (2.1). Then $z_1 = z_0$, by the proof of (a). By applying the second condition in (2.3), we can verify easily that

$$z_0 = \left(a + ba^\alpha\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^\alpha.$$

(d) Since $c^{\alpha-1}d \leqslant a^{\alpha-1}b$, it therefore follows from (a4) that, if there exists a solution of (2.1), then there exists a solution of (2.6) too. The remaining results come from (a) and the inequality

$$(a+bz)^\alpha - z \geqslant (c+dz)^\alpha - z, \quad z \in [0,\infty[.$$

$\square$

We introduce now some notations. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $S : X \to \mathscr{A}$ satisfy the condition (C). Suppose A is a nonempty element of $\mathscr{A}$. $(A, \mathscr{A}_A, \mu_A)$ means the following measure space: $\mathscr{A}_A$ is the trace of $\mathscr{A}$ in A, and μ_A is the restriction of μ to $\mathscr{A}_A$. The function S_A is defined on A by $S_A(x) := A \cap S(x)$. If $A = S(x_0)$ for some $x_0 \in X$, then $\mathscr{A}_{x_0} := \mathscr{A}_{S(x_0)}$, $\mu_{x_0} := \mu_{S(x_0)}$, and $S_{x_0} := S_{S(x_0)}$.

LEMMA 2.2. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $S : X \to \mathscr{A}$ satisfy the condition (C). If A is a nonempty element of $\mathscr{A}$, then S_A satisfies the condition (C).*

Proof. It is easy to check, and we omit the details. □

The following theorem is taken from [[7], Theorem 5].

THEOREM 2.3. *Let $(Y, \mathscr{B}, \nu)$ be a measure space, and let $S : Y \to \mathscr{B}$ satisfy the condition (C). Suppose $\nu(Y) > 0$, and $\nu(B) = 0$ for every measurable subset B of $N := \{x \in Y \mid \nu(S(x)) = 0\}$. If $p : Y \to \mathbb{R}$ is nonnegative and ν-integrable over $S(x)$ for every $x \in Y$, then for each $\varepsilon > 0$ there is $x_\varepsilon \in Y$ such that $\nu(S(x_\varepsilon)) > 0$ and $\int\limits_{S(x_\varepsilon)} p d\nu < \varepsilon$.*

The next result has been given in [[6], Lemma 5 (b)].

LEMMA 2.4. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let $S : X \to \mathscr{A}$ satisfy (C3), and let $A \in \mathscr{A}$ such that $S(x) \subset A$ for every $x \in A$. Suppose $p : A \to \mathbb{R}$ is μ-integrable over A, $q : A \to \mathbb{R}$ is μ-almost measurable on A, and there exists a measurable subset C of A such that $\mu(C)$ is σ-finite and $q(x) = 0$ for all $x \in A \setminus C$. Then the function*

$$x \to q(x) \int\limits_{S(x)} p d\mu, \quad x \in A$$

is μ-almost measurable on A.

3. The proofs of the main results

The proof of Theorem 1.1.

Proof. (a) It follows from the definition of D.
(b) The hypothesis implies that $a(x_0) > 0$, so that $\mu(S(x_0)) > 0$. Let

$$N := \{x \in S(x_0) \mid \mu(S(x)) = 0\}.$$

It is obvious that $N \subset D$. Hence if N contains a measurable set with positive μ-measure, then D also has this property, as required. Suppose now that $\mu(B) = 0$ for every measurable subset B of N. This condition and Lemma 2.2 ensure that Theorem 2.3 can be applied with the measure space $(S(x_0), \mathscr{A}_{x_0}, \mu_{x_0})$ and with the function $S_{x_0} : S(x_0) \to \mathscr{A}_{x_0}$, and hence there is $x_1 \in S(x_0)$ such that $\mu(S(x_1)) = \mu_{x_0}(S_{x_0}(x_1)) > 0$ and

$$b(x_1) < a^{1-\alpha}(x_0)\frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1}.$$

It therefore follows from the definition of $a(x)$ and $b(x)$ that

$$b(x)a^{\alpha-1}(x) \leqslant b(x_1)a^{\alpha-1}(x_1) \leqslant b(x_1)a^{\alpha-1}(x_0) < \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad x \in S(x_1).$$

This gives that $S(x_1) \subset D$, thus $S(x_1)$ is a required set.
(c) We separate the proof into four steps.

1. If $A \in \mathscr{A}$ such that $A \subset D_y$ and $S(x) \subset A$ for every $x \in D_y$, and if f, g, $y \in \mathscr{L}^{\alpha}(A)$, then

$$z(A) \leqslant (a(A) + b(A)z(A))^{\alpha}. \tag{3.1}$$

$z(A) = 0$ obviously implies (3.1). We assume now that $z(A) > 0$. Then by (1.2),

$$y^{\alpha}(x) \leqslant y^{\alpha-1}(x)f(x) + y^{\alpha-1}(x)g(x)\int_A y^{\alpha}d\mu, \quad x \in A. \tag{3.2}$$

Since

$$y^{\alpha-1} \in \mathscr{L}^{\frac{\alpha}{\alpha-1}}(A) \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\frac{\alpha}{\alpha-1}} = 1,$$

then $y^{\alpha-1}f$, $y^{\alpha-1}g \in \mathscr{L}^1(A)$. From (3.2) we therefore deduce that

$$\int_A y^{\alpha}d\mu \leqslant \int_A y^{\alpha-1}f\,d\mu + \int_A y^{\alpha-1}g\,d\mu \int_A y^{\alpha}d\mu,$$

and hence, by the Hölder's inequality,

$$\int_A y^{\alpha}d\mu \leqslant \left(\int_A y^{\alpha}d\mu\right)^{\frac{\alpha-1}{\alpha}} \left(\int_A f^{\alpha}d\mu\right)^{\frac{1}{\alpha}} + \left(\int_A y^{\alpha}d\mu\right)^{\frac{\alpha-1}{\alpha}} \left(\int_A g^{\alpha}d\mu\right)^{\frac{1}{\alpha}} \int_A y^{\alpha}d\mu.$$

Dividing both sides by $z^{\frac{\alpha-1}{\alpha}}(A)$, we have

$$z^{\frac{1}{\alpha}}(A) \leqslant a(A) + b(A)z(A),$$

and this is equivalent to (3.1).

2. Let $A \in \mathscr{A}$ such that $A \subset D_y$, $S(x) \subset A$ for every $x \in A$, f, g, $y \in \mathscr{L}^{\alpha}(A)$, and

$$b(A)a^{\alpha-1}(A) < \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1}. \tag{3.3}$$

If

$$z(x) \leqslant z_1(x) \quad \mu\text{-a.e. on } A,$$

then $z(A) \leqslant z_1(A)$.

Since

$$b(x)a^{\alpha-1}(x) \leqslant b(A)a^{\alpha-1}(A), \quad x \in A, \tag{3.4}$$

this, according to (3.3) and Lemma 2.1 (a), shows the existence of $z_1(A)$ and $z_1(x)$ for every $x \in A$.

If $z(A) = 0$, then the assertion is true, since, by Lemma 2.1 (a), $z_1(A) \geqslant 0$.

Now suppose $z(A) > 0$. Then (1.2) and Lemma 2.1 (d1) imply that

$$\begin{aligned} y^{\alpha}(x) &\leqslant y^{\alpha-1}(x)f(x) + y^{\alpha-1}(x)g(x)z_1(x) \\ &\leqslant y^{\alpha-1}(x)f(x) + y^{\alpha-1}(x)g(x)z_1(A) \quad \mu\text{-a.e. on } A. \end{aligned}$$

Hence an argument closely similar to that of the first part (the proof (3.2) $\Longrightarrow$ (3.1)) gives that

$$z(A) \leqslant (a(A)+b(A)z_1(A))^{\alpha}. \tag{3.5}$$

By using the upper bound for $z_1(A)$ established in Lemma 2.1 (c), and the inequality (3.3), we obtain that

$$\begin{aligned} z_1(A) &\leqslant \left(a(A)+b(A)a^{\alpha}(A)\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha} \\ &\leqslant \left(a(A)+a(A)\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{-1}\right)^{\alpha} = a^{\alpha}(A)\left(1-\frac{1}{\alpha}\right)^{-\alpha}. \end{aligned}$$

It therefore follows from (3.5) that

$$z(A) \leqslant \left(a(A)+b(A)a^{\alpha}(A)\left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha}.$$

The result can now be derived from (3.1), Lemma 2.1 (b), and the second part of Lemma 2.1 (c).

3. Let $A \in \mathscr{A}$ such that $A \subset D_y$, $S(x) \subset A$ for every $x \in A$, f, g, $y \in \mathscr{L}^{\alpha}(A)$, and

$$b(A)a^{\alpha-1}(A) < \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}. \tag{3.6}$$

Then

$$z(x) \leqslant z_1(x) \quad \mu\text{-a.e. on } A.$$

By part 1 applied to the set $S(x)$ $(x \in A)$, we have that

$$z(x) \leqslant (a(x)+b(x)z(x))^{\alpha}, \quad x \in A. \tag{3.7}$$

This, together with (3.4), (3.6), and Lemma 2.1 (b) gives that if $x \in A$ such that $z(x) \geqslant z_1(x)$, then $z(x) \geqslant z_2(x)$. Let

$$A_1 := \{x \in A \mid z(x) \leqslant z_1(x)\} \quad \text{and} \quad A_2 := \{x \in A \mid z(x) \geqslant z_2(x)\} = A \setminus A_1.$$

If $b(A) = 0$, then $b(x) = 0$ for every $x \in A$, and hence the inequality $z(x) \leqslant z_1(x)$, $x \in A$ can be obtained from (3.7), taking into account Lemma 2.1 (b2).

We now assume that $b(A) > 0$. Then (3.6) and Lemma 2.1 (a2) show that $z_2(A) > z_1(A)$, and therefore from Lemma 2.1 (d) we have

$$A_1 = \{x \in A \mid z(x) \leqslant z_1(A)\} \quad \text{and} \quad A_2 = \{x \in A \mid z(x) \geqslant z_2(A)\}. \tag{3.8}$$

Case 3.1. Suppose $\mu(A)$ is σ-finite.

By Lemma 2.4, the function z is μ-almost measurable on A, and thus there exists a subset B of A such that $B \in \mathscr{A}$, $\mu(A \setminus B) = 0$ and z is measurable on B. Let

$$B_1 := B \cap A_1 \quad \text{and} \quad B_2 := B \cap A_2.$$

(3.8) implies that B_1, $B_2 \in \mathscr{A}$. It is enough to show that $B_2 = \emptyset$. Suppose on the contrary that $B_2 \neq \emptyset$ holds. Consider the measure space $(B_2, \mathscr{A}_{B_2}, \mu_{B_2})$ and the function $S_{B_2} : B_2 \to \mathscr{A}_{B_2}$. First, we obviously have

$$\begin{aligned}\int\limits_{S_{B_2}(x)} y^\alpha d\mu &= \int\limits_{S(x)} y^\alpha d\mu - \int\limits_{S(x)\cap B_1} y^\alpha d\mu - \int\limits_{S(x)\cap(A\setminus B)} y^\alpha d\mu \\ &= \int\limits_{S(x)} y^\alpha d\mu - \int\limits_{S(x)\cap B_1} y^\alpha d\mu \geqslant z_2(A) - \int\limits_{S(x)\cap B_1} y^\alpha d\mu, \quad x \in B_2. \end{aligned} \tag{3.9}$$

If $S(x) \cap B_1 = \emptyset$ for some $x \in B_2$, then (3.9) gives that

$$\int\limits_{S_{B_2}(x)} y^\alpha d\mu \geqslant z_2(A) \geqslant z_2(A) - z_1(A) > 0. \tag{3.10}$$

Let $x \in B_2$ such that $S(x) \cap B_1 \neq \emptyset$. If $u \in S(x) \cap B_1$, then $S(u) \subset S(x) \cap B_1$, since $S(u) \subset A_1$ because of the definition of $z(u)$ and (3.8). This and the definition of A_1 yield that part 2 can be applied to the set $S(x) \cap B_1$, and thus we obtain

$$\int\limits_{S(x)\cap B_1} y^\alpha d\mu \leqslant z_1(S(x) \cap B_1) \leqslant z_1(A).$$

It follows from this and (3.9) that

$$\int\limits_{S_{B_2}(x)} y^\alpha d\mu \geqslant z_2(A) - z_1(A) > 0. \tag{3.11}$$

We have shown that (3.11) holds for every $x \in B_2$, hence $\mu(S_{B_2}(x)) > 0$, $x \in B_2$. But then we have from Theorem 2.3 that there is $x_0 \in B_2$ such that

$$\int\limits_{S_{B_2}(x_0)} y^\alpha d\mu < z_2(A) - z_1(A).$$

We can see that the assumption $b(A) > 0$ leads to a contradiction.

Case 3.2. Suppose $\mu(A)$ is not σ-finite.

Since f, g, $y \in \mathscr{L}^\alpha(A)$, there is a subset B of A such that $B \in \mathscr{A}$, $\mu(B)$ is σ-finite, and $f(x) = g(x) = y(x) = 0$ for every $x \in A \setminus B$. Considering the measure space $(B, \mathscr{A}_B, \mu_B)$, the function $S_B : B \to \mathscr{A}_B$, and the set B, the hypotheses of part 3 are satisfied. Since

$$\int\limits_{S_B(x)} f d\mu_B = \int\limits_{S(x)} f d\mu, \quad \int\limits_{S_B(x)} g d\mu_B = \int\limits_{S(x)} g d\mu, \quad \int\limits_{S_B(x)} y d\mu_B = \int\limits_{S(x)} g d\mu, \quad x \in B,$$

it follows from the case 3.1 that $z(x) \leqslant z_1(x)$ μ-a.e. on B.

If $x \in A \setminus B$ such that $S(x) \cap B = \emptyset$, then $z(x) = 0 \leqslant z_1(x)$. Suppose $x \in A \setminus B$ such that $S(x) \cap B \neq \emptyset$. Clearly

$$S_B(u) = S(u) \cap B \subset S(x) \cap B, \quad u \in S(x) \cap B,$$

thus considering the measure space $(B, \mathscr{A}_B, \mu_B)$, the function $S_B : B \to \mathscr{A}_B$, and the set $S(x) \cap B$, the hypotheses of part 2 are satisfied, and therefore it implies that

$$z(x) \leqslant z(S(x) \cap B) \leqslant z_1(S(x) \cap B) \leqslant z_1(x).$$

4. Proof of the inequality (1.6).

Let $x \in D$. By part 3 and part 2 with the set $S(x)$, the inequality $z(x) \leqslant z_1(x)$ holds, hence the first inequality in (1.6) follows from (1.2), while the second inequality in (1.6) can be obtained by Lemma 2.1 (c).

(d) The definition of D implies that $S(x) \subset D$ for every $x \in D$. We must show that for all $x \in D$ with $S(x) \neq \emptyset$, the α-th power of the function

$$\begin{aligned} u \to f(u)+g(u)\left(\left(\int_{S(u)} f^{\alpha} d\mu\right)^{\frac{1}{\alpha}} + \left(\int_{S(u)} g^{\alpha} d\mu\right)^{\frac{1}{\alpha}} \left(\int_{S(u)} f^{\alpha} d\mu\right)^{1-\frac{1}{\alpha}} \left(1-\frac{1}{\alpha}\right)^{-\alpha}\right)^{\alpha} \\ = f(u) + h(u), \quad u \in D \qquad (3.12) \end{aligned}$$

is μ-integrable over $S(x)$.

Let $x \in D$ with $S(x) \neq \emptyset$ be fixed. We prove first that the function (3.12) is μ-almost measurable on $S(x)$. It suffices to show that the functions

$$u \to g^{\alpha}(u) \int_{S(u)} f^{\alpha} d\mu, \quad u \in S(x) \qquad (3.13)$$

and

$$u \to g^{\alpha}(u) \int_{S(u)} g^{\alpha} d\mu \left(\int_{S(u)} f^{\alpha} d\mu\right)^{\alpha-1}, \quad u \in S(x) \qquad (3.14)$$

are μ-almost measurable on $S(x)$. This is an immediate consequence of Lemma 2.4 for the function (3.13). Since the function $g^{\frac{\alpha}{\alpha-1}}$ is μ-almost measurable on $S(x)$ and it vanishes on $S(x)$ except on a set of σ-finite μ-measure, the function

$$u \to \left(g^{\frac{\alpha}{\alpha-1}}(u) \int_{S(u)} f^{\alpha} d\mu\right)^{\alpha-1}, \quad u \in S(x)$$

is μ-almost measurable on $S(x)$, again by Lemma 2.4. Clearly this function also vanishes on $S(x)$ except on a set of σ-finite μ-measure, hence Lemma 2.4 implies that the function (3.14) is μ-almost measurable on $S(x)$.

We observe next that $h \in \mathscr{L}^{\alpha}(S(x))$. By using [[6], Lemma 1 (b)], we obtain that

$$h^{\alpha}(x) \leqslant \left(2^{\alpha-1}\left(g(u)\int\limits_{S(u)} f^{\alpha}d\mu + g(u)\int\limits_{S(u)} g^{\alpha}d\mu\left(\int\limits_{S(u)} f^{\alpha}d\mu\right)^{\alpha-1}\left(1-\frac{1}{\alpha}\right)^{-\alpha^2}\right)\right)^{\alpha}$$

$$\leqslant 2^{(\alpha-1)\alpha}2^{\alpha-1}\left(g^{\alpha}(u)\left(\int\limits_{S(x)} f^{\alpha}d\mu\right)^{\alpha}\right.$$

$$\left. + g^{\alpha}(u)\left(\int\limits_{S(x)} g^{\alpha}d\mu\right)^{\alpha}\left(\int\limits_{S(x)} f^{\alpha}d\mu\right)^{\alpha(\alpha-1)}\left(1-\frac{1}{\alpha}\right)^{-\alpha^3}\right), \quad u \in S(x),$$

and this implies the statement.

The proof of the theorem is now complete. □

The proof of Theorem 1.2.

Proof. (a) This can be confirmed exactly as Theorem 1.1 (d).
(b) It is obvious.
(c) and (d) follow from Theorem 1.1 (b) and (c). □

The proof of Theorem (1.3).

Proof. Define the following successive approximations:

$$\begin{aligned} y_0(x) &:= 0, \quad x \in D \\ y_{n+1}(x) &:= f(x) + g(x)\int\limits_{S(x)} y_n^{\alpha}d\mu, \quad x \in D. \end{aligned} \tag{3.15}$$

We show first that the sequence $(y_n)_{n=0}^{\infty}$ is well defined, which means that $y_n \in \mathscr{L}_{loc}^{\alpha}(D)$, $n \in \mathbb{N}$. Here the case $n = 0$ is immediate. Suppose then that $n \in \mathbb{N}$ for which the result holds, and let $x \in D$ with $S(x) \neq \emptyset$. It is enough to prove that the function

$$u \rightarrow g(u)\int\limits_{S(u)} y_n^{\alpha}d\mu, \quad u \in S(x) \tag{3.16}$$

belongs to $\mathscr{L}^{\alpha}(S(x))$. By Lemma (2.4), the function (3.16) is μ-almost measurable on $S(x)$, and therefore the inequality

$$\left(g(u)\int\limits_{S(u)} y_n^{\alpha}d\mu\right)^{\alpha} \leqslant g^{\alpha}(u)\left(\int\limits_{S(x)} y_n^{\alpha}d\mu\right)^{\alpha}, \quad u \in S(x)$$

yields that the function lies in $\mathscr{L}^{\alpha}(S(x))$.

We observe next that $(y_n)_{n=0}^{\infty}$ is increasing. The inequality $y_0 \leqslant y_1$ is simple. Suppose $n \in \mathbb{N}$ such that $y_n \leqslant y_{n+1}$. Then

$$y_{n+1}(x) = f(x) + g(x)\int\limits_{S(x)} y_n^{\alpha}d\mu \leqslant f(x) + g(x)\int\limits_{S(x)} y_{n+1}^{\alpha}d\mu = y_{n+2}(x), \quad x \in D.$$

Since

$$y_n(x) \leqslant y_{n+1}(x) = f(x) + g(x) \int\limits_{S(x)} y_n^\alpha d\mu, \quad x \in D,$$

y_n is a solution of (1.2), and by Theorem 1.1 (c), this gives

$$y_n(x) \leqslant f(x) + g(x) \left(a(x) + b(x) a^\alpha(x) \left(1 - \frac{1}{\alpha}\right)^{-\alpha} \right)^\alpha, \quad x \in D, \quad n \in \mathbb{N}.$$

It follows from Theorem 1.1 (d) that the upper bound for y_n belongs to $\mathscr{L}^\alpha_{loc}(D)$. We can see now that $(y_n)_{n=0}^\infty$ converges pointwise on D to a function $y \in \mathscr{L}^\alpha_{loc}(D)$ which is a solution of (1.3), by (3.15). □

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