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CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS ASSOCIATED WITH A CONVOLUTION STRUCTURE

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Abstract. Making use of a convolution structure, we introduce a new class of analytic functions $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ defined in the open unit disc and investigate its various characteristics. Further we obtained distortion bounds , extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$.

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic and univalent in the open disc $U=\{z:z\in\mathscr{C};\ |z|<1\}$. For functions $f\in A$ given by (1.1) and $g\in A$ given by $g(z)=z+\sum\limits_{k=2}^{\infty}b_kz^k$, we define the Hadamard product (or Convolution) of f and g by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k,\ \ z\in U. \tag{1.2}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in $A(n)$ such that $(f*g)(z)$ exists for any $f\in A(n)$, and for various choices of g we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (1.2), we consider the following examples.

(1) If

$$g(z)=z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}\dots(\alpha_l)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_m)_{k-1}}\frac{z^k}{(k-1)!}=z+\sum_{k=2}^{\infty}\Gamma_kz^k, \tag{1.3}$$

where

$$\Gamma_k=\frac{(\alpha_1)_{k-1}\dots(\alpha_p)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_q)_{k-1}}\frac{1}{(k-1)!},$$

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then the convolution (1.3) gives the Dziok–Srivastava operator [6]:

$$\Lambda(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z)f(z) \equiv \mathscr{H}_m^l(\alpha_1,\beta_1)f(z),$$

where $\alpha_1,\cdots,\alpha_l$; $\beta_1,\cdots,\beta_m$ are positive real numbers, $l \leqslant m+1$; $l,m \in \mathbb{N}\cup\{0\}$, and $(a)_k$ denotes the familiar Pochhammer symbol (or shifted factorial).

REMARK 1. When $l=1,\ m=1;\ \alpha_1=a,\ \alpha_2=1;\ \beta_1=c$, then the Dizok-Srivastava operator (1.4) corresponds to the operator due to Carlson-Shaffer operator [3] given by

$$\mathscr{L}(a,c)f(z) := (f*g)(z),$$

where

$$\mathscr{L}(a,c)f(z) \equiv zF(a,1;c;z)*f(z) := z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^k \quad (c \neq 0,-1,-2,\cdots), \quad (1.4)$$

and $F(a,b;c;z)$ is the well known Gaussian hypergeometric function.

REMARK 2. When $l=1$, $m=0$; $\alpha_1=\nu+1$, $\alpha_2=1$; $\beta_1=1$, then the the above Dizok-Srivastava operator yields the Ruscheweyh derivative operator [13] given by

$$D^{\nu}f(z) := (f*g)(z) = z+\sum_{k=2}^{\infty}\binom{\nu+k-1}{k-1}a_k z^k. \quad (1.5)$$

(2) Furthermore, if

$$g(z) = z+\sum_{k=2}^{\infty}k\left(\frac{k+l}{1+l}\right)^m z^k \quad (\sigma \geqslant 0;\ m \in \mathbb{Z}), \quad (1.6)$$

then the convolution (1.3) yields the operator $\mathscr{I}(l,m)f:\ A(n) \longrightarrow A(n)$ which was studied by Cho and Kim [5](see also [4]).

(3) Lastly, if

$$g(z) = z+\sum_{k=2}^{\infty}\left(\frac{k+l}{1+l}\right)^m z^k \quad (m \geqslant 0;\ l \in \mathbb{Z}), \quad (1.7)$$

then (1.3) gives the multiplier transformation $\mathscr{J}(l,m)f:\ A(n) \longrightarrow A(n)$, which was introduced by Cho and Srivastava [4].

REMARK 3. For $l=0$, the operator defined with (1.7) gives the *Sălăgean operator*

$$\mathscr{D}^m f(z) := z+\sum_{k=2}^{\infty}k^m z^k \quad (m \geqslant 0), \quad (1.8)$$

which was initially studied by Sălăgean [10].

For $\lambda \geqslant 0,\ 0 \leqslant \alpha < 1$ and $0<\beta \leqslant 1$, we let $\mathbb{P}_g(\lambda,\alpha,\beta,\gamma)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\left|\frac{\mathbb{J}_{g,\lambda}(z)-1}{2\gamma(\mathbb{J}_{g,\lambda}(z)-\alpha)-(\mathbb{J}_{g,\lambda}(z)-1)}\right| < \beta \quad (1.9)$$

where

$$\mathbb{J}_{g,\lambda}(z) = \frac{(1-\lambda)(f*g)+\lambda z(f*g)'}{z}, \tag{1.10}$$

$0<\gamma\leqslant 1,$ $(f*g)(z)$ is given by (1.2) and g is fixed function for all $z\in U$. We further let $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)=\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)\cap T,$ where

$$T \;:=\; \left\{ f\in A : f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\;\; z\in U \right\} \tag{1.11}$$

is a subclass of A introduced and studied by Silverman [12]. For various choices of $g(z)$ we get different operators and are listed below.

When $g(z)$ is as defined in (1.3), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{H}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{H,\lambda}(z) = \frac{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}{z}.$$

When $g(z)$ is as defined in (1.4), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{L}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{L,\lambda}(z) = \frac{(1-\lambda)L(a,c)f(z)+\lambda z(L(a,c)f(z))'}{z}.$$

When $g(z)$ is as defined in (1.5), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{R}\mathscr{T}^{\nu}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{\upsilon,\lambda}(z) = \frac{(1-\lambda)D^{\upsilon}f(z)+\lambda z(D^{\upsilon}f(z))'}{z}.$$

When $g(z)$ is as defined in (1.6), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{I}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{l,\lambda}(z) = \frac{(1-\lambda)\mathscr{I}(m,l)f(z)+\lambda z(\mathscr{I}(m,l))'}{z}.$$

When $g(z)$ is as defined in (1.7), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{J}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{n,\lambda}(z) = \frac{(1-\lambda)\mathscr{J}(l,m)f(z)+\lambda z(\mathscr{J}(l,m))'}{z}.$$

When $g(z)$ is as defined in (1.8), the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ reduces to the subclass $\mathscr{S}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ with (1.9) and

$$\mathbb{J}_{m,\lambda}(z) = \frac{(1-\lambda)\mathscr{D}^m f(z)+\lambda\mathscr{D}^{m+1}f(z)}{z}.$$

Furthermore, by suitably specializing the values of $g,$ $\alpha,$ $\beta,$ γ and $\lambda,$ the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ and the above subclasses reduce to the various subclasses introduced and studied in [1, 2, 7, 8, 9, 11, 14].

In the following section we obtain coefficient estimates and extreme points for the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$.

2. Coefficient Bounds

THEOREM 2.1. *Let the function f be defined by* (1.11)*. Then $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\, b_k \leqslant 2\beta\gamma(1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z)=z-\frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}z^k, \quad k\geqslant 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $|z|$,

$$\begin{aligned}
&\left|\mathbb{J}_{g,\lambda}(z)-1\right|-\beta\left|2\gamma(\mathbb{J}_{g,\lambda}(z)-\alpha)-(\mathbb{J}_{g,\lambda}(z)-1)\right|\\
&=\left|-\sum_{k=2}^{\infty}(1+\lambda(k-1))a_kb_kz^{k-1}\right|-\beta\left|2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)a_kb_kz^{k-1}\right|\\
&\leqslant\sum_{k=2}^{\infty}(1+\lambda(k-1))a_k\, b_k-2\beta\gamma(1-\alpha)+\sum_{k=2}^{\infty}(1+\lambda(k-1))\beta(2\gamma-1)a_k\, b_k\\
&=\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\, b_k-2\beta\gamma(1-\alpha)\\
&\leqslant 0, \quad \text{by}\,(2.1).
\end{aligned}$$

Hence, by maximum modulus theorem and (1.9), $f\in\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$. To prove the converse, assume that

$$\left|\frac{\mathbb{J}_{g,\lambda}(z)-1}{2\gamma(\mathbb{J}_{g,\lambda}(z)-\alpha)-(\mathbb{J}_{g,\lambda}(z)-1)}\right|=\left|\frac{-\sum_{k=2}^{\infty}(1+\lambda(k-1))a_kb_kz^{k-1}}{2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)a_kb_kz^{k-1}}\right|$$
$$\leqslant\beta, \quad z\in U.$$

Or, equivalently,

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty}(1+\lambda(k-1))a_kb_kz^{k-1}}{2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)a_kb_kz^{k-1}}\right\}<\beta. \tag{2.3}$$

Since $\operatorname{Re}(z)\leqslant|z|$ for all z. Choose values of z on the real axis so that $\mathbb{J}_{g,\lambda}(z)$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1$ through real values, we obtain the desired inequality (2.1). $\square$

COROLLARY 2.2. *If $f(z)$ of the form (1.11) is in $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$, then*

$$a_k \leqslant \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}, \quad k \geqslant 2, \tag{2.4}$$

with equality only for functions of the form (2.2).

Corresponding to the various subclasses which arise from the function class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$, by suitably choosing the function $g(z)$ as mentioned in (1.3) to (1.8), we arrive at the following corollaries giving the coefficient bound inequalities for these subclasses of functions.

COROLLARY 2.3. *A function $f \in \mathscr{H}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \Gamma_n(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k \leqslant 2\beta\gamma(1-\alpha).$$

where

$$\Gamma_n = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1},\ldots,(\alpha_l)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1},\ldots,(\beta_m)_{k-1}(1)_{k-1}} \tag{2.5}$$

REMARK 4. For specific choices of parameters l,m,α_1,β_1 (as mentioned in the Remarks 1 and 2), Corollary 2.3 would yield the coefficient bound inequalities for the subclasses of functions $\mathscr{L}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ and $\mathscr{R}\mathscr{T}^{\nu}(\lambda,\alpha,\beta,\gamma))$.

COROLLARY 2.4. *A function $f \in \mathscr{I}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]k\left(\frac{k+l}{1+l}\right)^m a_k \leqslant 2\beta\gamma(1-\alpha).$$

COROLLARY 2.5. *A function $f \in \mathscr{J}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]\left(\frac{k+l}{1+l}\right)^m a_k \leqslant 2\beta\gamma(1-\alpha).$$

REMARK 5. When $l=0$, Corollary 2.5 would give the coefficient bound inequality for the subclass of functions $\mathscr{S}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$.

THEOREM 2.6. **(Extreme Points)** *Let*

$$f_1(z) = z \quad \text{and}$$
$$f_k(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}z^k, \quad k \geqslant 2, \tag{2.6}$$

for $0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ \lambda \geqslant 0$ and $0 < \gamma \leqslant 1$. Then $f(z)$ is in the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{2.7}$$

where $\mu_k \geqslant 0$ *and* $\sum\limits_{k=1}^{\infty} \mu_k = 1$.

Proof. Suppose $f(z)$ can be written as in (2.7). Then

$$f(z) = z - \sum_{k=2}^{\infty} \mu_k \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k} z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}{2\beta\gamma(1-\alpha)} \mu_k \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}$$
$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leqslant 1.$$

Thus $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$. Conversely, let us have $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$. Then by using (2.4), we set

$$\mu_k = \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}{2\beta\gamma(1-\alpha)} a_k, \quad k \geqslant 2$$

and $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$. Then we have $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ and hence this completes the proof of Theorem 2.6. □

COROLLARY 2.7. *Let*

$$f_1(z) = z \text{ and}$$
$$f_k(z) = z - \frac{2\beta\gamma(1-\alpha)}{\Gamma_n(1+\lambda(k-1))[1+\beta(2\gamma-1)]} z^k, \quad k \geqslant 2, \tag{2.8}$$

for $0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ \lambda \geqslant 0$ *and* $0 < \gamma \leqslant 1$. *Then* $f(z)$ *is in the class* $\mathscr{HT}(\lambda,\alpha,\beta,\gamma)$ *if and only if it can be expressed in the form* $f(z) = \sum\limits_{k=1}^{\infty} \mu_k f_k(z)$, *where* $\mu_k \geqslant 0,\ \sum\limits_{k=1}^{\infty} \mu_k = 1$ *and* Γ_n *is given by (2.5).*

REMARK 6. For specific choices of parameters l, m, α_1, β_1 (as mentioned in the Remarks 1 and 2), Corollary 2.7 we can prove analogous results for the subclasses of functions $\mathscr{LT}(\lambda,\alpha,\beta,\gamma)$ and $\mathscr{RT}^{\nu}(\lambda,\alpha,\beta,\gamma)$. Further on lines similar to the above theorem one can easily prove the extreme points results for the classes $\mathscr{IT}(\lambda,\alpha,\beta,\gamma)$ and $\mathscr{JT}(\lambda,\alpha,\beta,\gamma)$.

3. Distortion Bounds

In this section we obtain distortion bounds for the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$.

THEOREM 3.1. *If $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$, then*

$$r - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}r^2 \leqslant |f(z)| \leqslant r + \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}r^2 \tag{3.1}$$

$$1 - \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}r \leqslant |f'(z)| \leqslant 1 + \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}r. \tag{3.2}$$

The bounds in (3.1) *and* (3.2) *are sharp, since the equalities are attained by the function*

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}z^2 \quad z = \pm r. \tag{3.3}$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} a_k \leqslant \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2} \tag{3.4}$$

Using (1.11) and (3.4), we obtain

$$|z| - |z|^2\sum_{k=2}^{\infty} a_k \leqslant |f(z)| \leqslant |z| + |z|^2\sum_{k=2}^{\infty} a_k$$

$$r - r^2\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2} \leqslant |f(z)| \leqslant r + r^2\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}. \tag{3.5}$$

Hence (3.1) follows from (3.5).

Further, since

$$\sum_{k=2}^{\infty} ka_k \leqslant \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]b_2}.$$

Hence (3.2) follows from

$$1 - r\sum_{k=2}^{\infty} ka_k \leqslant |f'(z)| \leqslant 1 + r\sum_{k=2}^{\infty} ka_k. \qquad \square$$

COROLLARY 3.2. *If $f \in \mathscr{H}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$, then*

$$r - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\Gamma_2}r^2 \leqslant |f(z)| \leqslant r + \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\Gamma_2}r^2 \tag{3.6}$$

$$1 - \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\Gamma_2}r \leqslant |f'(z)| \leqslant 1 + \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\Gamma_2}r. \tag{3.7}$$

The bounds in (3.6) *and* (3.7) *are sharp for*

$$f(z)=z-\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\Gamma_2}z^2 \tag{3.8}$$

where Γ_2 *is given by* (2.5)*.*

REMARK 7. For specific choices of parameters l,m,α_1,β_1 (as mentioned in the Remarks 1 and 2), Corollary 3.2 we can deduce analogous results for the subclasses of functions $\mathscr{L}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ and $\mathscr{R}\mathscr{T}^{v}(\lambda,\alpha,\beta,\gamma)$. Further on lines similar to the distortion theorem one can easily prove the distortion bounds for the classes $\mathscr{I}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$ and $\mathscr{J}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$.

COROLLARY 3.3. *If* $f\in\mathscr{S}\mathscr{T}(\lambda,\alpha,\beta,\gamma)$*, then*

$$r-\frac{\beta\gamma(1-\alpha)}{2^{m-1}(1+\lambda)[1+\beta(2\gamma-1)]}r^2 \leqslant |f(z)| \leqslant r+\frac{\beta\gamma(1-\alpha)}{2^{m-1}(1+\lambda)[1+\beta(2\gamma-1)]}r^2 \tag{3.9}$$

$$1-\frac{\beta\gamma(1-\alpha)}{2^{m-2}(1+\lambda)[1+\beta(2\gamma-1)]}r \leqslant |f'(z)| \leqslant 1+\frac{\beta\gamma(1-\alpha)}{2^{m-2}(1+\lambda)[1+\beta(2\gamma-1)]}r. \tag{3.10}$$

The bounds in (3.9) *and* (3.10) *are sharp for*

$$f(z)=z-\frac{\beta\gamma(1-\alpha)}{2^{m-1}(1+\lambda)[1+\beta(2\gamma-1)]}z^2. \tag{3.11}$$

4. Radius of Starlikeness and Convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ are given in this section.

THEOREM 4.1. *Let the function* $f(z)$ *defined by* (1.11) *belong to the class* $\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$*. Then* $f(z)$ *is close-to-convex of order* δ_1 $(0\leqslant\delta_1<1)$ *in the disc* $|z|<r_1$*, where*

$$r_1:=\left[\frac{(1-\delta_1)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,b_k}{2k\beta\gamma(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k\geqslant 2). \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ *given by* (2.6)*.*

Proof. Given $f\in T$ and f is close-to-convex of order δ_1, we have

$$|f'(z)-1|<1-\delta_1. \tag{4.2}$$

For the left hand side of (4.2) we have

$$|f'(z)-1| \leqslant \sum_{k=2}^{\infty} ka_k|z|^{k-1}.$$

The last expression is less than $1-\delta_1$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta_1} a_k|z|^{k-1} < 1.$$

Using the fact, that $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\, b_k}{2\beta\gamma(1-\alpha)} \leqslant 1.$$

We can say (4.2) is true if

$$\frac{k}{1-\delta_1}|z|^{k-1} \leqslant \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k}{2\beta\gamma(1-\alpha)}.$$

Or, equivalently,

$$|z|^{k-1} = \left[\frac{(1-\delta_1)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\, b_k}{2k\beta\gamma(1-\alpha)}\right]$$

which completes the proof. $\square$

THEOREM 4.2. *Let $f \in \mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$. Then*

(1) *f is starlike of order δ_1 $(0 \leqslant \delta_1 < 1)$ in the disc $|z| < r_2$; that is,* $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta_1, \quad (|z| < r_2\,;\, 0 \leqslant \delta_1 < 1)$, *where*

$$r_2 = \inf_{k \geqslant 2}\left\{\frac{(1-\delta_1)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\, b_k}{2\beta\gamma(1-\alpha)(k-\delta_1)}\right\}^{\frac{1}{k-1}}.$$

(2) *f is convex of order δ_1 $(0 \leqslant \delta_1 < 1)$ in the disc $|z| < r_3$, that is* $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta_1,\ (|z| < r_3; 0 \leqslant \delta_1 < 1)$, *where*

$$r_3 = \inf_{k \geqslant 2}\left\{\frac{(1-\delta_1)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\, b_k}{2\beta\gamma(1-\alpha)k(k-\delta_1)}\right\}^{\frac{1}{k-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.6).

Proof. Given $f \in T$ and f is starlike of order δ_1, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta_1. \tag{4.3}$$

For the left hand side of (4.3) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqslant \frac{\sum_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\,|z|^{k-1}}.$$

The last expression is less than $1-\delta_1$ if

$$\sum_{k=2}^{\infty}\frac{k-\delta_1}{1-\delta_1}a_k\,|z|^{k-1}<1.$$

Using the fact, that $f\in\mathbb{PT}_g(\lambda,\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\,b_k}{2\beta\gamma(1-\alpha)}<1.$$

We can say (4.3) is true if

$$\frac{k-\delta_1}{1-\delta_1}|z|^{k-1}<\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,b_k}{2\beta\gamma(1-\alpha)}.$$

Or, equivalently,

$$|z|^{k-1}<\frac{(1-\delta_1)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,b_k}{2\beta\gamma(1-\alpha)(k-\delta_1)}$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar the proof of (i). $\square$

REMARK 8. For specific choices of parameters l,m,α_1,β_1 we can deduce analogous results for the subclasses of functions introduced in this paper.

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