Journal of
Mathematical
Inequalities
Volume 3, Number 4 (2009), 567–576

CONVERGENCE RATE IN MULTIDIMENSIONAL IRREGULAR SAMPLING RESTORATION

ANDRIY YA. OLENKO AND TIBOR K. POGÁNY

Abstract. New magnitude of truncation error upper bound and convergence rate are obtained for Whittaker–Kotel'nikov–Shannon (WKS) sampling restoration sum for Bernstein function class $B^q_{\pi,d}$, $q \geqslant 1$, $d \in \mathbb{N}$, when the sampled functions decay rate is unknown. The case of multidimensional irregular sampling is discussed.

1. Introduction

The classical WKS sampling theorem has been extended to the case of nonuniform sampling by numerous authors. For detailed information on the theory and its various applications, we refer to [3, 6].

Most known irregular sampling results deal with Paley–Wiener functions which have $L^2(\mathbb{R})$ restrictions on the real line. It seems that the best known nonuniform WKS sampling results for entire functions in L^p–spaces were given in [4, 5, 19]. However, there are no explicit truncation error upper bounds in multidimensional WKS reconstructions in open literature. Recently the authors derived multidimensional L^p–WKS sampling theorems with precise truncation error estimates, see [11, 12] for more details and discussions.

In this paper we use methods developed in [11, 12] to investigate multidimensional irregular sampling in L^p-spaces. New magnitude of truncation error upper bound and convergence rate are obtained.

2. Multidimensional Plancherel–Pólya inequality

To prove the main theorem we need the multidimensional analog of the Plancherel–Pólya inequality, see [11].

Various multidimensional Plancherel–Pólya inequalities can be found e.g. in Triebel's book [18]; also, during last years several additional very far going generalizations of the multidimensional Plancherel–Pólya inequality were obtained, among others in [2] and [13]. Unfortunately, no explicit estimates of the Plancherel–Pólya constant appeared neither in these articles, nor in articles referenced therein including [14].

Mathematics subject classification (2000): Primary 94A20, 26D15; Secondary 30D15, 41A05.

Keywords and phrases: Whittaker–Kotel'nikov–Shannon sampling restoration formula; approximation/interpolation error level; truncation error upper bound; irregular sampling; multidimensional sampling; convergence rate.

The authors were supported by the La Trobe University Research Grant-501821 "Sampling, wavelets and optimal stochastic modelling"..

© ELEMENT, Zagreb
Paper JMI-03-55

In course to expose our estimate upon the multidimensional Plancherel–Pólya constant we make the following few conventions: **(i)** denote $\|\cdot\|_p$ the L_p-norm in finite case (while $\|\cdot\|_\infty \equiv \operatorname{ess\,sup}|\cdot|$), and **(ii)** hereinafter $B^r_{\boldsymbol{\sigma},d}$, $r>0$ denotes the Bernstein class [7] of d–variable entire functions of exponential type at most $\boldsymbol{\sigma}=(\sigma_1,\cdots,\sigma_d)$ coordinatewise whose restriction to $\mathbb{R}^d$ is in $L^r(\mathbb{R}^d)$.

THEOREM 1. [11, Theorem 1] *Let* $\mathfrak{T}=\{t_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $t_{\mathbf{n}}=(t_{n_1},...t_{n_d})$ *be real separated sequence, i.e.*

$$\inf_{n_\ell\neq m_\ell}|t_{n_\ell}-t_{m_\ell}|\geqslant\delta_\ell>0,\qquad \ell=\overline{1,d}\,.$$

Let $f\in B^r_{\boldsymbol{\sigma},d}$, $r\geqslant 1$. *Then*

$$\sum_{\mathbf{n}\in\mathbb{Z}^d}|f(t_{\mathbf{n}})|^r\leqslant\mathfrak{B}_{d,r}\|f\|_r^r\,,$$

where

$$\mathfrak{B}_{d,r}=\left(\frac{8}{r\pi}\right)^d\prod_{\ell=1}^d\frac{e^{r\delta_\ell\sigma_\ell/2}-1}{\sigma_\ell\delta_\ell^2}\,.$$

3. Multidimensional irregular sampling

In this section we introduce the multidimensional sampling theorem for $B^r_{\boldsymbol{\sigma},d}$ functional class. The theorem was proven in [12].

Let $\mathfrak{T}:=\{t_{\mathbf{n}}=\mathbf{n}+h_{\mathbf{n}}, mathfrakh:=(h_1,\cdots,h_n), \mathbf{N}:=(N_1,\cdots,N_d)\in\mathbb{N}^d$, while

$$\mathfrak{J}_{\mathbf{x}}:=\left\{\mathbf{n}:\bigwedge_{j=1}^d(|x_j-n_j|\leqslant N_j)\right\}$$

and

$$S(\mathbf{x},t_{\mathbf{n}})=\prod_{j=1}^d\frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})}\,,\tag{1}$$

where $G'_N(x,t)$ denotes a derivative with respect to t, being

$$G_N(x,t)=(t-h_0)\operatorname{sinc}(t)\prod_{\substack{|x-k|\leqslant N\\ k\neq 0}}\left(1-\frac{h_k}{t-k}\right)\frac{k}{t_k}\,,\tag{2}$$

$$\operatorname{sinc}(t):=\begin{cases}\dfrac{\sin(\pi t)}{\pi t} & \text{if } t\neq 0\,,\\ 1 & \text{if } t=0\,.\end{cases}$$

Denote $\mathbf{M}=(M_1,\cdots,M_d)$, $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_d)$, $\widetilde{M}:=\max_{j=\overline{1,d}}M_j$. Assume that $t_{n_j}=n_j+h_{n_j}$, $|h_{n_j}|\leqslant M_j$, $j=\overline{1,d}$; for all $\mathbf{n}\in\mathfrak{J}_{\mathbf{x}}$.

THEOREM 2. [12, Theorem 2] *Let* $f \in B^q_{\boldsymbol{\sigma},d}$, $q \geqslant 1$, $\sigma_j \leqslant \pi$ *for all* j, $\mathfrak{T} = \{t_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ *be real separated sequence with*

$$\widetilde{M} \leqslant \frac{1}{4} \quad \textit{for} \quad q=1 \quad \textit{and} \quad \widetilde{M} < \frac{1}{4q} \quad \textit{for} \quad 1<q<\infty. \tag{3}$$

Then the sampling expansion

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{t_n}) \prod_{j=1}^{d} \frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})}, \tag{4}$$

holds uniformly on each bounded $\mathbf{x}$*–subset of* $\mathbb{R}^d$. *Moreover, the series in* (4) *converges absolutely too.*

In this framework the sampling restoration procedure becomes of Lagrange–Yen type [1, 6, 15, 21].

For the general case of multidimensional irregular sampling with window canonical product sampling function $S(\mathbf{x},t_{\mathbf{n}})$ we have time–jittered nodes outside $\mathfrak{J}_{\mathbf{x}}$. Nonvanishing time-jitter $\mathfrak{h}$ outside $\mathfrak{J}_{\mathbf{x}}$ leads to functions $G_N(x,t)$ given by formulae different from (2), see [4]. However, in irregular sampling applications we would like to approximate $f(\mathbf{x})$ using only it's values from sample nodes $t_{\mathbf{n}}$ indexed by $\mathfrak{J}_{\mathbf{x}}$. Therefore we can try to use the truncated to $\mathfrak{J}_{\boldsymbol{x}}$ sampling approximation sum

$$Y_{\mathfrak{J}_{\mathbf{x}}}(f;\mathbf{x}) = \sum_{\mathbf{n}\in\mathfrak{J}_{\mathbf{x}}} f(t_{\mathbf{n}}) \prod_{j=1}^{d} \frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})} \tag{5}$$

with $G_N(x,t)$ (such that is given by (2)) even for arbitrary sample nodes outside $\mathfrak{J}_{\mathbf{x}}$. Under such assumptions the *truncation error*

$$\|T_{\mathbf{N},d}(f;\mathbf{x})\|_\infty = \|f(\mathbf{x}) - Y_{\mathfrak{J}_{\mathbf{x}}}(f;\mathbf{x})\|_\infty$$

coincides with truncation error for the case given by (1)–(4). One can see that $T_{\mathbf{N},d}(f;\mathbf{x})$ depends on nonvanishing $\mathfrak{h}$ in $\mathfrak{J}_{\mathbf{x}}$ due to multiplicative form of (4) and (5). Therefore, the multidimensional sampling problems are more difficult than the one–dimensional ones.

4. Truncation error upper bounds

In this section we obtain universal truncation bounds for multidimensional irregular sampling restoration procedure.

The most frequently appearing estimate of the truncation error is of the form

$$\|T_{\mathfrak{J}}(f;\mathbf{x})\|_\infty \leqslant \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\mathfrak{J}} |S(\mathbf{x},t_{\mathbf{n}})|^p\Big)^{1/p} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\mathfrak{J}} |f(t_{\mathbf{n}})|^q\Big)^{1/q} =: A_p B_q, \tag{6}$$

p,q being a conjugated Hölder pair, i.e. $1/p+1/q=1$.

To obtain a class of truncation error upper bounds when the decay rate of the initial signal function is not known one operates with the straightforward $B_q \leqslant C_{f,\mathfrak{T}} \|f\|_q$ where $C_{f,\mathfrak{T}}$ is suitable absolute constant. Thus (6) becomes

$$\|T_{\mathfrak{J}}(f;\mathbf{x})\|_\infty \leqslant C_{f,\mathfrak{T}} A_p \|f\|_q.$$

We are interested in estimates for A_p such that vanish with $|\mathfrak{J}_{\mathbf{x}}| \to \infty$. Therefore, the obtained upper bounds really will be universal for wide classes of $f(\mathbf{x})$ and $\mathfrak{T}$.

We will use the following two-sided bounds for a ratio of gamma functions, see [16, 17, 20].

LEMMA 1. *Let $z > 0$, $b \in (0,1)$. Then*

$$\frac{z}{(z+b)^{1-b}} \leqslant \frac{\Gamma(z+b)}{\Gamma(z)} \leqslant z^b. \tag{7}$$

THEOREM 3. *Let $\widetilde{M}$ satisfy* (3), *$f \in B^q_{\boldsymbol{\sigma},d}$, $q \geqslant 1$, $\sigma_j \leqslant \pi$ for all $j = \overline{1,d}$. Then we have*

$$\|T_{\mathbf{N},d}(f,\mathbf{x})\|_\infty \leqslant K_{\boldsymbol{\delta}}(\mathbf{N},\mathbf{M}) \cdot \|f\|_q \tag{8}$$

where

$$K_{\boldsymbol{\delta}}(\mathbf{N},\mathbf{M}) = \left(\frac{8}{q\pi^2}\right)^{d/q} \prod_{j=1}^{d} \frac{(e^{q\pi\delta_j/2}-1)^{1/q}}{\delta_j^{2/q}} \left(\sum_{k=1}^{d} \Big\{ C_1(N_k,M_k) \right.$$
$$\left. \times \prod_{\substack{j=1\\ j\neq k}}^{d} \Big(C_1(N_j,M_j) + C_2(M_j,\delta) C_3(N_j,M_j) \Big) \Big\} \right)^{1/p}, \tag{9}$$

and

$$C_1(N,M) := \frac{2^{Mp+2p+1}\pi^p (M+1/2)^p}{3^p \Gamma^{2p}(M+1/2)} \left(1 + \frac{N}{p-1}\right) \left(\frac{(1+N)^M (N-1/2)}{N(N-M-1/2)}\right)^p (N+3/2)^{2Mp};$$

$$C_2(M,\delta) := \left(\frac{3}{4M}\right)^p \left(\frac{\pi}{2}\right)^{2Mp-p} \left(\frac{e}{M\sqrt{\delta}}\right)^{4Mp};$$

$$C_3(N,M) := 2\,(M+3/2)^{4pM-p} + \frac{2\,(N+M+3/2)^{4pM-p+1} - 2\,(M+3/2)^{4pM-p+1}}{4pM-p+1}.$$

Proof. As we mentioned earlier, the function $Y_{\mathfrak{J}_{\mathbf{x}}}(f;\mathbf{x})$ does not depend on samples in $\mathfrak{T} \setminus \{t_{\mathbf{n}} : \mathbf{n} \in \mathfrak{J}_{\mathbf{x}}\}$ and we assume that outside $\mathfrak{J}_{\mathbf{x}}$ it is $\mathfrak{h} \equiv 0$. Therefore, the structure of $\{t_{n_j} : n_j \notin \mathfrak{J}_{x_j}\}$ becomes uniform $t_{n_j} \equiv n_j$. In this case Theorem 2 guarantees that $f(\mathbf{x})$ admits the representation (4), and the so evaluated model (6) gives us

$$\left|T_{\mathbf{N},d}(f;\mathbf{x})\right| \leqslant \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\mathfrak{J}_{\mathbf{x}}} \prod_{j=1}^{d} \Big| \frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})} \Big|^p \Big)^{1/p} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\mathfrak{J}_{\mathbf{x}}} |f(\mathbf{n})|^q \Big)^{1/q} =: A_p \cdot B_q\,.$$

The multiplicative structure of $S(\mathbf{x},t_{\mathbf{n}})$ enables to estimate A_p in the following way

$$\begin{aligned}
A_p^p &\leqslant \sum_{k=1}^{d} \sum_{n_k\in\mathbb{Z}\setminus\mathfrak{J}_{x_k}} \left|\frac{G_{N_k}(x_k,x_k)}{G'_{N_k}(x_k,n_k)(x_k-n_k)}\right|^p \prod_{\substack{j=1\\ j\neq k}}^{d} \sum_{n_j\in\mathbb{Z}} \left|\frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})}\right|^p \\
&= \sum_{k=1}^{d} \sum_{n_k\in\mathbb{Z}\setminus\mathfrak{J}_{x_k}} \left|\frac{G_{N_k}(x_k,x_k)}{G'_{N_k}(x_k,n_k)(x_k-n_k)}\right|^p \prod_{\substack{j=1\\ j\neq k}}^{d} \left(\sum_{n_j\in\mathbb{Z}\setminus\mathfrak{J}_{x_j}} \left|\frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,n_j)(x_j-n_j)}\right|^p \right. \\
&\qquad \left. + \sum_{n_j\in\mathfrak{J}_{x_j}} \left|\frac{G_{N_j}(x_j,x_j)}{G'_{N_j}(x_j,t_{n_j})(x_j-t_{n_j})}\right|^p \right). \qquad (10)
\end{aligned}$$

Let us estimate $\sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \left|\frac{G_N(x,x)}{G'_N(x,n)(x-n)}\right|^p$. Note, that due to our assumptions

$$|\psi_N(n,x)| := \left|\frac{G_N(x,x)}{G'_N(x,t_n)(x-t_n)}\right| = \left|\frac{\sin(\pi x)}{\pi(x-n)} \prod_{j\in\mathfrak{J}_x} \frac{(t_j-x)(j-n)}{(t_j-n)(j-x)}\right| \qquad n\in\mathbb{Z}\setminus\mathfrak{J}_x.$$

Hence

$$|\psi_N(n,x)| = \left|\operatorname{sinc}(x-j_x)\frac{(t_{j_x}-x)(j_x-n)}{(x-n)(t_{j_x}-n)} \prod_{\substack{|j-x|\leqslant N\\ j\neq j_x}} \frac{(t_j-x)(j-n)}{(t_j-n)(j-x)}\right|$$

where j_x denotes the index closest to x, i.e. $j_x-0.5\leqslant x<j_x+0.5$.

Due to $|h_{j_x}|\leqslant M$ we have

$$\left|\operatorname{sinc}(x-j_x)\,\frac{t_{j_x}-x}{x-n}\right| \leqslant \frac{M+1/2}{|x-n|} \qquad (11)$$

and

$$\left|\frac{j_x-n}{t_{j_x}-n}\right| \leqslant 1+\frac{M}{|x-n|-M-1/2} \leqslant 1+\frac{M}{N-M-1/2}. \qquad (12)$$

Due to $|h_{j_x}|\leqslant M$ and lemma 1 one concludes

$$\begin{aligned}
\prod_{\substack{|j-x|\leqslant N\\ j\neq j_x}} \left|\frac{t_j-x}{j-x}\right| &\leqslant \prod_{\substack{|j-x|\leqslant N\\ j\neq j_x}} \frac{|j-x|+M}{|j-x|} \leqslant \left(\prod_{j=0}^{N} \frac{j+1/2+M}{j+1/2}\right)^2 \\
&= \left(\frac{\Gamma(1/2)}{\Gamma(M+1/2)}\right)^2 \left(\frac{\Gamma(N+1+1/2+M)}{\Gamma(N+1+1/2)}\right)^2 \\
&\leqslant \frac{\pi}{\Gamma^2(M+1/2)}\,(N+3/2)^{2M}. \qquad (13)
\end{aligned}$$

Let $k(n,x) := \max(|j_x - n| - N, 1)$. Then by lemma 1 and $|h_{j_x}| \leqslant M \leqslant 1/4$ we have

$$\begin{aligned}
\prod_{\substack{|j-x|\leqslant N\\ j\neq j_x}} \left|\frac{j-n}{t_j-n}\right| &\leqslant \prod_{\substack{|j-x|\leqslant N\\ j\neq j_x}} \frac{|j-n|}{|j-n|-M} = \prod_{k=k(n,x)}^{k(n,x)+2N} \frac{k}{k-M} \\
&= \frac{\Gamma(k(n,x)+2N+1)}{\Gamma(k(n,x))} \frac{\Gamma(k(n,x)-M)}{\Gamma(k(n,x)+2N+1-M)} \\
&\leqslant (k(n,x)+2N+1-M)^M \frac{k(n,x)}{(k(n,x)-M)(k(n,x))^M} \\
&\leqslant \frac{4}{3}\left(\frac{k(n,x)+2N+1}{k(n,x)}\right)^M . \qquad (14)
\end{aligned}$$

Collecting all estimates (11)–(14), we deduce

$$\left|\psi_N(n,x)\right| \leqslant \frac{1}{|x-n|}\left(\frac{k(n,x)+2N+1}{k(n,x)}\right)^M \frac{4\pi(M+1/2)}{3\Gamma^2(M+1/2)} \frac{N-1/2}{N-M-1/2}(N+3/2)^{2M}$$

and hence

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \left|\frac{G_N(x,x)}{G_N'(x,n)(x-n)}\right|^p = \sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \left|\psi_N(n,x)\right|^p &\leqslant \sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \frac{1}{|x-n|^p}\left(\frac{k(n,x)+2N+1}{k(n,x)}\right)^{Mp} \\
&\times \left(\frac{4\pi(M+1/2)}{3\Gamma^2(M+1/2)} \frac{N-1/2}{N-M-1/2}\right)^p (N+3/2)^{2Mp}.
\end{aligned}$$

Thus, we proceed evaluating

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \frac{1}{|x-n|^p}\left(\frac{k(n,x)+2N+1}{k(n,x)}\right)^{Mp} &= \sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \frac{1}{|x-n|^p}\left(1+\frac{2N+1}{k(n,x)}\right)^{Mp} \\
&\leqslant \frac{2(2+2N)^{Mp}}{N^p} + \int_N^\infty \frac{2}{t^p}\left(1+\frac{2N+1}{t-N+1}\right)^{Mp} dt \\
&= \frac{2(2+2N)^{Mp}}{N^p} + 2(2N+1)^{Mp}\int_N^\infty \frac{1}{t^p}\left(\frac{1}{2N+1}+\frac{1}{t-N+1}\right)^{Mp} dt \\
&\leqslant \frac{2(2+2N)^{Mp}}{N^p} + \frac{2(2N+1)^{Mp}}{(p-1)N^{p-1}}\left(1+\frac{1}{2N+1}\right)^{Mp} \\
&= \frac{2^{Mp+1}(1+N)^{Mp}}{N^p}\left(1+\frac{N}{p-1}\right),
\end{aligned}$$

such that gives

$$\sum_{n\in\mathbb{Z}\setminus\mathfrak{J}_x} \left|\frac{G_N(x,x)}{G_N'(x,n)(x-n)}\right|^p \leqslant C_1(N,M).$$

Now, let us evaluate $\sum_{n\in\mathfrak{J}_x} |\psi_N(n,x)|^p$, the second addend in (10).

We can rewrite the function $G_N(x,t)$ into

$$G_N(x,t) = (t-h_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{\widetilde{t_n}}\right)\left(1-\frac{t}{\widetilde{t_{-n}}}\right),$$

where

$$\widetilde{t_n} = \begin{cases} t_n, & \text{if } |x-n| \leqslant N \\ n, & \text{else.} \end{cases}$$

Hence all results derived in [19] are valid for our function $G_N(x,t)$. By [19, Lemma 1.4.4 (a)] and [19, inequality (3.7)] we obtain

$$|\psi_N(n,x)| \leqslant \frac{3}{4M}\left(\frac{\pi}{2}\right)^{2M-1}\left(\frac{e}{M\sqrt{\delta}}\right)^{4M}(|x-t_n|+M+3/2)^{4M-1}.$$

Therefore

$$\begin{aligned}\sum_{n\in\mathfrak{I}_x}|\psi_N(n,x)|^p &\leqslant \left(\frac{3}{4M}\right)^p\left(\frac{\pi}{2}\right)^{2Mp-p}\left(\frac{e}{M\sqrt{\delta}}\right)^{4Mp}\sum_{n\in\mathfrak{I}_x}(|x-t_n|+M+3/2)^{4pM-p}\\ &= C_2(M,\delta)\sum_{n\in\mathfrak{I}_x}(|x-t_n|+M+3/2)^{4pM-p}.\end{aligned}$$

Finally for M satisfying (3) we have the estimate

$$\begin{aligned}\sum_{n\in\mathfrak{I}_x}(|x-t_n|+M+3/2)^{4pM-p} &\leqslant 2\left((M+3/2)^{4pM-p}+\int_0^N(t+M+3/2)^{4pM-p}\,dt\right)\\ &= 2\left((M+3/2)^{4pM-p}+\frac{(N+M+3/2)^{4pM-p+1}-(M+3/2)^{4pM-p+1}}{4pM-p+1}\right) = C_3(N,M),\end{aligned}$$

such that gives

$$\sum_{n\in\mathfrak{I}_x}\left|\frac{G_N(x,x)}{G_N'(x,t_n)(x-t_n)}\right|^p \leqslant C_2(M,\delta)C_3(N,M).$$

Therefore, by (10)

$$A_p^p \leqslant \sum_{k=1}^{d}\left(C_1(N_k,M_k)\prod_{\substack{j=1\\ j\neq k}}^{d}\Big(C_1(N_j,M_j)+C_2(M_j,\delta)C_3(N_j,M_j)\Big)\right). \tag{15}$$

To estimate B_q we use Theorem 1 with $\sigma_1=\cdots=\sigma_d=\pi$. This results in

$$B_q^q \leqslant \mathfrak{B}_{d,q} = \left(\frac{8}{q\pi^2}\right)^d\prod_{j=1}^{d}\frac{e^{q\pi\delta_j/2}-1}{\delta_j^2}.$$

Now, collecting all these involved estimates, we arrive at (8) and (9).

REMARK 1. Theorem 3 gives new truncation error upper bounds and the method to obtain them, compare with results in [12]. However, it has to be pointed out that there are many different bounds on the so-called Gautschi–Kershaw ratio of two gamma functions [17]. Applying instead of the Lemma 1 other estimates we obtain a set of truncation error upper bounds. We used (7) since its simplicity and elegance.

Denote here, and in what follows $\underline{\delta} := \min_{j=\overline{1,d}} \delta_j$, $\overline{\delta} := \max_{j=\overline{1,d}} \delta_j$.

COROLLARY 3.1. *Suppose that the conditions of Theorem* 3 *are satisfied. Then, we have*

$$\|T_{\mathbf{N},d}(f,\mathbf{x})\|_\infty \leqslant \widetilde{K}_{\underline{\delta},\overline{\delta}}(\mathbf{N},\widetilde{M})\,\|f\|_q$$

where

$$\widetilde{K}_{\underline{\delta},\overline{\delta}}(\mathbf{N},\widetilde{M}) = \left(\frac{8}{q\pi^2}\right)^{d/q} \left(\sum_{k=1}^{d} \left\{ C_1(N_k,\widetilde{M}) \prod_{\substack{j=1\\ j\neq k}}^{d} \Big(C_1(N_j,\widetilde{M}) + C_2(\widetilde{M},\underline{\delta}) C_3(N_j,\widetilde{M}) \Big) \right\} \right)^{1/p} \left(\max\left\{ \frac{e^{q\pi\underline{\delta}/2}-1}{\underline{\delta}^2}, \frac{e^{q\pi\overline{\delta}/2}-1}{\overline{\delta}^2} \right\} \right)^{d/q}. \tag{16}$$

Proof. If we make use of the estimate (15) for A_p with $\widetilde{M}$ instead of all M_j and $\underline{\delta}$ instead of all other δ_j, $\mathfrak{T}$ could contain some additional $\mathbf{t_n}$ which might causes only increasing A_p.

It was shown in [12] that

$$\mathfrak{B}_d{}^{1/d} \leqslant \max\left\{ \frac{8(e^{q\pi\underline{\delta}/2}-1)}{q\pi^2\underline{\delta}^2}, \frac{8(e^{q\pi\overline{\delta}/2}-1)}{q\pi^2\overline{\delta}^2} \right\}.$$

Collecting the involved estimates, we arrive at (16).

THEOREM 4. *Suppose that the conditions of Theorem* 3 *are satisfied. Let* $\widetilde{N} = \min_{j=\overline{1,d}} N_j \to \infty$ *in such way that* $\max_{k,j=\overline{1,d}} N_j/N_k = \mathscr{O}(1)$. *Then it holds*

$$\|T_{\mathbf{N},d}(f;\mathbf{x})\|_\infty = \mathscr{O}\big(\widetilde{N}^{3\widetilde{M}p-p+1}\big) \to 0.$$

Proof. According to definitions of $C_j,\ j=\overline{1,3}$ from Theorem 3, letting $\widetilde{N}\to\infty$, we get

$$C_1(N,M) = \mathscr{O}\big(N^{3Mp-p+1}\big); \quad C_2(M,\delta) = \mathscr{O}(1); \quad C_3(N,M) = \mathscr{O}\big(1+N^{4Mp-p+1}\big).$$

Having in mind these facts by (16) we deduce

$$\widetilde{K}^p_{\underline{\delta},\overline{\delta}}(\mathbf{N},\widetilde{M}) \sim \sum_{k=1}^{d} \left\{ \mathscr{O}\big(N_k^{3\widetilde{M}p-p+}\big) \prod_{\substack{j=1\\ j\neq k}}^{d} \Big(\mathscr{O}\big(N_j^{3\widetilde{M}p-p+1}\big) + \mathscr{O}\big(1+N_j^{4\widetilde{M}p-p+1}\big) \Big) \right\}$$

$$\sim \mathscr{O}\big(\widetilde{N}^{3\widetilde{M}p-p+1}\big) \prod_{\substack{j=1\\ N_j\neq\widetilde{N}}}^{d} \Big(\mathscr{O}\big(N_j^{3\widetilde{M}p-p+1}\big)+\mathscr{O}(1)\Big)\Big\} \sim \mathscr{O}\big(\widetilde{N}^{3\widetilde{M}p-p+1}\big),$$

when $\widetilde{N}\to\infty$ if $\widetilde{M}$ satisfies (3). The proof is complete.

REMARK 2. Theorem 4 improves the results on convergence rates derived in [12]. Namely, by specifying the sampling size numbers $\widetilde{N}$ and the irregular sampling deviation bound $\widetilde{M}$, involving the Gautschi–Kershaw inequality for the ratio of Gamma functions (Lemma 1), and Voss's upper bound on $\psi_N(n,x)$, we

- relax the conditions upon $\widetilde{M}$, with respect to the earlier condition

$$\widetilde{M} < \frac{1}{q(4d-1)};$$

- obtain a better rate of convergence, with respect to the earlier one

$$\mathscr{O}\big(\widetilde{N}^{3\widetilde{M}p-p+1}\cdot N^{4\widetilde{M}(d-1)}\big)$$

appearing in [12, Corollary 3.2].

5. Final remarks

Methods proposed by authors give an opportunity to obtain approximation error estimates for wide functional classes without strong assumptions on functions/signals decay rate behaviour.

In this article new magnitude of truncation error upper bounds and rates of convergence were obtained in $\|\cdot\|_\infty$–norm sampling theorem; the case of multidimensional irregular sampling was considered.

All presented results and the numerical simulations such that illustrate the achieved results open few new interesting and important problems:

1. To obtain sharp estimates in Theorem 3 (for uniform sampling and $p=2$ such sharp estimates were derived in [10]);
2. To obtain the best possible rate of convergence in Theorem 4;
3. To apply the obtained results to irregular sampling restoration for random fields, see [8, 9].

REFERENCES

[1] FLORNES K.M., LYUBARSKII YU. AND SEIP K., *A direct interpolation method for irregular sampling*, Appl. Comput. Harmon. Anal., **8**, 1 (2000), 113–121.

[2] HAN Y.–S., *Plancherel–Pólya type inequality on spaces of homogeneous type and its applications*, Proc. Amer. Math. Soc., **126**, 11 (1998), 3315–3327.

[3] HIGGINS J. R., *Sampling in Fourier and Signal Analysis: Foundations*, Clarendon Press, Oxford, 1996.
[4] HINSEN G., *Irregular sampling of bandlimited L^p-functions*, J. Approx. Theory, **72** (1993), 346–364.
[5] HINSEN G. AND KLÖSTERS D., *The sampling series as a limiting case of Lagrange interpolation*, Appl. Anal., **49** (1993), 49–60.
[6] MARVASTI F., ED., *Nonuniform Sampling: Theory and Practice*, Kluwer Academic Publishers, Boston, 2001.
[7] NIKOLSKIĬ S. M., *Approximation of functions of several variables and imbedding theorems*, Springer-Verlag, New York, 1975.
[8] OLENKO A. YA. AND POGÁNY T. K., *Direct Lagrange–Yen Type Interpolation of Random Fields*, Theor. Stoch. Proc., **9** (25), 3–4 (2003), 242–254.
[9] OLENKO A. YA. AND POGÁNY T. K., *Sharp exact upper bound for interpolation remainder of random processes*, Theor. Probab. Math. Statist., **71** (2004), 133–144.
[10] OLENKO A. YA. AND POGÁNY T. K., *On sharp exact bounds for remainders in multidimensional sampling theorem*, Sampl. Theory Signal Image Process, **6**, 3 (2007), 249–272.
[11] OLENKO A. YA. AND POGÁNY T. K., *Universal truncation error upper bounds in sampling restoration*, Georgian Math. J. (to appear).
[12] OLENKO A. YA. AND POGÁNY T. K., *Universal truncation error upper bounds in irregular sampling restoration*, Appl. Anal. (to appear).
[13] PESENSON I., *Plancherel–Pólya type inequalites for entire functions of exponential type in $L_p(\mathbb{R}^d)$*, J. Math. Anal. Appl., **330**, 2 (2007), 1194–1206.
[14] PLANCHEREL M. AND PÓLYA GY, *Fonctions entières et intégrales de Fourier multiples II*, Comment. Math. Helv., **10** (1937), 110–163.
[15] POGÁNY T. K., *Multidimensional Lagrange–Yen interpolation via Kotel'nikov–Shannon sampling formulas*, Ukr. Math. J., **50**, 11 (2003), 1810–1827.
[16] QI FENG, *A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality*, J. Comput. and Applied Math., **206**, 2 (2007), 1007–1014.
[17] QI FENG, *Bounds for the ratio of two Gamma functions*, RGMIA Res. Rep. Coll., **11**, 3 (2008), 1–63.
[18] TRIEBEL H., *Theory of Function spaces, II*, Birkhäuser Verlag, Basel, 1992.
[19] VOSS J. J., *Irregular sampling: error analysis, applications, and extensions*, In CD-ROM Nonuniform Sampling: Theory and Practice, Kluwer Academic Publishers, Boston, 2001.
[20] WENDEL J., *Note on the gamma function*, Amer. Math. Monthly, **55** (1948), 563–564.
[21] YEN J. L., *On nonuniform sampling of bandwidth limited signals*, IRE Trans. Circuit Theory, CT-3 (1956), 251–257.

(Received October 28, 2008)

Andriy Ya. Olenko
Department of Mathematics and Statistics
La Trobe University
Victoria, 3086
Australia
e-mail: a.olenko@latrobe.edu.au

Tibor K. Pogány
Faculty of Maritime Studies
University of Rijeka
51000 Rijeka
Studentska 2
Croatia
e-mail: poganj@brod.pfri.hr